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Hybrid third derivative block method for the solution of general second order initial value problems with generalized one step point

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Abstract. This paper deals with two-step hybrid block method with one generalized off-step points for solving second order initial value problem. In derivation of this method, power series of order nine are interpolated at the first two step points while its second and third derivatives are collocated at all point in the selected interval. The new developed method is employed to solve several problems of second order initial value problems. Convergence analysis of the new method alongside numerical procedure is established. The performance of the proposed method is found to be more accurate than existing method available in the literature when solving the same problems.

2010 Mathematics Subject Classifications: 65L05, 65L06, 65L20

Key Words and Phrases: Block method, Hybrid method, Second order differential equation, Collocation and Interpolation, Two off step points

1. Introduction

This article considered the solution to the general second order initial value problem (IVPs) of the form

$$y^{''} = f(x, y, y^{'}),\ y(a) = \delta_0,\ y^{'}(a) = \delta_1.\ x \in [a, b]. \tag{1}$$

Equation (1) occurs in different fields of applied mathematics, among which are elasticity, fluid mechanics, and quantum mechanics as well as in engineering and physics. The existence and uniqueness of the solution for these equations have been discussed in [11] . In general, finding the exact solutions of these equations is not easy. Over the years,

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different numerical methods have been developed in order to approximate the solution of equation (1). Among these methods are block method, linear multistep method, hybrid method and rung-kutta method, see [6], [7],[10],[5]. Recently, some efforts have been made to develop hybrid block method for solving (1) directly, see [8], [2], [9]. However, these methods are focused on to specific off-step points.
Therefore, in this work, we are going to develop two-step hybrid block method with generalized one off-step points for solving equation (1) directly using high derivative approach. Hence, the errors of the methods can be reduced by judicious choice of the off step pints.

2. Development of the Method

In this section, two step hybrid block method with one generalized off-step points $i.e$ x_{n+s} for solving (1) is derived. Let the approximate solution of (1) to be the power series polynomial of the form:

$$y(x) = \sum_{i=0}^{2r+1} a_i \left(\frac{x - x_n}{h}\right)^i . \tag{2}$$

where $x \in [x_n, x_{n+2}]$ for $n = 0, 1, 2, ..., N-1$ with r represent the number of points and $h = x_n - x_{n-1}$ is a constant step size of partition of interval $[a, b]$ which is division as $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$.

Differentiating (2) twice and thrice yields

$$y^{''}(x) = f(x, y, y^{'}) = \sum_{i=2}^{q+d-1} \frac{i(i-1)}{h^2} a_i \left(\frac{x - x_n}{h}\right)^{i-2} . \tag{3}$$

$$y^{'''}(x) = d(x, y, y^{'}, f) = \sum_{i=3}^{2r+1} \frac{i(i-1)(i-2)}{h^3} a_i \left(\frac{x - x_n}{h}\right)^{i-3} . \tag{4}$$

Interpolating (2) at x_n, x_{n+1} and collocating (3) and (4) at all points in the selected interval produces ten equations which can be written in matrix of the form:

$$AX = B \tag{5}$$

where

$$A=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2} & \frac{72}{h^2}\\
0 & 0 & \frac{2}{h^2} & \frac{6s}{h^2} & \frac{12s^2}{h^2} & \frac{20s^3}{h^2} & \frac{30s^4}{h^2} & \frac{42s^5}{h^2} & \frac{56s^6}{h^2} & \frac{72s^7}{h^2}\\
0 & 0 & \frac{2}{h^2} & \frac{12}{h^2} & \frac{48}{h^2} & \frac{160}{h^2} & \frac{480}{h^2} & \frac{1344}{h^2} & \frac{3584}{h^2} & \frac{9216}{h^2}\\
0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} & \frac{210}{h^3} & \frac{336}{h^3} & \frac{504}{h^3}\\
0 & 0 & 0 & \frac{6}{h^3} & \frac{24s}{h^3} & \frac{60s^2}{h^3} & \frac{120s^3}{h^3} & \frac{210s^4}{h^3} & \frac{336s^5}{h^3} & \frac{504s^6}{h^3}\\
0 & 0 & 0 & \frac{6}{h^3} & \frac{48}{h^3} & \frac{240}{h^3} & \frac{960}{h^3} & \frac{3360}{h^3} & \frac{10752}{h^3} & \frac{32256}{h^3}
\end{pmatrix}$$

$$X=\left[\ a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,\ \right]^T$$

$$B=\left[\ y_n,y_{n+1},f_n,f_{n+1},f_{n+s},f_{n+2},d_{n+1},d_{n+s},d_{n+2}\ \right]^T$$

Gaussian elimination method is Employed on (2) to give the unknown values of $a_i's, i = 0(1)9$. Substituting these values into equation (2) produces the following a continuous implicit scheme

$$y(x)=\sum_{i=0}^{1}\alpha_i(x)y_{n+i}+\sum_{i=0}^{2}[\beta_i(x)f_{n+i}+\zeta_i d_{n+i}]+\beta_s(x)f_{n+s}+\zeta_s d_{n+s} \tag{6}$$

The first derivative of equation (6)with respect to x gives

$$y^{'}(x)=\sum_{i=0}^{1}\alpha_i^{'}(x)y_{n+i}+\sum_{i=0}^{2}\left[\beta_i^{'}(x)f_{n+i}+\zeta_i^{'}d_{n+i}\right]+\beta_s(x)f_{n+s}+\zeta_s^{'}(x)d_{n+s} \tag{7}$$

where

$$\begin{aligned}
&d_{n+i}=f^{'}_{n+i}, \quad i\in(0,s,1,2)\\
&\alpha_0=1-\frac{(x-x_n)}{h}\\
&\alpha_1=\frac{(x-x_n)}{h}\\
&\beta_0=-\frac{(x-x_n)(41h-96hs-315hs^2+1560hs^3-2520s^3x+2520s^3x_n))}{(5040s^3}\\
&\quad+\frac{(3s+2)(x-x_n)^9}{(288h^7s^3}-\frac{3(x-x_n)^8(2s^2+7s+4)}{(224h^6s^3)}-\frac{(x-x_n)^4(23s^2+24s+12)}{(48h^2s^2)}\\
&\quad+\frac{(x-x_n)^7(3s^3+36s^2+57s+26)}{(168h^5s^3}+\frac{(x-x_n)^5(33s^3+72s^2+48s+8)}{(80h^3s^3}
\end{aligned}$$

$$-\frac{(x-x_n)^6(17s^3+78s^2+75s+24)}{(120h^4s^3)}$$

$$\beta_1 = \frac{(x-x_n)^9}{(36h^7(s-1)^3} - \frac{h(x-x_n)(420s^3-1260s^2+945s-233)}{(2520(s-1)^3} - \frac{(3s+11)(x-x_n)^8}{(56h^6(s-1)^3}$$
$$+\frac{(9s+10)(x-x_n)^7}{21h^5(s-1)^3} - \frac{(x-x_n)^6(-s^3+3s^2+36s+12)}{30h^4(s-1)^3} + \frac{s^2(x-x_n)^4(s-3)}{3h^2(s-1)^3}$$
$$+\frac{s(x-x_n)^5(-s^2+3s+6))}{(5h^3(s-1)^3)}$$

$$\beta_s = \frac{(x-x_n)^7(-21s^3+6s^2+60s-26)}{21h^5s^3(s-1)^3(s-2)^3} - \frac{(x-x_n)^5(21s^3-39s^2+6s+4)}{5h^3s^3(s-1)^3(s-2)^3}$$
$$+\frac{(x-x_n)^8(7s^3+21s^2-66s+24)}{56h^6s^3(s-1)^3(s-2)^3} - \frac{(x-x_n)^6(-91s^3+123s^2+66s-48)}{30h^4s^3(s-1)^3(s-2)^3}$$
$$-\frac{h(x-x_n)(735s^3-1821s^2+1122s-164)}{2520s^3(s-1)^3(s-2)^3} - \frac{(x-x_n)^9(3s^2-6s+2)}{36h^7s^3(s-1)^3(s-2)^3}$$
$$+\frac{(x-x_n)^4(7s^2-15s+6)}{(3h^2s^2(s-1)^3(s-2)^3}$$

$$\beta_2 = \frac{(x-x_n)^7(-3s^3-18s^2+51s+44)}{(168h^5(s-2)^3} - \frac{(3s-8)(x-x_n)^9)}{(288h^7(s-2)^3}$$
$$-\frac{(x-x_n)^8(-6s^2+3s+34)}{(224h^6(s-2)^3} - \frac{(x-x_n)^6(-13s^3+81s+18)}{(120h^4(s-2)^3}$$
$$-\frac{h(x-x_n)(120s^3-405s^2+276s-61)}{(5040(s-2)^3} + \frac{s(x-x_n)^5(-17s^2+30s+36)}{(80h^3(s-2)^3}$$
$$+\frac{s^2(7s-18)(x-x_n)^4}{(48h^2(s-2)^3)}$$

$$\zeta_0 = \frac{(x-x_n)(-354h^2s^2+210h^2s-41h^2+1680s^2x^2-3360s^2xx_n+1680s^2x_n^2)}{(10080s^2)}$$
$$+\frac{(x-x_n)^7(s^2+12s+13)}{(168h^4s^2)} - \frac{(3s+2)(x-x_n)^4)}{(12hs)} - \frac{(x-x_n)^6(3s^2+13s+6)}{(60h^3s^2)}$$
$$+\frac{(x-x_n)^5(13s^2+24s+4)}{(80h^2s^2)} - \frac{(x-x_n)^8(s+3)}{(112h^5s^2)} + \frac{(x-x_n)^9}{(288h^6s^2)}$$

$$\zeta_1 = -\frac{s^2(x-x_n)^4)}{(3h(s-1)^2} + \frac{h^2(x-x_n)(96s^2-87s+23)}{(1260(s-1)^2} + \frac{(x-x_n)^7(s^2+10s+8)}{42h^4(s-1)^2}$$
$$-\frac{(2s+5)(x-x_n)^8)}{56h^5(s-1)^2} - \frac{(x-x_n)^6(5s^2+16s+4)}{30h^3(s-1)^2} + \frac{2s(x-x_n)^5(s+1)}{(5h^2(s-1)^2)}$$

$$+\frac{(x-x_n)^9}{(72h^6(s-1)^2}$$

$$\begin{aligned}\zeta_s = {} & \frac{(x-x_n)^9}{72h^6s^2(s-1)^2(s-2)^2} - \frac{(x-x_n)^4}{3hs(s-1)^2(s-2)^2} + \frac{h^2(105s-41)(x-x_n)}{2520s^2(s-1)^2(s-2)^2} \\ & - \frac{(x-x_n)^8(s+6)}{56h^5s^2(s-1)^2(s-2)^2} + \frac{(3s+1)(x-x_n)^5}{5h^2s^2(s-1)^2(s-2)^2} + \frac{(6s+13)(x-x_n)^7}{42h^4s^2(s-1)^2(s-2)^2} \\ & - \frac{(13s+12)(x-x_n)^6)}{(30h^3s^2(s-1)^2(s-2)^2)}\end{aligned}$$

$$\begin{aligned}\zeta_2 = {} & -\frac{(x-x_n)^8(s+2)}{112h^5(s-2)^2} - \frac{s^2(x-x_n)^4}{24h(s-2)^2} + \frac{h^2(x-x_n)(66s^2-54s+13)}{10080(s-2)^2} \\ & + \frac{(x-x_n)^7(s^2+8s+5)}{168h^4(s-2)^2} - \frac{(x-x_n)^6(2s^2+5s+1)}{60h^3(s-2)^2} + \frac{s(5s+4)(x-x_n)^5}{80h^2(s-2)^2} \\ & + \frac{(x-x_n)^9}{(288h^6(s-2)^2}\end{aligned}$$

Equation (6) is evaluated at the non-interpolating point x_{n+s} and x_{n+2} while Equation (7) is evaluated at all points to give the discrete schemes and its derivative. The discrete scheme and its derivatives are then combined in a matrix form as below

$$KY_M = BR_1 + h^2\left[C_nR_2 + C_{n+1}R_3\right] + h^3\left[D_nR_4 + D_{n+1}R_5\right] \tag{8}$$

where

$$K = \begin{pmatrix} -s & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ \frac{-1}{h} & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{h} & 0 & 0 & 1 & 0 & 0 \\ \frac{-1}{h} & 0 & 0 & 0 & 1 & 0 \\ \frac{-1}{h} & 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ Y_M = \begin{pmatrix} y_{n+1} \\ y_{n+s} \\ y_{n+2} \\ y'_{n+1} \\ y'_{n+s} \\ y'_{n+2} \end{pmatrix},\ B = \begin{pmatrix} 1-s & 0 \\ -1 & 0 \\ \frac{-1}{h} & -1 \\ \frac{-1}{h} & 0 \\ \frac{-1}{h} & 0 \\ \frac{-1}{h} & 0 \end{pmatrix},\ R_1 = \begin{bmatrix} y_n \\ y'_n \end{bmatrix}$$

$$C_n = \begin{pmatrix} \frac{-(s-1)(-15s^8+128s^7-358s^6+140s^5+1148s^4-2380s^3+740s^2+110s-82)}{10080s^2} \\ \frac{(112s^3-87s^2+29)}{840s^3} \\ \frac{(-1560s^3+315s^2+96s-41)}{5040s^3h} \\ \frac{-(-1122s^3+522s^2+123s-92)}{10080s^3h} \\ \frac{(45s^9-378s^8+1098s^7-840s^6-2016s^5+5040s^4-3120s^3+630s^2+192s-82)}{10080s^3h} \\ \frac{(792s^3-837s^2+96s+343)}{5040s^3h} \end{pmatrix},\ R_2 = \begin{bmatrix} f_n \end{bmatrix},$$

$$D_n = \begin{pmatrix} \frac{(s-1)(5s^7-49s^6+185s^5-319s^4+185s^3+185s^2-169s+41)}{10080s} \\ \frac{(42s^2-58s+29)}{1680s^2} \\ \frac{-(354s^2-210s+41)}{10080s^2h} \\ \frac{(96s^2-87s+23)}{5040s^2h} \\ \frac{(15s^8-144s^7+546s^6-1008s^5+840s^4-354s^2+210s-41)}{10080s^2h} \\ \frac{(318s^2-558s+343)}{10080s^2h} \end{pmatrix}, \; R_3 = \begin{bmatrix} d_n \end{bmatrix}, \quad R_4 = \begin{bmatrix} f_{n+1} \\ f_{n+s} \\ f_{n+2} \end{bmatrix},$$

$$C_{n+1} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \\ C_{41} & C_{42} & C_{43} \\ C_{51} & C_{52} & C_{53} \\ C_{61} & C_{62} & C_{63} \end{pmatrix}, \; R_5 = \begin{pmatrix} d_{n+1} \\ d_{n+s} \\ d_{n+2} \end{pmatrix}, \; D_{n+1} = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \\ D_{41} & D_{42} & D_{43} \\ D_{51} & D_{52} & D_{53} \\ D_{61} & D_{62} & D_{63} \end{pmatrix},$$

The elements of C_{n+1} and D_{n+1} are shown in Appendix A. Multiplying (8) with the inverse of K gives

$$K^{(0)}Y_M = \bar{B}R_1 + h^2\left[\bar{C}_nR_2 + \bar{C}_{n+1}R_3\right] + h^3\left[\bar{D}_nR_4 + \bar{D}_{n+1}R_5\right] \tag{9}$$

where $K^{(0)}$ is 6×6 identity matrix and

$$\bar{B} = \begin{pmatrix} 1 & h \\ 1 & sh \\ 1 & 2h \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad \bar{C}n = \begin{pmatrix} \frac{(1560s^3-315s^2-96s+41)}{5040s^3} \\ \frac{s^2(15s^5-143s^4+486s^3-498s^2-1008s+3528)}{10080} \\ \frac{(237s^3-72s^2-12s+16)}{315s^3} \\ \frac{(1414s^3-384s^2-105s+58)}{3360s^3h} \\ \frac{s(15s^5-126s^4+366s^3-280s^2-672s+1680)}{3360h} \\ \frac{(49s^3-24s^2+8)}{105s^3h} \end{pmatrix}$$

$$\bar{D}n = \begin{pmatrix} \frac{(354s^2-210s+41)}{10080s^2} \\ \frac{s^3(5s^4-54s^3+234s^2-504s+504)}{10080} \\ \frac{2(15s^2-12s+4)}{315s^2} \\ \frac{(182s^2-128s+29)}{3360s^2h} \\ \frac{s^2(5s^4-48s^3+182s^2-336s+280)}{3360h} \\ \frac{(7s^2-8s+4)}{105s^2h} \end{pmatrix}, \; \bar{C}_{n+1} = \begin{pmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} \\ \bar{C}_{31} & \bar{C}_{32} & \bar{C}_{33} \\ \bar{C}_{41} & \bar{C}_{42} & \bar{C}_{43} \\ \bar{C}_{51} & \bar{C}_{52} & \bar{C}_{53} \\ \bar{C}_{61} & \bar{C}_{62} & \bar{C}_{63} \end{pmatrix}, \; \bar{D}_{n+1} = \begin{pmatrix} \bar{D}_{11} & \bar{D}_{12} & \bar{D}_{13} \\ \bar{D}_{21} & \bar{D}_{22} & \bar{D}_{23} \\ \bar{D}_{31} & \bar{D}_{32} & \bar{D}_{33} \\ \bar{D}_{41} & \bar{D}_{42} & \bar{D}_{43} \\ \bar{D}_{51} & \bar{D}_{52} & \bar{D}_{53} \\ \bar{D}_{61} & \bar{D}_{62} & \bar{D}_{63} \end{pmatrix}$$

The elements of $\bar{C}_{n+1}$ and $\bar{D}_{n+1}$ are presented in Appendix B

3. Properties of the Method

3.1. Order of the Methhod

Extending to[7], the linear difference operator L associated with main block of (9) is defined as

$$L[y(x);h] = K^{(0)}Y_M - \bar{B}R_1 - h^2\left[\bar{C}_nR_2 + \bar{C}_{n+1}R_3\right] - h^3\left[\bar{D}_nR_4 + \bar{C}_{n+1}R_5\right] \tag{10}$$

where $y(x)$ is an arbitrary test function continuously differentiable on $[a, b]$. in finding the order of main block $\hat{Y}_M = [y_{n+1}, y_{n+s}, y_{n+2}]^T$. Expanding Y_M, R_3 and R_5 Components in Taylors series about x_n then collecting its terms in powers of h gives

$$L[y(x), h] = \bar{C_0} y(x) + \bar{C_1} h y^{'}(x) + \bar{C_2} h^2 y^{''}(x) + \cdots \tag{11}$$

Definition 3.1 Hybrid block method (9) and associated linear operator (10)are said to be of order p, if $\bar{C_0} = \bar{C_1} = \bar{\ }_2 = \cdots = \bar{C}_{p+2} = 0$ and $\bar{C}_{p+2} \neq 0$ with error vector constants $\bar{C}_{p+2}$.

Employing Definition (3.1)into equation (9) yields

$$\begin{bmatrix} \begin{array}{c} \sum_{j=0}^{\infty} \frac{(1)^j h^j}{j!} y_n^j - y_n - h y_n^{'} - \frac{h^2(1560s^3-315s^2-96s+41)}{5040s^3} y_n^{''} \\ \frac{(1260s^2-420s^3-945s+233)}{(2520(s-1)^3)} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} - \frac{(735s^3-1821s^2+1122s-164)}{(2520s^3(s-1)^3(s-2)^3)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+2}}{j!} y_n^{j+2} \\ - \frac{((120s^3-405s^2+276s-61))}{(5040(s-2)^3)} \sum_{j=0}^{\infty} \frac{(2)^j h^{j+2}}{j!} y_n^{j+2} - \frac{h^3(354s^2-210s+41)}{10080s^2} y_n^{'''} \\ + \frac{((96s^2-87s+23))}{(1260(s-1)^2)} \sum_{j=0}^{\infty} \frac{(1)^j h^{j+3}}{j!} y_n^{j+3} + \frac{((105s-41))}{(2520s^2(s-1)^2(s-2)^2)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+3}}{j!} y_n^{j+3} \\ - \frac{(s^6(5s^3-36s^2+90s-84))}{(10080(s-2)^2)} \sum_{j=0}^{\infty} \frac{(2)^j h^{j+3}}{j!} y_n^{j+3} \end{array} \\ \\ \begin{array}{c} \sum_{j=0}^{\infty} \frac{(s)^j h^j}{j!} y_n^j - y_n - sh y_n^{'} - \frac{h^2 s^2(15s^5-143s^4+486s^3-498s^2-1008s+3528)}{10080} y_n^{''} \\ - \frac{(s^6(19s^3-171s^2+528s-504))}{(2520(s-1)^3)} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\ - \frac{(s^2(105s^6-1155s^5+5254s^4-12636s^3+16872s^2-11592s+3024))}{(2520(s-1)^3(s-2)^3)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+2}}{j!} y_n^{j+2} \\ + \frac{(s^6(15s^4-157s^3+612s^2-1086s+756))}{(10080(s-2)^3)} \sum_{j=0}^{\infty} \frac{(2)^j h^{j+2}}{j!} y_n^{j+2} - \frac{2h^3(15s^2-12s+4)}{315s^2} y_n^{'''} \\ - \frac{(s^6(5s^3-45s^2+144s-168))}{(2520(s-1)^2)} \sum_{j=0}^{\infty} \frac{(1)^j h^{j+3}}{j!} y_n^{j+3} \\ + \frac{(s^3(5s^4-45s^3+156s^2-252s+168))}{(1260(s-1)^2(s-2)^2)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+3}}{j!} y_n^{j+3} + \frac{((66s^2-54s+13))}{(10080(s-2)^2)} \sum_{j=0}^{\infty} \frac{(2)^j h^{j+3}}{j!} y_n^{j+3} \end{array} \\ \\ \begin{array}{c} \sum_{j=0}^{\infty} \frac{(2)^j h^j}{j!} y_n^j - y_n - (2h) y_n^{'} - \frac{h^2(237s^3-72s^2-12s+16)}{315s^3} y_n^{''} \\ - \frac{(16(21s^3-63s^2+54s-14))}{(315(s-1)^3)} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} - \frac{(16(21s^3-57s^2+42s-8))}{(315s^3(s-1)^3(s-2)^3)} \sum_{j=0}^{\infty} \frac{s^j h^{j+2}}{j!} y_n^{j+2} \\ + \frac{(16(3s^2-2))}{(315(s-1)^2)} \sum_{j=0}^{\infty} - \frac{(2)^j h^{j+2}}{j!} y_n^{j+2} - \frac{2(15s^2-12s+4)}{315s^2} y_n^{'''} \\ - \frac{((57s^3-270s^2+408s-224))}{(315(s-2)^3)} \sum_{j=0}^{\infty} \frac{(1)^j h^{j+3}}{j!} y_n^{j+3} \\ + \frac{(16(3s-2))}{(315s^2(s-1)^2(s-2)^2)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+3}}{j!} y_n^{j+3} + \frac{(4(3s^2-6s+4))}{(315(s-2)^2)} \sum_{j=0}^{\infty} \frac{(2)^j h^{j+3}}{j!} y_n^{j+3} \end{array} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives, the new method of order $[8, 8, 8]^T$ together with error vector constant

$$\begin{bmatrix} \frac{(484227031934875729920s^2-378158253511045808128s+87622034350120359039)}{468576814262740546328985600} \\ \frac{(s^6(s^4-10s^3+39s^2-72s+56))}{33868800} \\ \frac{(3s^2-4s+2)}{793800} \end{bmatrix} \text{which is true for all} s \in (1, 2)$$

3.2. Zero Stability

According to [4], the block method is said to be zero stable if the roots of the first characteristic function $\pi(x)$ satisfies that $|x_z| \leq 1$, and if $|x_z| = 1$ then, the multiplicity of x_z must not greater than two.
In order to find the zero-stability of $\hat{Y}_M$, $\pi(x) = |x\,I - \hat{B}| = 0$ is solved where I and $\hat{B}$ are 3×3 identity matrix and coefficients matrix of y_n respectively. This is demonstrated below

$$\begin{aligned}
\pi(x) &= |x\,I - \hat{B}^{[2]_2}| \\
&= \left| x \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| \\
&= x^2(x-1)
\end{aligned}$$

whose solutions are $x = 0, 0, 1$. Hence, our method is zero stable for all $s \in (1, 2)$

3.3. Consistency

Two step hybrid block method (9) is said to be consistent if its order greater than or equal one i.e. $P \geq 1$. Therefore, our method is consistent for all $s \in (1, 2)$.

3.4. Convergence

Theorem 3.1 (Henrici, 1962)**.** *Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent, [10].*

Since our method is zero stable and consistent, this implies that it is convergent for all $s \in (0, 1)$.

4. Numerical Results

In this section, we test the effectiveness and validity of our newly derived scheme in equation (12) by applying it to some second order differential equations. All calculations and programs are carried out with the aid of Maple 13 software.

In finding the accuracy of our methods, the following second order ODEs are examined. The new block methods solved the same problems the existing methods solved in order to compare results in terms of error.

Problem 1: $y^{''} - 2y^3 = 0, \quad y(01) = 1, \quad y^{'}(1) = -1,\ h = 0.1.$
Exact solution: $y(x) = \frac{1}{x}$

Problem 2: $y^{''} + \frac{6}{x}y^{'} + \frac{4}{x^2}y = 0,\ y(1) = 1,\ y^{'}(1) = 1,\ h = \frac{1}{320}.$

Exact solution: $y(x) = \frac{5}{3x} - \frac{2}{3x^2}$

Problem 3: $y'' - x(y')^2 = 0, \quad y(0) = 1,\ y'(0) = \frac{1}{2},\ h = \frac{1}{320}.$
Exact solution: $y(x) = 1 + \frac{1}{2} ln\left(\frac{2+x}{2-x}\right)$

Table 1: Comparison of the new method with [12] for solving Problem 1, $h = \frac{1}{10}$

x	Exact solution	Computed solution in new method $s = \frac{3}{2}$	Error in our method	Errors in [12]
1.1	0.909090909090909060	0.909090908924874320	$1.660347E^{-10}$	$1.55450E^{-6}$
1.2	0.833333333333333370	0.833333332509376020	$8.239573E^{-10}$	$3.10356E^{-6}$
1.3	0.769230769230769160	0.769230086014952350	$6.832158E^{-7}$	$2.25690E^{-6}$
1.4	0.714285714285714300	0.714284324837971660	$1.389448E^{-6}$	$2.38050E^{-6}$
1.5	0.666666666666666630	0.666664386589143400	$2.280078E^{-6}$	$2.63360E^{-6}$

Table 2: Comparison of the new method with[3] for solving problem 2, $h = \frac{1}{320}$

x	Exact solution	Computed solution in new method $s = \frac{3}{2}$	Error in our method	Errors in [3]
1.003125	1.003076525857696100	1.003076525857589300	$1.0680E^{-13}$	$8.30E^{-8}$
1.006250	1.006057503083516400	1.006057503083044100	$4.7228E^{-13}$	$1.16E^{-6}$
1.009375	1.008944995088837600	1.008944995106134400	$1.7296E^{-11}$	$6.63E^{-6}$
1.012500	1.011741018167988700	1.011741018202483700	$3.4495E^{-11}$	$9.49E^{-6}$
1.015625	1.014447542686413900	1.014447542754808500	$6.8394E^{-11}$	$1.95E^{-6}$

Table 3: Comparison of the new method with[1] for solving problem 3, $h = \frac{1}{100}$

x	Exact solution	Computed solution in new method $s = \frac{3}{2}$	Error in our method	Errors in [1]
0.1	1.0500417292784914	1.0500417292784907	$6.661338E^{-16}$	$9.992E^{-15}$
0.2	1.1003353477310753	1.1003353477310667	$8.881784E^{-15}$	$8.149E^{-14}$
0.3	1.1511404359364665	1.1511404359364179	$4.884981E^{-14}$	$4.700E^{-12}$
0.4	1.2027325540540816	1.2027325540539073	$5.506706E^{-13}$	$1.637E^{-12}$
0.5	1.2554128118829946	1.2554128118825090	$4.862777E^{-13}$	$4.664E^{-12}$

5. Conclusion

Hybrid block method with step length two and one generalized off-step point $s \in (1, 2)$ has been developed. The new method was established to be convergent with order, at least, eight for all equations in the proposed block. The derived method able to solve both linear and non-linear second ODEs problems without converting to the equivalents system of first order ODEs. The generated results, as appear in the tables 1-3, shown that the derived methods are notable better than those methods in literature.

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Appendix A

$$C_{11} = \frac{-s(-19s^7 + 152s^6 - 376s^5 + 128s^4 + 128s^3 + 548s^2 - 712s + 233)}{2520(s-1)^2}$$

$$C_{12} = \frac{(105s^9 + 1050s^8 + 4204s^7 - 8432s^6 + 8440s^5 + 3152s^4 - 128s^3 - 863s^2 + 958s - 164)}{2520s^2(s-1)^2(s-2)^3}$$

$$C_{13} = -\frac{s(s-1)(15s^8 - 142s^7 + 470s^6 - 616s^5 + 140s^4 + 140s^3 + 380s^2 - 430s + 122)}{10080(s-2)^3}$$

$$C_{21} = \frac{(308s^2 - 616s + 221)}{420(s-1)^2}$$

$$C_{22} = \frac{29(7s^2 - 14s + 4)}{420s^3(s-1)^2(s-2)^3}$$

$$C_{23} = \frac{(112s^3 - 585s^2 + 996s - 577)}{840(s-2)^3}$$

$$C_{31} = \frac{-(420s^3 - 1260s^2 + 945s - 233)}{2520h(s-1)^3}$$

$$C_{32} = \frac{-(735s^3 - 1821s^2 + 1122s - 164)}{2520s^3h(s-1)^3(s-2)^3}$$

$$C_{33} = \frac{-(120s^3 - 405s^2 + 276s - 61)}{5040h(s-2)^3}$$

$$C_{41} = \frac{(924s^3 - 2772s^2 + 2511s - 745)}{2520h(s-1)^3}$$
$$C_{42} = \frac{(609s^3 - 1581s^2 + 1074s - 184)}{2520s^3h(s-1)^3(s-2)^3}$$
$$C_{43} = \frac{(222s^3 - 810s^2 + 699s - 190)}{10080h(s-2)^3}$$
$$C_{51} = \frac{-(-54s^8 + 432s^7 - 1176s^6 + 1008s^5 + 420s^3 - 1260s^2 + 945s - 233)}{2520h(s-1)^3}$$
$$C_{52} = \frac{(630s^10 - 6300s^9 + 25884s^8 - 55872s^7 + 66696s^6 - 41328s^5)}{2520s^3h(s-1)^3(s-2)^3} + \frac{(10080s^4 - 735s^3 + 1821s^2 - 1122s164)}{2520s^3h(s-1)^3(s-2)^3}$$
$$C_{53} = \frac{-(45s^9 - 432s^8 + 1530s^7 - 2436s^6 + 1512s^5 + 240s^3 - 810s^2 + 552s - 122)}{10080h(s-2)^3}$$
$$C_{61} = \frac{(2268s^3 - 6804s^2 + 5967s - 1303)}{2520h(s-1)^3}$$
$$C_{62} = \frac{(1953s^3 - 6243s^2 + 5790s - 1372)}{2520s^3h(s-1)^3(s-2)^3}$$
$$C_{63} = \frac{(2232s^3 - 12555s^2 + 23340s - 14531)}{5040h(s-2)^3}$$
$$D_{11} = \frac{-(s(-5s^7 + 40s^6 - 104s^5 + 64s^4 + 64s^3 + 64s^2 - 128s + 46))}{(2520s - 2520)}$$
$$D_{12} = \frac{-((10s^7 - 80s^6 + 232s^5 - 272s^4 + 64s^3 + 64s^2 + 64s - 41))}{(2520s(s-1)(s-2)^2)}$$
$$D_{13} = \frac{-(s(s-1)(-5s^7 + 31s^6 - 59s^5 + 25s^4 + 25s^3 + 25s^2 - 41s + 13))}{(10080(s-2)^2)}$$
$$D_{21} = \frac{-29}{(210s - 210)}$$
$$D_{22} = \frac{-29}{(420s^2(s-1)(s-2)^2)}$$
$$D_{23} = \frac{-(42s^2 - 110s + 81)}{(1680(s-2)^2)}$$

$$D_{31} = \frac{((96s^2 - 87s + 23))}{(1260h(s-1)^2)}$$

$$D_{32} = \frac{(105s - 41)}{(2520s^2h(s-1)^2(s-2)^2)}$$

$$D_{33} = \frac{(66s^2 - 54s + 13)}{(10080h(s-2)^2)}$$

$$D_{41} = \frac{-(228s^2 - 282s + 95)}{(2520h(s-1)^2)}$$

$$D_{42} = \frac{-(87s - 46)}{(2520s^2h(s-1)^2(s-2)^2)}$$

$$D_{43} = \frac{-(30s^2 - 33s + 10)}{(5040h(s-2)^2)}$$

$$D_{51} = \frac{(15s^8 - 120s^7 + 336s^6 - 336s^5 + 192s^2 - 174s + 46)}{(2520h(s-1)^2)}$$

$$D_{52} = \frac{-(45s^8 - 360s^7 + 1092s^6 - 1512s^5 + 840s^4 - 105s + 41)}{(2520s^2h(s-1)^2(s-2)^2)}$$

$$D_{53} = \frac{(15s^8 - 96s^7 + 210s^6 - 168s^5 + 66s^2 - 54s + 13)}{(10080h(s-2)^2)}$$

$$D_{61} = \frac{(96s^2 - 471s + 407)}{(1260h(s-1)^2)}$$

$$D_{62} = \frac{-(279s - 343)}{(2520s^2h(s-1)^2(s-2)^2)}$$

$$D_{63} = \frac{-(606s^2 - 1866s + 1523)}{(10080h(s-2)^2)}$$

Appendix B

$$\bar{C}_{11} = \frac{(420s^3 - 1260s^2 + 945s - 233)}{2520(s-1)^3}$$

$$\bar{C}_{12} = \frac{(735s^3 - 1821s^2 + 1122s - 164)}{2520s^3(s-1)^3(s-2)^3}$$

$$\bar{C}_{13} = \frac{(120s^3 - 405s^2 + 276s - 61)}{5040(s-2)^3)}$$

$$\bar{C}_{21} = \frac{(s^6(19s^3 - 171s^2 + 528s - 504))}{2520(s-1)^3}$$

$$\bar{C}_{22} = \frac{(s^2(105s^6 - 1155s^5 + 5254s^4 - 12636s^3 + 16872s^2 - 11592s + 3024))}{2520(s-1)^3(s-2)^3}$$

$$\bar{C}_{23} = \frac{-(s^6(15s^4 - 157s^3 + 612s^2 - 1086s + 756))}{10080(s-2)^3)}$$

$$\bar{C}_{31} = \frac{(16(21s^3 - 63s^2 + 54s - 14))}{315(s-1)^3}$$

$$\bar{C}_{32} = \frac{(16(21s^3 - 57s^2 + 42s - 8))}{315s^3(s-1)^3(s-2)^3}$$

$$\bar{C}_{33} = \frac{(57s^3 - 270s^2 + 408s - 224)}{315(s-2)^3)}$$

$$\bar{C}_{41} = \frac{(224s^3 - 672s^2 + 576s - 163))}{420h(s-1)^3}$$

$$\bar{C}_{42} = \frac{(224s^3 - 567s^2 + 366s - 58)}{420s^3(s-1)^3(s-2)^3}$$

$$\bar{C}_{43} = \frac{(154s^3 - 540s^2 + 417s - 104)}{3360h(s-2)^3)}$$

$$\bar{C}_{51} = \frac{s^5(9s^3 - 72s^2 + 196s - 168)}{420h(s-1)^3}$$

$$\bar{C}_{52} = \frac{s(105s^6 - 1050s^5 + 4314s^4 - 9312s^3 + 11116s^2 - 6888s + 1680)}{420h(s-1)^3(s-2)^3}$$

$$\bar{C}_{53} = \frac{-(s^5(15s^4 - 144s^3 + 510s^2 - 812s + 504))}{3360h(s-2)^3)}$$

$$\bar{C}_{61} = \frac{(16(7s^2 - 14s + 4))}{105h(s-1)^2}$$

$$\bar{C}_{62} = \frac{(16(7s^2 - 14s + 4))}{105s^3h(s-1)^2(s-2)^3}$$

$$\bar{C}_{63} = \frac{(49s^3 - 270s^2 + 492s - 304)}{105h(s-2)^3)}$$

$$\bar{D}_{11} = \frac{-(96s^2 - 87s + 23)}{(1260(s-1)^2)}$$

$$\bar{D}_{12} = \frac{-(105s - 41)}{(2520s^2(s-1)^2(s-2)^2)}$$

$$\bar{D}_{13} = \frac{-(66s^2 - 54s + 13)}{(10080(s-2)^2)}$$

$$\bar{D}_{21} = \frac{s^6(5s^3 - 45s^2 + 144s - 168)}{(2520(s-1)^2)}$$

$$\bar{D}_{22} = \frac{-s^3(5s^4 - 45s^3 + 156s^2 - 252s + 168)}{(1260(s-1)^2(s-2)^2)}$$

$$\bar{D}_{23} = \frac{s^6(5s^3 - 36s^2 + 90s - 84)}{(10080(s-2)^2)}$$

$$\bar{D}_{31} = \frac{-(16(3s^2 - 2))}{(315(s-1)^2)}$$

$$\bar{D}_{32} = \frac{-(16(3s - 2))}{(315s^2(s-1)^2(s-2)^2)}$$

$$\bar{D}_{33} = \frac{-(4(3s^2 - 6s + 4))}{(315(s-2)^2)}$$

$$\bar{D}_{41} = \frac{-(140s^2 - 152s + 47)}{(840h(s-1)^2)}$$

$$\bar{D}_{42} = \frac{-(64s - 29)}{(840s^2h(s-1)^2(s-2)^2)}$$

$$\bar{D}_{43} = \frac{-(42s^2 - 40s + 11)}{(3360h(s-2)^2)}$$

$$\bar{D}_{51} = \frac{s^5(5s^3 - 40s^2 + 112s - 112)}{(840h(s-1)^2)}$$

$$\bar{D}_{52} = \frac{-s^2(15s^4 - 120s^3 + 364s^2 - 504s + 280)}{(840h(s-1)^2(s-2)^2)}$$

$$\bar{D}_{53} = \frac{s^5(5s^3 - 32s^2 + 70s - 56)}{(3360h(s-2)^2)}$$

$$\bar{D}_{61} = \frac{-32}{(105h(s-1))}$$

$$\bar{D}_{62} = \frac{-(16h^2)}{(105s^2(s-1)(s-2)^2)}$$

$$\bar{D}_{63} = \frac{-(7s^2 - 20s + 16)}{(105h(s-2)^2)}$$