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On Topologies Induced by Graphs Under Some Unary and Binary Operations

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Abstract. Let $G = (V(G), E(G))$ be any simple undirected graph. The *open hop neighborhood of* $v \in V(G)$ is the set $N^2_G(v) = \{u \in V(G) : d_G(u, v) = 2\}$. Then G induces a topology τ_G on $V(G)$ with base consisting of sets of the form $F^2_G[A] = V(G)\backslash N^2_G[A]$, where $N^2_G[A] = A \cup \{v \in V(G) : N^2_G(v) \cap A \neq \emptyset\}$ and A ranges over all subsets of $V(G)$. In this paper, we describe the topologies induced by the complement of a graph, the join, the corona, the composition and the Cartesian product of graphs.

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Key Words and Phrases: Join, Corona, Lexicographic product, Cartesian product, Open hop neighborhood

1. Introduction

Let $G = (V(G), V(H))$ be any simple undirected graph. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path joining u and v. Let $v \in V(G)$. The *neighborhood* of v is the set $N(v)$ consisting of all $u \in V(G)$ which are adjacent with v and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For any $A \subseteq V(G)$, $N(A) = \{x : xa \in E(G)$ for some $a \in A\}$ is called the *neighborhood of* A and $N[A] = N(A) \cup A$ is called the *closed neighborhood of* A. Moreover, for each $v \in V(G)$, the *open hop neighborhood of* v is the set $N^2_G(v) = \{u \in V(G) : d_G(u, v) = 2\}$ and the *closed hop neighborhood of* v is the set $N^2_G[v] = \{v\} \cup N^2_G(v)$. Also, for any $A \subseteq V(G)$, $N^2_G(A) = \{v \in V(G) : N^2_G(v) \cap A \neq \emptyset\}$ is called the *open hop neighborhood of* A and the set $N^2_G[A] = A \cup N^2_G(A)$ is the called *closed hop neighborhood of* A. Denote by $F^2_G[A]$ the *complement* of $N^2_G[A]$, i.e., $F^2_G[A] = V(G)\backslash N^2_G[A]$.

In 1983, Diesto and Gervacio in [5] proved that given a simple graph $G = (V(G), E(G))$, G induces a topology on $V(G)$, denoted by τ_G, with base consisting of sets of the form

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$F_G(A) = V(G)\backslash N_G(A)$, where $N_G(A) = A \cup \{x : xa \in E$ for some $a \in A\}$ and A ranges over all subsets of $V(G)$. Their construction was further investigated in [2], [3] and [6]. In particular, Canoy and Lemence in [2] described the topologies induced by the complement of a graph, the join of graphs, composition and Cartesian product of graphs.

In [1], Canoy and Gimeno presented another way of constructing a topology τ_G from a connected graph G by considering the family $\Omega(G) = \{F_G^2[A] : A \subseteq V(G)\}$ where $F_G^2[A] = \{x \in V(G) : x \notin A$ and $d_G(x,a) \neq 2$ for all $a \in A\}$. They showed that this family is a base for some topology τ_G on $V(G)$. This construction is also studied by Nianga et al, in [4] for any graph G. It is also shown that the family $\mathcal{B}_G = \{F_G^2[A] : A \subseteq V(G)\}$ and $\mathcal{S}_G = \{F_G^2[v] : v \in V(G)\}$ are, respectively, base and subbase for the topology τ_G on $V(G)$.

Concepts on Graph Theory and Topology are taken from [7] and [8], respectively.

2. Results

Definition 1. *The complement of graph G, denoted by $\overline{G}$ is the graph with $V(G) = V(\overline{G})$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$, where $u, v \in V(G) = V(\overline{G})$.*

Theorem 1. *Let G be any graph and $\overline{G}$ its complement. Then for each $v \in V(G)$,*

$$F_{\overline{G}}^2[v] = \begin{cases} F_G[v] \cup \left[\bigcap_{u \in F_G[v]} N_G(u)\right], & \text{if } F_G[v] \neq \emptyset \\ N_G(v), & \text{if } F_G[v] = \emptyset. \end{cases} \tag{1}$$

Proof. Let G be any graph and $\overline{G}$ its complement. Let $v \in V(G)$ and set $A = \cap_{u \in F_G[v]} N_G(u)$. Suppose $F_G[v] = \emptyset$. Then $N_G(v) = V(G)\backslash\{v\}$. Hence, v is an isolated vertex in $\overline{G}$. Thus, $F_{\overline{G}}^2[v] = N_G(v)$. Suppose $F_G[v] \neq \emptyset$. Let $u \in F_G[v]$. Then $u \neq v$ and $u \notin N_G(v)$. Hence, $u \neq v$ and $u \in N_{\overline{G}}(v)$. Thus, $u \in F_{\overline{G}}^2[v]$. Next, let $w \in A$. Then $w \in N_G(u)$ for all $u \in F_G[v]$. Since $u \notin N_G(v)$, it follows that $w \neq v$. Also, $w \notin N_{\overline{G}}(u)$ for all $u \in N_{\overline{G}}(v)$. It implies that $d_{\overline{G}}(w, v) \neq 2$. Hence, $w \in F_{\overline{G}}^2[v]$. Consequently, $F_G[v] \cup [\cap_{u \in F_G[v]} N_G(u)] \subseteq F_{\overline{G}}^2[v]$. Next, let $x \in F_{\overline{G}}^2[v]$. Then $x \neq v$ and $x \notin N_{\overline{G}}^2(v)$. If $x \in F_G[v]$, then we are done. Suppose $x \notin F_G[v]$. Then $x \in N_G(v)$. Suppose further that there exists $u \in F_G[v]$ such that $x \notin N_G(u)$. Thus, $u \in N_{\overline{G}}(v)$ and $x \in N_{\overline{G}}(u)$. Also, since $x \in N_G(v)$, $x \notin N_{\overline{G}}(v)$. Thus, $d_{\overline{G}}(x, v) = 2$, that is, $x \in N_{\overline{G}}^2(v)$, a contradiction. Therefore, $x \in N_G(u)$ for all $u \in F_G[v]$. This shows that $x \in A$. Accordingly, $F_{\overline{G}}^2[v] \subseteq F_G[v] \cup [\cap_{u \in F_G[v]} N_G(u)]$. This establishes the desired equality. □

Theorem 2. *Let G be any graph and $\overline{G}$ its complement. If v is an isolated vertex of G (or of $\overline{G}$), then $\{v\} \in \tau_G \cap \tau_{\overline{G}}$.*

Proof. Suppose v is an isolated vertex of G (or of $\overline{G}$). Then $\{v\} = F_G^2[V(G)\backslash\{v\}] = F_{\overline{G}}^2[V(\overline{G})\backslash\{v\}]$ and so, $\{v\} \in \mathcal{B}_G$ and $\{v\} \in \mathcal{B}_{\overline{G}}$. Thus, $\{v\} \in \tau_G$ and $\{v\} \in \tau_{\overline{G}}$. Therefore, $\{v\} \in \tau_G \cap \tau_{\overline{G}}$. □

Remark 1. *The converse of theorem* 17 *is not true.*

Consider $G = P_5 = [a,b,c,d,e]$. Then $\{e\} = F_G^2[a,b]$ and $\{e\} = F_{\overline{G}}[a,c]$. However, e is not an isolated vertex of G nor of $\overline{G}$.

Definition 2. *The join $G_1 + G_2$ of graphs G_1 and G_2 is the graph G with $V(G) = V(G_1) \cup V(G_2)$ and*

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and v} \in \text{V(G}_2)\}.$$

Theorem 3. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs and let $\emptyset \neq A \subseteq V(G)$ and $\emptyset \neq B \subseteq V(H)$. Then*

(*i*) $F_{G+H}^2[A] = V(H) \cup [\cap_{a \in A} N_G(a)]$;

(*ii*) $F_{G+H}^2[B] = V(G) \cup [\cap_{b \in B} N_H(b)]$ *and*

(*iii*) $F_G^2[\emptyset] = V(G) \cup V(H)$.

Proof. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. Let $\emptyset \neq A \subseteq V(G)$ and $\emptyset \neq B \subseteq V(H)$.

(*i*) Note that

$$N_{G+H}^2[A] = A \cup \{v \in V(G+H) : d_{G+H}(v,a) = 2 \text{ for some } a \in A\}.$$

Since $V(H) \subseteq N_{G+H}(A)$,

$$\begin{aligned} N_{G+H}^2[A] &= A \cup \{v \in V(G) : d_{G+H}(v,a) = 2 \text{ for some } a \in A\} \\ &= A \cup \{v \in V(G) : d_G(v,a) \neq 1 \text{ for some } a \in A\}. \end{aligned}$$

Hence,

$$F_{G+H}^2[A] = V(H) \cup [\cap_{a \in A} N_G(a)].$$

(*ii*) Similarly,

$$F_{G+H}^2[B] = V(G) \cup [\cap_{b \in B} N_H(b)].$$

(*iii*) Clearly, $F_{G+H}^2[\emptyset] = V(G) \cup V(H)$. □

Remark 2. *Let G be any graph and let $A_1, A_2 \subseteq V(G)$. Then*

$$N_G^2[A_1 \cup A_2] = N_G^2[A_1] \cup N_G^2[A_2].$$

Theorem 4. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. Then for any $A \subseteq V(G+H)$ such that $A \cap V(G) = A_G \neq \emptyset$ and $A \cap V(H) = A_H \neq \emptyset$,*

$$F_{G+H}^2[A] = F_{G+H}^2[A_G] \cap F_{G+H}^2[A_H].$$

Proof. Let $A \subseteq V(G+H)$. Suppose $A \cap V(G) = A_G \neq \emptyset$ and $A \cap V(H) = A_H \neq \emptyset$. Then $x \in F^2_{G+H}[A]$ if and only if $x \notin N^2_{G+H}[A]$. By Remark 2, $x \in F^2_{G+H}[A]$ if and only if $x \in F^2_{G+H}[A_G] \cap F^2_{G+H}[A_H]$. □

The next theorem follows from Theorem 3 (i) and (ii).

Corollary 1. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. Then for any $v \in V(G) \cup V(H)$,*

$$F^2_{G+H}[v] = \begin{cases} V(H) \cup N_G(v), & \text{if } v \in V(G) \\ V(G) \cup N_G(v), & \text{if } v \in V(H). \end{cases} \tag{2}$$

Definition 3. *The corona $G \circ H$ of graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies H and then forming the sum $\langle v \rangle + H^v = v + H^v$ for each $v \in V(G)$, where H^v is a copy of H corresponding to the vertex v.*

Theorem 5. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. Then for any $a \in V(G \circ H)$,*

$$F^2_{G\circ H}[a] = \begin{cases} F^2_G[a] \cup \left[\bigcup_{v \in V(G)\setminus N_G(a)} V(H^v) \right], & \text{if } a \in V(G) \\ N_{H^w}(a) \cup [V(G)\setminus N_G(w)] \cup \left[\bigcup_{v \in V(G)\setminus\{w\}} V(H^v) \right], & \text{if } a \in V(H^w) \end{cases} \tag{3}$$

Proof. Let $x \in F^2_{G\circ H}[a]$. Then $x \neq a$ and $x \notin N^2_{G\circ H}(a)$. Consider the following cases:

Case 1. Suppose $a \in V(G)$. If $x \in V(G)$, then $x \notin N^2_G(a)$ since $x \notin N^2_{G\circ H}(a)$. Hence, $x \in F^2_G[a]$. Suppose $x \notin V(G)$. Let $u \in V(G)$ such that $x \in V(H^u)$. If $u = a$, then $x \in V(H^u)$ and $u \in V(G)\setminus N_G(a)$. Suppose $u \neq a$. Since $x \notin N^2_G(a)$ and $d_{G\circ H}(a, y) = 2$ for all $y \in V(H^z)$ with $z \in N_G(a)$, it follows that $u \in V(G)\setminus N_G(a)$. Thus,

$$F^2_{G\circ H}[a] \subseteq F^2_G[a] \cup \left[\cup_{v\in V(G)\setminus N_G(a)} V(H^v) \right] = X.$$

Now, let $w \in X$. If $w \in F^2_G[a]$, then $w \notin N^2_G[a]$. Hence, $w \notin N^2_{G\circ H}[a]$. This implies that $w \in F^2_{G\circ H}[a]$. Suppose $w \in \cup_{v\in V(G)\setminus N_G(a)} V(H^v)$. Then there exists $v \in V(G)\setminus N_G(a)$ such that $w \in V(H^v)$. It follows that $w \neq a$ and $d_{G\circ H}(w, a) \neq 2$. Thus, $w \in F^2_{G\circ H}[a]$. Therefore,

$$F^2_G[a] \cup \left[\cup_{v\in V(G)\setminus N_G(a)} V(H^v) \right] \subseteq F^2_{G\circ H}[a].$$

Case 2. Suppose $a \in V(H^w)$ for some $w \in V(G)$. If $x = w$, then $x \in V(G)\setminus N_G(w)$. Suppose $x \neq w$. If $x \in V(G)$, then $d_G(x, w) \neq 1$ because $d_{G\circ H}(x, a) \neq 2$. Hence, $x \in V(G)\setminus N_G(w)$. Suppose $x \in V(H^q)$ for some $q \in V(G)$. If $q = w$, then $x \in V(H^w)$. Since $x \neq a$ and $a \in V(H^w)$, $x \in N_{H^w}(a)$ (otherwise, $d_{G\circ H}(a, x) = 2$). Suppose $q \neq w$. Then $x \in V(H^q)$ and $q \in V(G)\setminus\{w\}$. Thus,

$$z \in N_{H^w}(a) \cup [V(G)\setminus N_G(w)] \cup \left[\cup_{v\in V(G)\setminus\{w\}} V(H^v) \right] = Y.$$

Suppose now that $p \in Y$. If $p \in N_{H^w}(a)$, then $d_{G\circ H}(p,a) = d_{H^w}(p,a) = 1$. Hence, $p \in F^2_{G\circ H}[a]$. If $p \in V(G)\backslash N_G(w)$, then $d_{G\circ H}(p,w) = d_G(p,w) \neq 1$. Hence, $p \neq a$ and $d_{G\circ H}(a,p) \neq 2$. This implies that $p \in F^2_{G\circ H}[a]$. Finally, if $p \in \cup_{v\in V(G)\backslash\{w\}} V(H^v)$, then there exists $r \in V(G)\backslash\{w\}$ such that $p \in V(H^r)$. Since

$$d_{G\circ H}(a,p) = d_{G\circ H}(a,w) + d_{G\circ H}(r,w) + d_{G\circ H}(r,p) = 2 + d_{G\circ H}(r,w) \geq 3,$$

it follows that $p \in F^2_{G\circ H}[a]$. Therefore,

$$N_{H^w}(a) \cup [V(G)\backslash N_G(w)] \cup \left[\cup_{v\in V(G)\backslash\{w\}} V(H^v)\right] \subseteq F^2_{G\circ H}[a].$$

Accordingly, the desired equality follows. □

Definition 4. *The lexicographic product (composition) of graphs G and H, denoted by $G[H]$, is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u,v)(u',v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.*

Theorem 6. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be any two graphs and let $(v,a) \in V(G[H])$. Then*

$$F^2_{G[H]}[(v,a)] = \left(F^2_G[v] \times V(H)\right) \cup \left(\{v\} \times F^2_H[a]\right).$$

Proof. Note that $(x,q) \in F^2_{G[H]}[(v,a)]$ if and only if $(x,q) \neq (v,a)$ and $d_{G[H]}((x,q),(v,a)) \neq 2$. Consider the following cases:

Case 1. Suppose $x = v$. Then $q \neq a$. Since

$$d_{G[H]}((v,q),(v,a)) = d_H(a,q) \neq 2, q \in F^2_H[a],$$

$q \in F^2_H[a]$. Hence, $(x,q) \in \{v\} \times F^2_H[a]$.

Case 2. Suppose $x \neq v$. Then

$$d_G(x,v) = d_{G[H]}((x,q),(v,a)) \neq 2.$$

Hence, $x \in F^2_G[v]$ and $(x,q) \in F^2_G[v] \times V(H)$. Therefore,

$$F^2_{G[H]}[(v,a)] \subseteq \left(F^2_G[v] \times V(H)\right) \cup \left(\{v\} \times F^2_H[a]\right).$$

Next, let $(w,p) \in F^2_G[v] \times V(H)$. Then $w \in F^2_G[v]$, that is, $w \neq v$ and $d_G(w,v) \neq 2$. It follows that $(w,p) \neq (v,a)$ and

$$d_{G[H]}((w,p),(v,a)) = d_G(w,v) \neq 2.$$

This shows that $(w,p) \in F^2_{G[H]}[(v,a)]$. Hence, $F^2_G[v] \times V(H) \subseteq F_{G[H]}[(v,a)]$. Finally, let $(z,t) \in \{v\} \times F^2_H[a]$. Then $z = v$ and $t \in F^2_H[a]$. Hence, $t \neq a$ and $d_H(a,t) \neq 2$. Consequently, $(z,t) \neq (v,a)$ and

$$d_{G[H]}((z,t),(v,a)) = d_H(a,t) \neq 2,$$

showing that $(z,t) \in F^2_{G[H]}[a]$. Thus, $\{v\} \times F^2_H[a] \subseteq F^2_{G[H]}[(v,a)]$. This establishes the desired equality. □

Definition 5. *The Cartesian Product of two graphs* G_1 *and* G_2 *denoted by* $G_1 \square G_2$ *is a graph with* $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ *and two vertices* $a = (u_1, u_2)$ *and* $b = (v_1, v_2)$ *are adjacent in* $G_1 \square G_2$ *if and only if either* $u_1 = v_1$ *and* $u_2 v_2 \in E(G_2)$ *or* $u_2 = v_2$ *and* $u_1 v_1 \in E(G_1)$.

Theorem 7. *Let* $K = G \square H = (V(K), E(K))$, *where* $G = (V(G), E(G))$ *and* $H = (V(H), E(H))$. *Then for each* $(v, a) \in V(K)$,

$$F_K^2[(v,a)] = \left[F_G^2[v] \times \{a\}\right] \cup \left[\{v\} \times F_H^2[a]\right] \cup \left[F_G[v] \times V(H) \backslash \{a\}\right] \cup \left[N_G(v) \times F_G[a]\right].$$

Proof. Let $(v, a) \in V(K) = V(G \square H)$ and $(x, q) \in F_K^2[(v, a)]$. Then $(v, a) \neq (x, q)$ and $d_K((v, a), (x, q)) \neq 2$. Now, consider the following cases:

Case 1. Assume that $x = v$. Then $q \neq a$ and $d_H(q, a) = d_K((x, q), (x, a)) \neq 2$ and so, $q \in F_H^2[a]$. Hence, $(x, q) \in \{v\} \times F_H^2[a]$.

Case 2. Assume that $x \neq v$.

Subcase 1. Let $q = a$. Then $d_G(x, v) = d_K((x, q), (v, q)) \neq 2$ and thus, $x \in F_G^2[v]$. It follows that $(x, q) \in F_G^2[v] \times \{a\}$.

Subcase 2. Let $q \neq a$. Suppose that $x \in N_G(v)$. If $q \in N_H(a)$, then

$$d_K((x, q), (v, a)) = d_G(x, v) + d_H(q, a) = 2,$$

a contradiction. Thus, $q \in V(H) \backslash N_H[a]$. Hence, $(x, q) \in N_G(v) \times F_G[a]$. Suppose $x \notin N_G(v)$. Then $x \in F_G[v]$. Hence, $(x, q) \in F_G[v] \times V(H) \backslash \{a\}$. Therefore,

$$F_K^2[(v,a)] \subseteq \left[F_G^2[v] \times \{a\}\right] \cup \left[\{v\} \times F_H^2[a]\right] \cup \left[F_G[v] \times V(H) \backslash \{a\}\right] \cup \left[N_G(v) \times F_G[a]\right].$$

Next, let $(v, p) \in \{v\} \times F_H^2[a]$. Then $p \neq a$ and $d_H(a, p) \neq 2$. Hence, $(v, p) \neq (v, a)$ and $d_K((v, p), (v, a)) = d_H(a, p) \neq 2$, that is, $(v, p) \in F_K^2[(v, a)]$. If $(x, a) \in F_G^2[v] \times \{a\}$, then $x \neq v$ and $d_G(x, v) \neq 2$. Hence, $(x, a) \neq (v, a)$ and $d_K((v, a), (x, a)) = d_G(x, v) \neq 2$, that is, $(x, a) \in F_K^2[(v, a)]$. Now, $(y, b) \in N_G(v) \times F_H[a]$ implies $d_G(y, v) = 1$ and $d_H(b, a) \geq 2$. It follows that $(y, b) \neq (v, a)$ and

$$d_K((y, b), (v, a)) = d_G(y, v) + d_H(b, a) \geq 3.$$

Hence, $(y, b) \in F_K^2[(v, a)]$. Finally, $(z, t) \in [F_G[v] \times V(H) \backslash \{a\}]$ implies $d_G(z, v) \geq 2$ and $d_H(t, a) \geq 1$. This means that $(z, t) \neq (v, a)$ and

$$d_K((z, t), (v, a)) = d_G(z, v) + d_H(t, a) \geq 3.$$

Thus, $(z, t) \in F_K^2[(v, a)]$. Therefore,

$$\left[F_G^2[v] \times \{a\}\right] \cup \left[\{v\} \times F_H^2[a]\right] \cup \left[F_G[v] \times V(H) \backslash \{a\}\right] \cup \left[N_G(v) \times F_G[a]\right] \subseteq F_K^2[(v,a)].$$

This establishes the desired equality. □

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