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Direct product of finite intuitionistic anti fuzzy normal subrings over non-associative rings

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Abstract. Shal et. al [17], have introduced the concept of intuitionistic fuzzy normal subrings over a non-associative ring. In this paper, we investigate the concept of intuitionistic anti fuzzy normal subrings over non-associative rings and give some properties of such subrings

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1. Introduction

In 1972, ageneralization of commutative semigroups has been established by Kazim et. al [9]. In ternary commutative law: $abc = cba$, they introduced the braces on the left side of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$, is called the left invertive law. A groupoid S is said to be a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$.

In [7] (resp. [5]), a groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [9], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial in [15] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Kamran [8], extended the notion of LA-semigroup to the left almost group (LA-group). An LA-semigroup G is said to be a left almost group, if there exists left identity $e \in G$ such that $ea = a$ for all $a \in G$ and for every $a \in G$ there exists $b \in G$ such that $ba = e$.

Shah et. al [18], discussed the left almost ring (LA-ring) of finitely nonzero functions which is a generalization of commutative semigroup ring. By a left almost ring, we mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all

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$a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. In fact an LA-ring is a non-associative and non-commutative ring.

A non-empty subset A of an LA-ring R is an LA-subring of an LA-ring R if $a - b$ and $ab \in A$ for all $a, b \in A$. A is a left (resp. right) ideal of R if $(A, +)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R.

After the introduction of fuzzy set by Zadeh [22], several researchers explored on the generalization of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [1, 2], as a generalization of the notion of fuzzy set.

Sherwood [20], introduced the concept of product of fuzzy subgroups. After this, further study on this concept continued by Osman [11, 12] and Ray [16]. Zaid [23], gave the idea of normal fuzzy subgroups.

An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$ [1, 2].

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ in X can be identified to be an ordered pair (μ_A, γ_A) in $I^X \times I^X$, where I^X is the set of all functions from X to $[0, 1]$. For the sake of simplicity, we will use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

Intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring have been defined in [3, 6]. Palaniappan et al [13, 14], explored the notions of homomorphism, antihomomorphism of intuitionistic fuzzy normal subrings and also discussed some properties of intuitionistic fuzzy normal subrings. Moreover intuitionistic fuzzy ring and its homomorphism image have been investigated by Yan [21]. Shal et al [17], introduced the concept of intuitionistic fuzzy normal subrings over a non-associative ring (LA-ring).

We define the direct product of intuitionistic fuzzy sets A_1 and A_2 of LA-rings R_1 and R_2, respectively and investigate the some basic properties of intuitionistic anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$.

We define the direct product of intuitionistic fuzzy sets $A_1, A_2, ..., A_n$ of LA-rings $R_1, R_2, ..., R_n$, respectively and examine the some fundamental properties of intuitionistic anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Specifically we show that:

Let $X = A \times B$ and $Y = C \times D$ be two LA-subrings of an LA-ring $R_1 \times R_2$. Then $X \cap Y$ is an LA-subring of an LA-ring $R_1 \times R_2$ if and only if the intuitionistic anti characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = X \cap Y$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

Let $A = A_1 \times A_2 \times ... \times A_n$ and $B = B \times B_2 \times ... \times B_n$ be two LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Then $A \cap B$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ if and only if the intuitionistic anti characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = A \cap B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Let A and B be intuitionistic fuzzy sets of LA-rings R_1 and R_2 with left identities e_1 and e_2, respectively and $A \times B$ be an intuitionistic anti fuzzy normal LA-subring of an

LA-ring $R_1 \times R_2$. Then the following conditions are true.

1. If $\mu_A(x) \geq \mu_B(e_2)$ and $\gamma_A(x) \leq \gamma_B(e_2)$, for all $x \in R_1$, then A is an intuitionistic anti fuzzy normal LA-subring of R_1.

2. If $\mu_B(x) \geq \mu_A(e_1)$ and $\gamma_B(x) \leq \gamma_A(e_1)$, for all $x \in R_2$, then B is an intuitionistic anti fuzzy normal LA-subring of R_2.

2. Direct Product of Intuitionistic Anti Fuzzy Normal LA-subrings

We define the direct product of intuitionistic fuzzy sets A_1, A_2 of LA-rings R_1, R_2, respectively and examine the some fundamental properties of direct product of intuitionistic anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$.

Let μ_1 and μ_2 be fuzzy subsets of LA-rings R_1 and R_2, respectively. The direct product of fuzzy subsets μ_1 and μ_2 is denoted by $\mu_1 \times \mu_2$ and defined by $(\mu_1 \times \mu_2)(x_1, x_2) = min\{\mu_1(x_1), \mu_2(x_2)\}$.

A fuzzy subset $\mu_1 \times \mu_2$ of an LA-ring $R_1 \times R_2$ is to be a fuzzy LA-subring of $R_1 \times R_2$ if

1. $(\mu_1 \times \mu_2)(x - y) \geq min\{\mu_1(x), \mu_2(y)\}$,
2. $(\mu_1 \times \mu_2)(xy) \geq min\{\mu_1(x), \mu_2(y)\}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

A fuzzy subset $\mu_1 \times \mu_2$ of an LA-ring $R_1 \times R_2$ is to be an anti fuzzy LA-subring of $R_1 \times R_2$ if

1. $(\mu_1 \times \mu_2)(x - y) \leq max\{\mu_1(x), \mu_2(y)\}$
2. $(\mu_1 \times \mu_2)(xy) \leq max\{\mu_1(x), \mu_2(y)\}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

A fuzzy LA-subring of an LA-ring $R_1 \times R_2$ is to be a fuzzy normal LA-subring of $R_1 \times R_2$ if $(\mu_1 \times \mu_2)(xy) = (\mu_1 \times \mu_2)(yx)$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$. Similarly for anti fuzzy normal LA-subring.

Let A and B be intuitionistic fuzzy sets of LA-rings R_1 and R_2, respectively. The direct product of A and B is denoted by $A \times B$ and defined by $A \times B = \{((x, y), \mu_{A\times B}(x, y), \gamma_{A\times B}(x, y)) \mid$ for all $x \in R_1$ and $y \in R_2\}$, where $\mu_{A\times B}(x, y) = max\{\mu_A(x), \mu_B(y)\}$ and $\gamma_{A\times B}(x, y) = min\{\gamma_A(x), \gamma_B(y)\}$.

An intuitionistic fuzzy set (IFS) $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ of an LA-ring $R_1 \times R_2$ is an intuitionistic anti fuzzy LA-subring (IAFLSR) of $R_1 \times R_2$ if

1. $\mu_{A\times B}(x - y) \leq max\{\mu_{A\times B}(x), \mu_{A\times B}(y)\}$,
2. $\mu_{A\times B}(xy) \leq max\{\mu_{A\times B}(x), \mu_{A\times B}(y)\}$,
3. $\gamma_{A\times B}(x - y) \geq min\{\gamma_{A\times B}(x), \gamma_{A\times B}(y)\}$,
4. $\gamma_{A\times B}(xy) \geq min\{\gamma_{A\times B}(x), \gamma_{A\times B}(y)\}$, for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

An intuitionistic anti fuzzy LA-subring $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ of an LA-ring $R_1 \times R_2$ is an intuitionistic anti fuzzy normal LA-subring (IAFNLSR) of $R_1 \times R_2$ if $\mu_{A\times B}(xy) = \mu_{A\times B}(yx)$ and $\gamma_{A\times B}(xy) = \gamma_{A\times B}(yx)$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

Let $A \times B$ be a non-empty subset of an LA-ring $R_1 \times R_2$. The intuitionistic anti characteristic function of $A \times B$ is denoted by $\chi_{A\times B} = \langle \mu_{\chi_{A\times B}}, \gamma_{\chi_{A\times B}} \rangle$ and defined by

$$\mu_{\chi_{A\times B}}(x) = \begin{cases} 0 \text{ if } x \in A \times B \\ 1 \text{ if } x \notin A \times B \end{cases} \text{ and } \gamma_{\chi_{A\times B}}(x) = \begin{cases} 1 \text{ if } x \in A \times B \\ 0 \text{ if } x \notin A \times B \end{cases}$$

Lemma 1. *[17, Lemma 4.2] If A and B are LA-subrings of LA-rings R_1 and R_2, respectively, then $A \times B$ is an LA-subring of an LA-ring $R_1 \times R_2$ under the same operations defined as in $R_1 \times R_2$.*

Proposition 1. *Let A and B be LA-subrings of LA-rings R_1 and R_2, respectively. Then $A \times B$ is an LA-subring of an LA-ring $R_1 \times R_2$ if and only if the intuitionistic anti characteristic function $\chi_C = \langle \mu_{\chi_C}, \gamma_{\chi_C} \rangle$ of $C = A \times B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Proof. Let $C = A \times B$ be an LA-subring of an LA-ring $R_1 \times R_2$ and $a = (a_1, a_2), b = (b_1, b_2) \in R_1 \times R_2$. If $a, b \in C = A \times B$, then by definition of intuitionistic anti characteristic function $\mu_{\chi_C}(a) = 0 = \mu_{\chi_C}(b)$ and $\gamma_{\chi_C}(a) = 1 = \gamma_{\chi_C}(b)$. Since $a - b$ and $ab \in C$, C being an LA-subring of $R_1 \times R_2$. This implies that

$$
\begin{aligned}
\mu_{\chi_C}(a-b) &= 0 = 0 \vee 0 = \mu_{\chi_C}(a) \vee \mu_{\chi_C}(b), \\
\mu_{\chi_C}(ab) &= 0 = 0 \vee 0 = \mu_{\chi_C}(a) \vee \mu_{\chi_C}(b), \\
\gamma_{\chi_C}(a-b) &= 1 = 1 \wedge 1 = \gamma_{\chi_C}(a) \wedge \gamma_{\chi_C}(b), \\
\gamma_{\chi_C}(ab) &= 1 = 1 \wedge 1 = \gamma_{\chi_C}(a) \wedge \gamma_{\chi_C}(b).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mu_{\chi_C}(a-b) &\leq max\{\mu_{\chi_C}(a), \mu_{\chi_C}(b)\}, \\
\mu_{\chi_C}(ab) &\leq max\{\mu_{\chi_C}(a), \mu_{\chi_C}(b)\}, \\
\gamma_{\chi_C}(a-b) &\geq min\{\gamma_{\chi_C}(a), \gamma_{\chi_C}(b)\}, \\
\gamma_{\chi_C}(ab) &\geq min\{\gamma_{\chi_C}(a), \gamma_{\chi_C}(b)\}.
\end{aligned}
$$

As ab and $ba \in C$, by definition we have $\mu_{\chi_C}(ab) = 0 = \mu_{\chi_C}(ba)$ and $\gamma_{\chi_C}(ab) = 1 = \gamma_{\chi_C}(ba)$, i.e., $\mu_{\chi_C}(ab) = \mu_{\chi_C}(ba)$ and $\gamma_{\chi_C}(ab) = \gamma_{\chi_C}(ba)$. Similarly, we have

$$
\begin{aligned}
\mu_{\chi_C}(a-b) &\leq max\{\mu_{\chi_C}(a), \mu_{\chi_C}(b)\},\ \mu_{\chi_C}(ab) \leq max\{\mu_{\chi_C}(a), \mu_{\chi_C}(b)\}, \\
\gamma_{\chi_C}(a-b) &\geq min\{\gamma_{\chi_C}(a), \gamma_{\chi_C}(b)\},\ \gamma_{\chi_C}(ab) \geq min\{\gamma_{\chi_C}(a), \gamma_{\chi_C}(b)\}, \\
\gamma_{\chi_C}(ab) &= \gamma_{\chi_C}(ba),\ \gamma_{\chi_C}(ab) = \gamma_{\chi_C}(ba),
\end{aligned}
$$

when $a, b \notin C$. Hence the intuitionistic anti characteristic function $\chi_C = \langle \mu_{\chi_C}, \gamma_{\chi_C} \rangle$ of C is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

Conversely, suppose that the intuitionistic anti characteristic function $\chi_C = \langle \mu_{\chi_C}, \gamma_{\chi_C} \rangle$ of $C = A \times B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. Let $a, b \in C = A \times B$, then by definition, we have $\mu_{\chi_C}(a) = 0 = \mu_{\chi_C}(b)$ and $\gamma_{\chi_C}(a) = 1 = \gamma_{\chi_C}(b)$. By our supposition

$$
\begin{aligned}
\mu_{\chi_C}(a-b) &\leq \mu_{\chi_C}(a) \vee \mu_{\chi_C}(b) = 0 \vee 0 = 0, \\
\mu_{\chi_C}(ab) &\leq \mu_{\chi_C}(a) \vee \mu_{\chi_C}(b) = 0 \vee 0 = 0, \\
\gamma_{\chi_C}(a-b) &\geq \gamma_{\chi_C}(a) \wedge \gamma_{\chi_C}(b) = 1 \wedge 1 = 1,
\end{aligned}
$$

$$\gamma_{\chi_C}(ab) \geq \gamma_{\chi_C}(a) \wedge \gamma_{\chi_C}(b) = 1 \wedge 1 = 1.$$

Thus $\mu_{\chi_C}(a-b) = 0 = \mu_{\chi_C}(ab)$ and $\gamma_{\chi_C}(a-b) = 1 = \gamma_{\chi_C}(ab)$, i.e., $a-b$ and $ab \in C$. Hence C is an LA-subring of an LA-ring $R_1 \times R_2$.

Lemma 2. *If $X = A \times B$ and $Y = C \times D$ are two LA-subrings of an LA-ring $R_1 \times R_2$, then their intersection $X \cap Y$ is also an LA-subring of an LA-ring $R_1 \times R_2$.*

Proof. Straight forward.

Theorem 1. *Let $X = A \times B$ and $Y = C \times D$ be two LA-subrings of an LA-ring $R_1 \times R_2$. Then $X \cap Y$ is an LA-subring of an LA-ring $R_1 \times R_2$ if and only if the intuitionistic anti characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = X \cap Y$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Proof. Let $Z = X \cap Y$ be an LA-subring of an LA-ring $R_1 \times R_2$ and $a = (a_1, a_2), b = (b_1, b_2) \in R_1 \times R_2$. If $a, b \in Z = X \cap Y$, then by definition of intuitionistic anti characteristic function $\mu_{\chi_Z}(a) = 0 = \mu_{\chi_Z}(b)$ and $\gamma_{\chi_Z}(a) = 1 = \gamma_{\chi_Z}(b)$. Since $a-b$ and $ab \in Z,$ Z being an LA-subring of an LA-ring $R_1 \times R_2$. This means that

$$\begin{aligned}
\mu_{\chi_Z}(a-b) &= 0 = 0 \vee 0 = \mu_{\chi_Z}(a) \vee \mu_{\chi_Z}(b), \\
\mu_{\chi_Z}(ab) &= 0 = 0 \vee 0 = \mu_{\chi_Z}(a) \vee \mu_{\chi_Z}(b), \\
\gamma_{\chi_Z}(a-b) &= 1 = 1 \wedge 1 = \gamma_{\chi_Z}(a) \wedge \gamma_{\chi_Z}(b), \\
\gamma_{\chi_Z}(ab) &= 1 = 1 \wedge 1 = \gamma_{\chi_Z}(a) \wedge \gamma_{\chi_Z}(b).
\end{aligned}$$

Thus

$$\begin{aligned}
\mu_{\chi_Z}(a-b) &\leq max\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\}, \\
\mu_{\chi_Z}(ab) &\leq max\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\}, \\
\gamma_{\chi_Z}(a-b) &\geq min\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\}, \\
\gamma_{\chi_Z}(a-b) &\geq min\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\}.
\end{aligned}$$

As ab and $ba \in Z$, by definition we get $\mu_{\chi_Z}(ab) = 0 = \mu_{\chi_Z}(ba)$ and $\gamma_{\chi_Z}(ab) = 1 = \gamma_{\chi_Z}(ba)$, i.e., $\mu_{\chi_Z}(ab) = \mu_{\chi_Z}(ba)$ and $\gamma_{\chi_Z}(ab) = \gamma_{\chi_Z}(ba)$. Similarly, we have

$$\begin{aligned}
\mu_{\chi_Z}(a-b) &\leq max\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\},\ \mu_{\chi_Z}(ab) \leq max\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\}, \\
\gamma_{\chi_Z}(a-b) &\geq min\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\},\ \gamma_{\chi_Z}(ab) \geq min\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\}, \\
\gamma_{\chi_Z}(ab) &= \gamma_{\chi_Z}(ba),\ \gamma_{\chi_Z}(ab) = \gamma_{\chi_Z}(ba),
\end{aligned}$$

when $a, b \notin Z$. Hence the intuitionistic anti characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of Z is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

Conversely, assume that the intuitionistic anti characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = X \cap Y$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

Let $a, b \in Z = X \cap Y$, then by definition, we have $\mu_{\chi_Z}(a) = 0 = \mu_{\chi_Z}(b)$ and $\gamma_{\chi_Z}(a) = 1 = \gamma_{\chi_Z}(b)$. By our assumption

$$\begin{aligned}
\mu_{\chi_Z}(a-b) &\leq \mu_{\chi_Z}(a) \vee \mu_{\chi_Z}(b) = 0 \vee 0 = 0, \\
\mu_{\chi_Z}(ab) &\leq \mu_{\chi_Z}(a) \vee \mu_{\chi_Z}(b) = 0 \vee 0 = 0, \\
\gamma_{\chi_Z}(a-b) &\geq \gamma_{\chi_Z}(a) \wedge \gamma_{\chi_Z}(b) = 1 \wedge 1 = 1, \\
\gamma_{\chi_Z}(ab) &\geq \gamma_{\chi_Z}(a) \wedge \gamma_{\chi_Z}(b) = 1 \wedge 1 = 1.
\end{aligned}$$

Thus $\mu_{\chi_Z}(a-b) = 0 = \mu_{\chi_Z}(ab)$ and $\gamma_{\chi_Z}(a-b) = 1 = \gamma_{\chi_Z}(ab)$, i.e., $a-b$ and $ab \in Z$. Hence Z is an LA-subring of an LA-ring $R_1 \times R_2$.

Corollary 1. *Let $\{C_i\}_{i\in I} = \{A_i \times B_i\}_{i\in I}$ be a family of LA-subrings of an LA-ring $R_1 \times R_2$. Then $C = \cap C_i$ is an LA-subring of an LA-ring $R_1 \times R_2$ if and only if the intuitionistic anti characteristic function $\chi_C = \langle \mu_{\chi_c}, \gamma_{\chi_c} \rangle$ of $C = \cap C_i$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Lemma 3. *If A and B are intuitionistic anti fuzzy normal LA-subrings of LA-rings R_1 and R_2, respectively, then $A \times B$ is also an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Proof. Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in R_1\}$ and $B = \{(y, \mu_B(y), \gamma_B(y)) \mid y \in R_2\}$ be intuitionistic anti fuzzy normal LA-subrings of LA-rings R_1 and R_2, respectively. Now $A \times B = \{((x, y), \mu_{A\times B}(x, y), \gamma_{A\times B}(x, y)) \mid \text{for all } x \in R_1 \text{ and } y \in R_2\}$, where

$$\mu_{A\times B}(x, y) = max\{\mu_A(x), \mu_B(y) \text{ and } \gamma_{A\times B}(x, y) = min\{\gamma_A(x), \gamma_B(y)\}.$$

We have to show that $A \times B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned}
\mu_{A\times B}((a, b) - (c, d)) &= \mu_{A\times B}(a - c, b - d) \\
&= max\{\mu_A(a - c), \mu_B(b - d)\} \\
&= \mu_A(a - c) \vee \mu_B(b - d) \\
&\leq \{\mu_A(a) \vee \mu_A(c)\} \vee \{\mu_B(b) \vee \mu_B(d)\} \\
&= \mu_A(a) \vee \{\mu_A(c) \vee \mu_B(b)\} \vee \mu_B(d) \\
&= \mu_A(a) \vee \{\mu_B(b) \vee \mu_A(c)\} \vee \mu_B(d) \\
&= \{\mu_A(a) \vee \mu_B(b)\} \vee \{\mu_A(c) \vee \mu_B(d)\} \\
&= \mu_{A\times B}(a, b) \vee \mu_{A\times B}(c, d)
\end{aligned}$$

and

$$\begin{aligned}
\mu_{A\times B}((a, b) \circ (c, d)) &= \mu_{A\times B}(a \circ c, b \circ d) \\
&= max\{\mu_A(a \circ c), \mu_B(b \circ d)\} \\
&= \mu_A(a \circ c) \vee \mu_B(b \circ d)
\end{aligned}$$

$$\begin{aligned} &\leq \{\mu_A(a) \vee \mu_A(c)\} \vee \{\mu_B(b) \vee \mu_B(d)\} \\ &= \mu_A(a) \vee \{\mu_A(c) \vee \mu_B(b)\} \vee \mu_B(d) \\ &= \mu_A(a) \vee \{\mu_B(b) \vee \mu_A(c)\} \vee \mu_B(d) \\ &= \{\mu_A(a) \vee \mu_B(b)\} \vee \{\mu_A(c) \vee \mu_B(d)\} \\ &= \mu_{A\times B}(a,b) \vee \mu_{A\times B}(c,d). \end{aligned}$$

Thus

$$\begin{aligned} \mu_{A\times B}((a,b)-(c,d)) &\leq \mu_{A\times B}(a,b) \vee \mu_{A\times B}(c,d) \\ \text{and } \mu_{A\times B}((a,b)\circ(c,d)) &\leq \mu_{A\times B}(a,b) \vee \mu_{A\times B}(c,d). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \gamma_{A\times B}((a,b)-(c,d)) &\geq \gamma_{A\times B}(a,b) \wedge \gamma_{A\times B}(c,d) \\ \text{and } \gamma_{A\times B}((a,b)\circ(c,d)) &\geq \gamma_{A\times B}(a,b) \wedge \gamma_{A\times B}(c,d). \end{aligned}$$

Therefore $A \times B$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned} \mu_{A\times B}((a,b)\circ(c,d)) &= \mu_{A\times B}(ac,bd) \\ &= max\{\mu_A(ac), \mu_B(bd)\} \\ &= max\{\mu_A(ca), \mu_B(db)\} \\ &= \mu_{A\times B}(ca,db) = \mu_{A\times B}((c,d)\circ(a,b)). \end{aligned}$$

Similarly, $\gamma_{A\times B}((a,b)\circ(c,d)) = \gamma_{A\times B}((c,d)\circ(a,b))$. Hence $A \times B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

Proposition 2. *If $X = A \times B$ and $Y = C \times D$ are two intuitionistic anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$, then their intersection $X \cap Y$ is also an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Proof. Let $X = A\times B = \{((x_1,x_2), \mu_{A\times B}(x_1,x_2), \gamma_{A\times B}(x_1,x_2)) \mid \text{for all } (x_1,x_2) \in R_1\times R_2\}$ and $Y = C \times D = \{((y_1,y_2), \mu_{C\times D}(y_1,y_2), \gamma_{C\times D}(y_1,y_2)) \mid \text{for all } (y_1,y_2) \in R_1 \times R_2\}$ be two intuitionistic anti fuzzy normal LA-subrings of an LA-ring $R_1\times R_2$. Let $Z = X\cap Y$ and $Z = \{((z_1,z_2), \mu_Z(z_1,z_2), \gamma_Z(z_1,z_2)) \mid (z_1,z_2) \in R_1 \times R_2\}$, where

$$\begin{aligned} \mu_Z(z_1,z_2) &= \mu_{X\cap Y}(z_1,z_2) = max\{\mu_X(z_1,z_2), \mu_Y(z_1,z_2)\} \\ \text{and } \gamma_Z(z_1,z_2) &= \gamma_{X\cap Y}(z_1,z_2) = min\{\gamma_X(z_1,z_2), \gamma_Y(z_1,z_2)\}. \end{aligned}$$

Now

$$\begin{aligned} \mu_Z((z_1,z_2)-(z_3,z_4)) &= \mu_{X\cap Y}((z_1,z_2)-(z_3,z_4)) \\ &= max\{\mu_X((z_1,z_2)-(z_3,z_4)), \mu_Y((z_1,z_2)-(z_3,z_4))\} \\ &\leq \{\mu_X(z_1,z_2) \vee \mu_X(z_3,z_4)\} \vee \{\mu_Y(z_1,z_2) \vee \mu_Y(z_3,z_4)\} \end{aligned}$$

$$\begin{aligned}
&= \{\mu_X(z_1,z_2) \vee \{\mu_X(z_3,z_4) \vee \mu_Y(z_1,z_2)\} \vee \mu_Y(z_3,z_4)\} \\
&= \{\mu_X(z_1,z_2) \vee \{\mu_Y(z_1,z_2) \vee \mu_X(z_3,z_4)\} \vee \mu_Y(z_3,z_4)\} \\
&= \{\mu_X(z_1,z_2) \vee \mu_Y(z_1,z_2)\} \vee \{\mu_X(z_3,z_4) \vee \mu_Y(z_3,z_4)\} \\
&= max\{\mu_{X\cap Y}(z_1,z_2), \mu_{X\cap Y}(z_3,z_4)\} \\
&= max\{\mu_Z(z_1,z_2), \mu_Z(z_3,z_4)\}
\end{aligned}$$

and

$$\begin{aligned}
\mu_Z((z_1,z_2)\circ(z_3,z_4)) &= \mu_{X\cap Y}((z_1,z_2)\circ(z_3,z_4)) \\
&= max\{\mu_X((z_1,z_2)\circ(z_3,z_4)), \mu_Y((z_1,z_2)\circ(z_3,z_4))\} \\
&\leq \{\mu_X(z_1,z_2) \vee \mu_X(z_3,z_4)\} \vee \{\mu_Y(z_1,z_2) \vee \mu_Y(z_3,z_4)\} \\
&= \{\mu_X(z_1,z_2) \vee \{\mu_X(z_3,z_4) \vee \mu_Y(z_1,z_2)\} \vee \mu_Y(z_3,z_4)\} \\
&= \{\mu_X(z_1,z_2) \vee \{\mu_Y(z_1,z_2) \vee \mu_X(z_3,z_4)\} \vee \mu_Y(z_3,z_4)\} \\
&= \{\mu_X(z_1,z_2) \vee \mu_Y(z_1,z_2)\} \vee \{\mu_X(z_3,z_4) \vee \mu_Y(z_3,z_4)\} \\
&= max\{\mu_{X\cap Y}(z_1,z_2), \mu_{X\cap Y}(z_3,z_4)\} \\
&= max\{\mu_Z(z_1,z_2), \mu_Z(z_3,z_4)\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mu_Z((z_1,z_2)-(z_3,z_4)) &\leq max\{\mu_Z(z_1,z_2), \mu_Z(z_3,z_4)\} \\
\text{and } \mu_Z((z_1,z_2)\circ(z_3,z_4)) &\leq max\{\mu_Z(z_1,z_2), \mu_Z(z_3,z_4)\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\gamma_Z((z_1,z_2)-(z_3,z_4)) &\geq min\{\gamma_Z(z_1,z_2), \gamma_Z(z_3,z_4)\} \\
\text{and } \gamma_Z((z_1,z_2)\circ(z_3,z_4)) &\geq min\{\gamma_Z(z_1,z_2), \gamma_Z(z_3,z_4)\}.
\end{aligned}$$

Therefore $Z = (\mu_Z, \gamma_Z)$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned}
\mu_Z((z_1,z_2)\circ(z_3,z_4)) &= \mu_{X\cap Y}(z_1z_3, z_2z_4) \\
&= max\{\mu_X(z_1z_3, z_2z_4), \mu_Y(z_1z_3, z_2z_4)\} \\
&= max\{\mu_X(z_3z_1, z_4z_2), \mu_Y(z_3z_1, z_4z_2)\} \\
&= \mu_{X\cap Y}(z_3z_1, z_4z_2) \\
&= \mu_Z((z_3,z_4)\circ(z_1,z_2)).
\end{aligned}$$

Similarly, $\gamma_Z((z_1,z_2)\circ(z_3,z_4)) = \gamma_Z((z_3,z_4)\circ(z_1,z_2))$. Hence $Z = X \cap Y$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

Corollary 2. *If* $\{C_i\}_{i\in I} = \{A_i \times B_i\}_{i\in I}$ *is a family of intuitionistic anti fuzzy normal LA-subrings of an LA-ring* $R_1 \times R_2$*, then* $C = \cap C_i$ *is also an intuitionistic anti fuzzy normal LA-subring of an LA-ring* $R_1 \times R_2$.

Theorem 2. *If $X = A\times B$ and $Y = C\times D$ are intuitionistic anti fuzzy normal LA-subrings of LA-rings $R' = R_1 \times R_2$ and $R'' = R_3 \times R_4$, respectively, then $Z = X \times Y$ is also an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R'\times R'' = (R_1\times R_2)\times(R_3\times R_4)$.*

Proof. Let $X = A \times B = \{((x_1, x_2), \mu_{A\times B}(x_1, x_2), \gamma_{A\times B}(x_1, x_2)) \mid \text{for all } (x_1, x_2) \in R_1 \times R_2\}$ and $Y = C \times D = \{((y_1, y_2), \mu_{C\times D}(y_1, y_2), \gamma_{C\times D}(y_1, y_2)) \mid \text{for all } (y_1, y_2) \in R_3 \times R_4\}$ be intuitionistic anti fuzzy normal LA-subrings of LA-rings $R' = R_1 \times R_2$ and $R'' = R_3 \times R_4$, respectively. Let $Z = X \times Y$ and $Z = \{((z', z''), \mu_Z(z', z''), \gamma_Z(z', z'')) \mid (z', z'') = ((z_1, z_2), (z_3, z_4)) \in R' \times R''\}$, where

$$\begin{aligned} \mu_Z(z', z'') &= \mu_{X\times Y}((z_1, z_2), (z_3, z_4)) = max\{\mu_X(z_1, z_2), \mu_Y(z_3, z_4)\}, \\ \text{and } \gamma_Z(z', z'') &= \gamma_{X\times Y}((z_1, z_2), (z_3, z_4)) = min\{\gamma_X(z_1, z_2), \gamma_Y(z_3, z_4)\}. \end{aligned}$$

Now

$$\begin{aligned} & \mu_Z(((z_1, z_2), (z_3, z_4)) - ((z_5, z_6), (z_7, z_8))) \\ = \;& \mu_{X\times Y}(((z_1, z_2), (z_3, z_4)) - ((z_5, z_6), (z_7, z_8))) \\ = \;& \mu_{X\times Y}(((z_1, z_2) - (z_5, z_6)), ((z_3, z_4) - (z_7, z_8))) \\ = \;& max\{\mu_X((z_1, z_2) - (z_5, z_6)), \mu_Y((z_3, z_4) - (z_7, z_8))\} \\ \leq \;& max\{(\mu_X(z_1, z_2) \vee \mu_X(z_5, z_6)), (\mu_Y(z_3, z_4) \vee \mu_Y(z_7, z_8))\} \\ = \;& ((\mu_X(z_1, z_2) \vee \mu_Y(z_5, z_6)) \vee (\mu_X(z_3, z_4) \vee \mu_Y(z_7, z_8))) \\ = \;& ((\mu_X(z_1, z_2) \vee \mu_Y(z_3, z_4)) \vee (\mu_X(z_5, z_6) \vee \mu_Y(z_7, z_8))) \\ = \;& max\{(\mu_X(z_1, z_2) \vee \mu_Y(z_3, z_4)), (\mu_X(z_5, z_6) \vee \mu_Y(z_7, z_8))\} \\ = \;& max\{\mu_{X\times Y}((z_1, z_2), (z_3, z_4)), \mu_{X\times Y}((z_5, z_6), (z_7, z_8))\} \\ = \;& max\{\mu_Z((z_1, z_2), (z_3, z_4)), \mu_Z((z_5, z_6), (z_7, z_8))\}. \end{aligned}$$

and

$$\begin{aligned} & \mu_Z(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \\ = \;& \mu_{X\times Y}(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \\ = \;& \mu_{X\times Y}(((z_1, z_2) \circ (z_5, z_6)), ((z_3, z_4) \circ (z_7, z_8))) \\ = \;& max\{\mu_X((z_1, z_2) \circ (z_5, z_6)), \mu_Y((z_3, z_4) \circ (z_7, z_8))\} \\ \leq \;& max\{(\mu_X(z_1, z_2) \vee \mu_X(z_5, z_6)), (\mu_Y(z_3, z_4) \vee \mu_Y(z_7, z_8))\} \\ = \;& ((\mu_X(z_1, z_2) \vee \mu_Y(z_5, z_6)) \vee (\mu_X(z_3, z_4) \vee \mu_Y(z_7, z_8))) \\ = \;& ((\mu_X(z_1, z_2) \vee \mu_Y(z_3, z_4)) \vee (\mu_X(z_5, z_6) \vee \mu_Y(z_7, z_8))) \\ = \;& max\{(\mu_X(z_1, z_2) \vee \mu_Y(z_3, z_4)), (\mu_X(z_5, z_6) \vee \mu_Y(z_7, z_8))\} \\ = \;& max\{\mu_{X\times Y}((z_1, z_2), (z_3, z_4)), \mu_{X\times Y}((z_5, z_6), (z_7, z_8))\} \\ = \;& max\{\mu_Z((z_1, z_2), (z_3, z_4)), \mu_Z((z_5, z_6), (z_7, z_8))\}. \end{aligned}$$

Similarly

$$\gamma_Z(((z_1, z_2), (z_3, z_4)) - ((z_5, z_6), (z_7, z_8)))$$

$$\begin{aligned} &\geq \; min\{\gamma_Z((z_1,z_2),(z_3,z_4)),\gamma_Z((z_5,z_6),(z_7,z_8))\} \\ &\quad \text{and } \gamma_Z(((z_1,z_2),(z_3,z_4))\circ((z_5,z_6),(z_7,z_8))) \\ &\geq \; min\{\gamma_Z((z_1,z_2),(z_3,z_4)),\gamma_Z((z_5,z_6),(z_7,z_8))\} \end{aligned}$$

Thus $Z=(\mu_Z,\gamma_Z)$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R\prime\times R''$. Now

$$\begin{aligned} & \mu_Z(((z_1,z_2),(z_3,z_4))\circ((z_5,z_6),(z_7,z_8))) \\ = \; & \mu_{X\times Y}(((z_1,z_2)\circ(z_5,z_6)),((z_3,z_4)\circ(z_7,z_8))) \\ = \; & max\{\mu_X((z_1,z_2)\circ(z_5,z_6)),\mu_Y((z_3,z_4)\circ(z_7,z_8))\} \\ = \; & max\{\mu_X((z_5,z_6)\circ(z_1,z_2)),\mu_Y((z_7,z_8)\circ(z_3,z_4))\} \\ = \; & \mu_{X\times Y}(((z_5,z_6)\circ(z_1,z_2)),((z_7,z_8)\circ(z_3,z_4))) \\ = \; & \mu_Z(((z_5,z_6),(z_7,z_8))\circ((z_1,z_2),(z_3,z_4))). \end{aligned}$$

Similarly

$$\begin{aligned} & \gamma_Z(((z_1,z_2),(z_3,z_4))\circ((z_5,z_6),(z_7,z_8))) \\ = \; & \gamma_Z(((z_5,z_6),(z_7,z_8))\circ((z_1,z_2),(z_3,z_4))). \end{aligned}$$

Hence $Z=X\times Y$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R\prime\times R''$.

Proposition 3. *If an IFS $A\times B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$, then $\Box A\times B=\left(\mu_{A\times B},\overline{\mu}_{A\times B}\right)$ (resp. $\Diamond A\times B=\left(\overline{\gamma}_{A\times B},\gamma_{A\times B}\right)$) is also an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$.*

Proof. Let $A\times B$ be an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$. We have to show that $\Box A\times B=\left(\mu_{A\times B},\overline{\mu}_{A\times B}\right)$ is also an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$. Now

$$\begin{aligned} \overline{\mu}_{A\times B}((x_1,x_2)-(y_1,y_2)) &= 1-\mu_{A\times B}((x_1,x_2)-(y_1,y_2)) \\ &\geq 1-max\left\{\mu_{A\times B}(x_1,x_2),\mu_{A\times B}(y_1,y_2)\right\} \\ &= min\left\{1-\mu_{A\times B}(x_1,x_2),1-\mu_{A\times B}(y_1,y_2)\right\} \\ &= min\{\overline{\mu}_{A\times B}(x_1,x_2),\overline{\mu}_{A\times B}(y_1,y_2)\}. \\ \text{and } \overline{\mu}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) &= 1-\mu_{A\times B}((x_1,x_2)\circ(y_1,y_2)) \\ &\geq 1-max\left\{\mu_{A\times B}(x_1,x_2),\mu_{A\times B}(y_1,y_2)\right\} \\ &= min\left\{1-\mu_{A\times B}(x_1,x_2),1-\mu_{A\times B}(y_1,y_2)\right\} \\ &= min\{\overline{\mu}_{A\times B}(x_1,x_2),\overline{\mu}_{A\times B}(y_1,y_2)\}. \end{aligned}$$

Thus $\Box A\times B=\left(\mu_{A\times B},\overline{\mu}_{A\times B}\right)$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1\times R_2$. Now

$$\overline{\mu}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) \;=\; 1-\mu_{A\times B}((x_1,x_2)\circ(y_1,y_2))$$

$$\begin{aligned} &= 1-\mu_{A\times B}((y_1,y_2)\circ(x_1,x_2)) \\ &= \overline{\mu}_{A\times B}((y_1,y_2)\circ(x_1,x_2)). \end{aligned}$$

Hence $\Box A\times B=\left(\mu_{A\times B},\overline{\mu}_{A\times B}\right)$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$.

Corollary 3. *An IFS $A\times B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$ if and only if $\Box A\times B=\left(\mu_{A\times B},\overline{\mu}_{A\times B}\right)$ (resp. $\Diamond A\times B=\left(\overline{\gamma}_{A\times B},\gamma_{A\times B}\right)$) is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$.*

Theorem 3. *An IFS $A\times B=(\mu_{A\times B},\gamma_{A\times B})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$ if and only if the fuzzy subsets $\mu_{A\times B}$ and $\overline{\gamma}_{A\times B}$ are anti fuzzy normal LA-subrings of an LA-ring $R_1\times R_2$.*

Proof. Let $A\times B=(\mu_{A\times B},\gamma_{A\times B})$ be an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$. This implies that $\mu_{A\times B}$ is an anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$. We have to show that $\overline{\gamma}_{A\times B}$ is also an anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$. Now

$$\begin{aligned} \overline{\gamma}_{A\times B}((x_1,x_2)-(y_1,y_2)) &= 1-\gamma_{A\times B}((x_1,x_2)-(y_1,y_2)) \\ &\leq 1-min\{\gamma_{A\times B}(x_1,x_2),\gamma_{A\times B}(y_1,y_2)\} \\ &= max\{1-\gamma_{A\times B}(x_1,x_2),1-\gamma_{A\times B}(y_1,y_2)\} \\ &= max\{\overline{\gamma}_{A\times B}(x_1,x_2),\overline{\gamma}_{A\times B}(y_1,y_2)\}. \\ \text{and } \overline{\gamma}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) &= 1-\gamma_{A\times B}((x_1,x_2)\circ(y_1,y_2)) \\ &\leq 1-min\{\gamma_{A\times B}(x_1,x_2),\gamma_{A\times B}(y_1,y_2)\} \\ &= max\{1-\gamma_{A\times B}(x_1,x_2),1-\gamma_{A\times B}(y_1,y_2)\} \\ &= max\{\overline{\gamma}_{A\times B}(x_1,x_2),\overline{\gamma}_{A\times B}(y_1,y_2)\}. \end{aligned}$$

Thus $\overline{\gamma}_{A\times B}$ is an anti fuzzy LA-subring of an LA-ring $R_1\times R_2$. Now

$$\begin{aligned} \overline{\gamma}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) &= 1-\gamma_{A\times B}((x_1,x_2)\circ(y_1,y_2)) \\ &= 1-\gamma_{A\times B}((y_1,y_2)\circ(x_1,x_2)) \\ &= \overline{\gamma}_{A\times B}((y_1,y_2)\circ(x_1,x_2)). \end{aligned}$$

Hence $\overline{\gamma}_{A\times B}$ is an anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$.

Conversely, suppose that $\mu_{A\times B}$ and $\overline{\gamma}_{A\times B}$ are anti fuzzy normal LA-subrings of an LA-ring $R_1\times R_2$. We have to show that $A\times B=(\mu_{A\times B},\gamma_{A\times B})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2$. Now

$$\begin{aligned} 1-\gamma_{A\times B}((x_1,x_2)-(y_1,y_2)) &= \overline{\gamma}_{A\times B}((x_1,x_2)-(y_1,y_2)) \\ &\leq max\{\overline{\gamma}_{A\times B}(x_1,x_2),\overline{\gamma}_{A\times B}(y_1,y_2)\} \\ &= max\{1-\gamma_{A\times B}(x_1,x_2),1-\gamma_{A\times B}(y_1,y_2)\} \end{aligned}$$

$$\begin{aligned} & = 1 - min\{\gamma_{A\times B}(x_1,x_2), \gamma_{A\times B}(y_1,y_2)\} \\ \text{and } 1-\gamma_{A\times B}((x_1,x_2)\circ(y_1,y_2)) & = \overline{\gamma}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) \\ & \leq max\{\overline{\gamma}_{A\times B}(x_1,x_2), \overline{\gamma}_{A\times B}(y_1,y_2)\} \\ & = max\{1-\gamma_A(x_1,x_2), 1-\gamma_A(y_1,y_2)\} \\ & = 1-min\{\gamma_A(x_1,x_2), \gamma_A(y_1,y_2)\}. \end{aligned}$$

Thus $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned} 1-\gamma_{A\times B}((x_1,x_2)\circ(y_1,y_2)) & = \overline{\gamma}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) \\ & = \overline{\gamma}_{A\times B}((y_1,y_2)\circ(x_1,x_2)) \\ & = 1-\gamma_A((y_1,y_2)\circ(x_1,x_2)). \end{aligned}$$

Hence $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

Theorem 4. *An IFS $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$ if and only if the fuzzy subsets $\overline{\mu}_{A\times B}$ and $\gamma_{A\times B}$ are fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$.*

Proof. Let $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ be an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. This means that $\gamma_{A\times B}$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. We have to show that $\overline{\mu}_{A\times B}$ is also a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned} \overline{\mu}_{A\times B}((x_1,x_2)-(y_1,y_2)) & = 1-\mu_{A\times B}((x_1,x_2)-(y_1,y_2)) \\ & \geq 1-max\{\mu_{A\times B}(x_1,x_2), \mu_{A\times B}(y_1,y_2)\} \\ & = min\{1-\mu_{A\times B}(x_1,x_2), 1-\mu_{A\times B}(y_1,y_2)\} \\ & = min\{\overline{\mu}_{A\times B}(x_1,x_2), \overline{\mu}_{A\times B}(y_1,y_2)\}. \\ \text{and } \overline{\mu}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) & = 1-\mu_{A\times B}((x_1,x_2)\circ(y_1,y_2)) \\ & \geq 1-max\{\mu_A(x_1,x_2), \mu_A(y_1,y_2)\} \\ & = min\{1-\mu_A(x_1,x_2), 1-\mu_A(y_1,y_2)\} \\ & = min\{\overline{\mu}_{A\times B}(x_1,x_2), \overline{\mu}_{A\times B}(y_1,y_2)\}. \end{aligned}$$

Thus $\overline{\mu}_{A\times B}$ is a fuzzy LA-subring of an LA-ring $R_1 \times R_2$.

$$\begin{aligned} \overline{\mu}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) & = 1-\mu_A((x_1,x_2)\circ(y_1,y_2)) \\ & = 1-\mu_A((y_1,y_2)\circ(x_1,x_2)) \\ & = \overline{\mu}_{A\times B}((y_1,y_2)\circ(x_1,x_2)). \end{aligned}$$

Hence $\overline{\mu}_{A\times B}$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

Conversely, assume that $\overline{\mu}_{A\times B}$and $\gamma_{A\times B}$ are fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$. We have to show that $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned}
1-\mu_{A\times B}((x_1,x_2)-(y_1,y_2)) &= \overline{\mu}_{A\times B}((x_1,x_2)-(y_1,y_2)) \\
&\geq min\{\overline{\mu}_{A\times B}(x_1,x_2),\overline{\mu}_{A\times B}(y_1,y_2)\} \\
&= min\{1-\mu_{A\times B}(x_1,x_2),1-\mu_{A\times B}(y_1,y_2)\} \\
&= 1-max\{\mu_{A\times B}(x_1,x_2),\mu_{A\times B}(y_1,y_2)\} \\
\text{and } 1-\mu_{A\times B}((x_1,x_2)\circ(y_1,y_2)) &= \overline{\mu}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) \\
&\geq min\{\overline{\mu}_{A\times B}(x_1,x_2),\overline{\mu}_{A\times B}(y_1,y_2)\} \\
&= min\{1-\mu_{A\times B}(x_1,x_2),1-\mu_{A\times B}(y_1,y_2)\} \\
&= 1-max\{\mu_{A\times B}(x_1,x_2),\mu_{A\times B}(y_1,y_2)\}.
\end{aligned}$$

Thus $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned}
1-\mu_{A\times B}((x_1,x_2)\circ(y_1,y_2)) &= \overline{\mu}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) \\
&= \overline{\mu}_{A\times B}((y_1,y_2)\circ(x_1,x_2)) \\
&= 1-\mu_{A\times B}((y_1,y_2)\circ(x_1,x_2)).
\end{aligned}$$

Hence $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

Lemma 4. *Let A and B be intuitionistic fuzzy sets of LA-rings R_1 and R_2 with left identities e_1 and e_2, respectively. If $A \times B$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1 \times R_2$, then at least one of the following two statements must hold.*

1. $\mu_A(x) \geq \mu_B(e_2)$ *and* $\gamma_A(x) \leq \gamma_B(e_2)$, *for all* $x \in R_1$.
2. $\mu_B(x) \geq \mu_A(e_1)$ *and* $\gamma_B(x) \leq \gamma_A(e_1)$, *for all* $x \in R_2$.

Proof. Let $A \times B$ be an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1 \times R_2$. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a and b in R_1 and R_2, respectively such that

$$\begin{aligned}
\mu_A(a) &\leq \mu_B(e_2) \text{ and } \gamma_A(a) \geq \gamma_B(e_2), \\
\mu_B(b) &\leq \mu_A(e_1) \text{ and } \gamma_B(b) \geq \gamma_A(e_1).
\end{aligned}$$

Thus

$$\begin{aligned}
\mu_{A\times B}(a,b) &= max\{\mu_A(a),\mu_B(b)\} \\
&\leq max\{\mu_A(e_1),\mu_B(e_2)\} \\
&= \mu_{A\times B}(e_1,e_2) \\
\text{and } \gamma_{A\times B}(a,b) &= min\{\gamma_A(a),\gamma_B(b)\}
\end{aligned}$$

$$\begin{array}{cl} \geq & min(\gamma_A(e_1), \gamma_B(e_2)) \\ = & \gamma_{A\times B}(e_1, e_2). \end{array}$$

This implies that $A \times B$ is not an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1 \times R_2$. Hence either $\mu_A(x) \geq \mu_B(e_2)$ and $\gamma_A(x) \leq \gamma_B(e_2)$, for all $x \in R_1$ or $\mu_B(x) \geq \mu_A(e_1)$ and $\gamma_B(x) \leq \gamma_A(e_1)$, for all $x \in R_2$.

Theorem 5. *Let A and B be intuitionistic fuzzy sets of LA-rings R_1 and R_2 with left identities e_1 and e_2, respectively and $A \times B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. Then the following conditions are true.*

1. *If $\mu_A(x) \geq \mu_B(e_2)$ and $\gamma_A(x) \leq \gamma_B(e_2)$, for all $x \in R_1$, then A is an intuitionistic anti fuzzy normal LA-subring of R_1.*

2. *If $\mu_B(x) \geq \mu_A(e_1)$ and $\gamma_B(x) \leq \gamma_A(e_1)$, for all $x \in R_2$, then B is an intuitionistic anti fuzzy normal LA-subring of R_2.*

Proof. 1. Let $\mu_A(x) \geq \mu_B(e_2)$ and $\gamma_A(x) \leq \gamma_B(e_2)$ for all $x \in R_1$, and $y \in R_1$. We have to show that A is an intuitionistic anti fuzzy normal LA-subring of an LA-ring R_1. Now

$$\begin{array}{rcl} \mu_A(x-y) & = & \mu_A(x+(-y)) \\ & = & max\{\mu_A(x+(-y)), \mu_B(e_2+(-e_2))\} \\ & = & \mu_{A\times B}(x+(-y), e_2+(-e_2)) \\ & = & \mu_{A\times B}((x,e_2)+(-y,-e_2)) \\ & = & \mu_{A\times B}((x,e_2)-(y,e_2)) \\ & \leq & \mu_{A\times B}(x,e_2) \vee \mu_{A\times B}(y,e_2) \\ & = & max\{max\{\mu_A(x), \mu_B(e_2)\}, max\{\mu_A(y), \mu_B(e_2)\}\} \\ & = & \mu_A(x) \vee \mu_A(y) \end{array}$$

and

$$\begin{array}{rcl} \mu_A(xy) & = & max\{\mu_A(xy), \mu_B(e_2e_2)\} \\ & = & \mu_{A\times B}(xy, e_2e_2) \\ & = & \mu_{A\times B}((x,e_2)\circ(y,e_2)) \\ & \leq & \mu_{A\times B}(x,e_2) \vee \mu_{A\times B}(y,e_2) \\ & = & max\{max\{\mu_A(x), \mu_B(e_2)\}, max\{\mu_A(y), \mu_B(e_2)\}\} \\ & = & \mu_A(x) \vee \mu_A(y). \end{array}$$

Similarly, we have

$$\gamma_A(x-y) \geq min\{\gamma_A(x), \gamma_A(y)\} \text{ and } \gamma_A(xy) \geq min\{\gamma_A(x), \gamma_A(y)\}.$$

Thus A is an intuitionistic anti fuzzy LA-subring of an LA-ring R_1. Now

$$\mu_A(xy) \quad = \quad max\{\mu_A(xy), \mu_B(e_2e_2)\}$$

$$\begin{aligned}
&= \mu_{A\times B}\,(xy, e_2e_2)\\
&= \mu_{A\times B}\,((x, e_2)\circ(y, e_2))\\
&= \mu_{A\times B}\,((y, e_2)\circ(x, e_2))\\
&= \mu_{A\times B}(yx, e_2e_2)\\
&= max\{\mu_A(yx), \mu_B(e_2e_2)\}\\
&= \mu_A(yx).
\end{aligned}$$

Similarly, $\gamma_B(xy) = \gamma_B(yx)$. Hence A is an intuitionistic anti fuzzy normal LA-subring of an LA-ring R_1. 2. is same as 1.

3. Direct Product of Finite Intuitionistic Anti Fuzzy Normal LA-subrings

We define the direct product of intuitionistic fuzzy sets $A_1, A_2, ..., A_n$ of LA-rings $R_1, R_2, ..., R_n$, respectively and examine the some fundamental properties of direct product of intuitionistic anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Let $\mu_1, \mu_2, ..., \mu_n$ be fuzzy subsets of LA-rings $R_1, R_2, ..., R_n$, respectively. The direct product of fuzzy subsets $\mu_1, \mu_2, ..., \mu_n$ is denoted by $\mu_1 \times \mu_2 \times ... \times \mu_n$ and defined by $(\mu_1 \times \mu_2 \times ... \times \mu_n)(x_1, x_2, ..., x_n) = min\{\mu_1(x_1), \mu_2(x_2), ..., \mu_n(x_n)\}$.

A fuzzy subset $\mu_1 \times \mu_2 \times ... \times \mu_n$ of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is to be a fuzzy LA-subring of $R_1 \times R_2 \times ... \times R_n$ if

1. $(\mu_1 \times \mu_2 \times ... \times \mu_n)(x - y) \geq min\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x), (\mu_1 \times \mu_2 \times ... \times \mu_n)(y)\}$,

2. $(\mu_1 \times \mu_2 \times ... \times \mu_n)(xy) \geq min\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x), (\mu_1 \times \mu_2 \times ... \times \mu_n)(y)\}$ for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$.

A fuzzy subset $\mu_1 \times \mu_2 \times ... \times \mu_n$ of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is to be an anti fuzzy LA-subring of $R_1 \times R_2 \times ... \times R_n$ if

1. $(\mu_1 \times \mu_2 \times ... \times \mu_n)(x - y) \leq max\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x), (\mu_1 \times \mu_2 \times ... \times \mu_n)(y)\}$,

2. $\mu_1 \times \mu_2 \times ... \times \mu_n(xy) \leq max\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x), (\mu_1 \times \mu_2 \times ... \times \mu_n)(y)\}$ for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$.

A fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is said to be a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$ if $(\mu_1 \times \mu_2 \times ... \times \mu_n)(xy) = (\mu_1 \times \mu_2 \times ... \times \mu_n)(yx)$ for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$. Similarly for anti fuzzy normal LA-subring.

Let $A_1, A_2, ..., A_n$ be intuitionistic fuzzy sets of LA-rings $R_1, R_2, ..., R_n$, respectively. The direct product of intuitionistic fuzzy sets $A_1, A_2..., A_n$ is denoted by $A_1 \times A_2 \times ... \times A_n$ and defined by $A_1 \times A_2 \times ... \times A_n = \{(x, \mu_{A_1\times A_2\times...\times A_n}(x), \gamma_{A_1\times A_2\times...\times A_n}(x)) \mid$ for all $x = (x_1, x_2, ..., x_n) \in R_1 \times R_2 \times ... \times R_n\}$, where

$$\begin{aligned}
\mu_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n) &= max\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\}\\
\text{and } \gamma_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n) &= min\{\gamma_{A_1}(x_1), \gamma_{A_2}(x_2), ..., \gamma_{A_n}(x_n)\}.
\end{aligned}$$

An intuitionistic fuzzy set (IFS) $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1 \times A_2 \times ... \times A_n}, \gamma_{A_1 \times A_2 \times ... \times A_n})$ of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is to be an intuitionistic anti fuzzy LA-subring (IAFLSR) of $R_1 \times R_2 \times ... \times R_n$ if

1. $\mu_{A_1 \times A_2 \times ... \times A_n}(x-y) \leq max\{\mu_{A_1 \times A_2 \times ... \times A_n}(x), \mu_{A_1 \times A_2 \times ... \times A_n}(y)\}$,
2. $\mu_{A_1 \times A_2 \times ... \times A_n}(xy) \leq max\{\mu_{A_1 \times A_2 \times ... \times A_n}(x), \mu_{A_1 \times A_2 \times ... \times A_n}(y)\}$,
3. $\gamma_{A_1 \times A_2 \times ... \times A_n}(x-y) \geq min\{\gamma_{A_1 \times A_2 \times ... \times A_n}(x), \gamma_{A_1 \times A_2 \times ... \times A_n}(y)\}$,
4. $\gamma_{A_1 \times A_2 \times ... \times A_n}(xy) \geq min\{\gamma_{A_1 \times A_2 \times ... \times A_n}(x), \gamma_{A_1 \times A_2 \times ... \times A_n}(y)\}$, for all $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$.

An intuitionistic anti fuzzy LA-subring $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1 \times A_2 \times ... \times A_n}, \gamma_{A_1 \times A_2 \times ... \times A_n})$ of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is said to be an intuitionistic anti fuzzy normal LA-subring (IAFNLSR) of $R_1 \times R_2 \times ... \times R_n$ if

1. $\mu_{A_1 \times A_2 \times ... \times A_n}(xy) = \mu_{A_1 \times A_2 \times ... \times A_n}(yx)$
2. $\gamma_{A_1 \times A_2 \times ... \times A_n}(xy) = \gamma_{A_1 \times A_2 \times ... \times A_n}(yx)$ for all $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$.

Let $A_1 \times A_2 \times ... \times A_n$ be a non-empty subset of an LA-ring $R = R_1 \times R_2 \times ... \times R_n$. The intuitionistic anti characteristic function of $A = A_1 \times A_2 \times ... \times A_n$ is denoted by $\chi_{A_1 \times A_2 \times ... \times A_n} = \langle \mu_{\chi_{A_1 \times A_2 \times ... \times A_n}}, \gamma_{\chi_{A_1 \times A_2 \times .. \times A_n}} \rangle$ and defined by

$$\mu_{\chi_A}(x) = \begin{cases} 0 \text{ if } x \in A \\ 1 \text{ if } x \notin A \end{cases} \quad \text{and } \gamma_{\chi_A}(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

[17] If R_1, R_2 are LA-rings, then direct product $R_1 \times R_2$ of R_1 and R_2 is an LA-ring with pointwise addition '+' and multiplication '$\circ$' defined as $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \circ (c, d) = (ac, bd)$, respectively for every $(a, b), (c, d) \in R_1 \times R_2$. Likewise the direct product $R = \times_{i \in \Omega} R_i$ of a family of LA-rings $\{R_i : i \in \Omega\}$ has the structure of an LA-ring with the operations of addition and multiplication defined as

$$\begin{aligned} a + b &= (a_1, a_2, a_3, ...) + (b_1, b_2, b_3, ...) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3, ...) \\ \text{and } a \circ b &= (a_1, a_2, a_3, ...) \circ (b_1, b_2, b_3, ...) \\ &= (a_1 b_1, a_2 b_2, a_3 b_3, ...) \end{aligned}$$

for all $a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n) \in R$.

Lemma 5. *If $A_1, A_2, ..., A_n$ are LA-subrings of LA-rings $R_1, R_2, ..., R_n$, respectively, then $A_1 \times A_2 \times ... \times A_n$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ under the same operations defined as in [17].*

Proof. Straight forward.

Lemma 6. *Let $A_1, A_2, ..., A_n$ be LA-subrings of LA-rings $R_1, R_2, ..., R_n$, respectively. Then $A_1 \times A_2 \times ... \times A_n$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ if and only if the intuitionistic anti characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of $A = A_1 \times A_2 \times ... \times A_n$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $A = A_1 \times A_2 \times ... \times A_n$ be an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ and $a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n) \in R_1 \times R_2 \times ... \times R_n$. If $a, b \in A = A_1 \times A_2 \times ... \times A_n$, then by definition of intuitionistic anti characteristic function $\mu_{\chi_A}(a) = 0 = \mu_{\chi_A}(b)$ and $\gamma_{\chi_A}(a) = 1 = \gamma_{\chi_A}(b)$. Since $a - b$ and $ab \in A$, A being an LA-subring. This implies that

$$\begin{aligned} \mu_{\chi_A}(a-b) &= 0 = 0 \vee 0 = \mu_{\chi_A}(a) \vee \mu_{\chi_A}(b), \\ \mu_{\chi_A}(ab) &= 0 = 0 \vee 0 = \mu_{\chi_A}(a) \vee \mu_{\chi_A}(b), \\ \gamma_{\chi_A}(a-b) &= 1 = 1 \wedge 1 = \gamma_{\chi_A}(a) \wedge \gamma_{\chi_A}(b), \\ \gamma_{\chi_A}(ab) &= 1 = 1 \wedge 1 = \gamma_{\chi_A}(a) \wedge \gamma_{\chi_A}(b). \end{aligned}$$

Thus

$$\begin{aligned} \mu_{\chi_A}(a-b) &\leq max\{\mu_{\chi_A}(a), \mu_{\chi_A}(b)\}, \\ \mu_{\chi_A}(ab) &\leq max\{\mu_{\chi_A}(a), \mu_{\chi_A}(b)\}, \\ \gamma_{\chi_A}(a-b) &\geq min\{\gamma_{\chi_A}(a), \gamma_{\chi_A}(b)\}, \\ \gamma_{\chi_A}(ab) &\geq min\{\gamma_{\chi_A}(a), \gamma_{\chi_A}(b)\}. \end{aligned}$$

As ab and $ba \in A$, so $\mu_{\chi_A}(ab) = 0 = \mu_{\chi_A}(ba)$ and $\gamma_{\chi_A}(ab) = 1 = \gamma_{\chi_A}(ba)$, i.e., $\mu_{\chi_A}(ab) = \mu_{\chi_A}(ba)$ and $\gamma_{\chi_A}(ab) = \gamma_{\chi_A}(ba)$. Similarly, we have

$$\begin{aligned} \mu_{\chi_A}(a-b) &\leq max\{\mu_{\chi_A}(a), \mu_{\chi_A}(b)\},\ \mu_{\chi_A}(ab) \leq max\{\mu_{\chi_A}(a), \mu_{\chi_A}(b)\}, \\ \gamma_{\chi_A}(a-b) &\geq min\{\gamma_{\chi_A}(a), \gamma_{\chi_A}(b)\},\ \gamma_{\chi_A}(ab) \geq min\{\gamma_{\chi_A}(a), \gamma_{\chi_A}(b)\}, \\ \mu_{\chi_A}(ab) &= \mu_{\chi_A}(ba),\ \gamma_{\chi_A}(ab) = \gamma_{\chi_A}(ba), \end{aligned}$$

when $a, b \notin A$. Hence the intuitionistic anti characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of $A = A_1 \times A_2 \times ... \times A_n$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Conversely, suppose that the intuitionistic anti characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of $A = A_1 \times A_2 \times ... \times A_n$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. We have to show that $A = A_1 \times A_2 \times ... \times A_n$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Let $a, b \in A$, where $a = (a_1, a, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$, by definition, we have $\mu_{\chi_A}(a) = 0 = \mu_{\chi_A}(b)$ and $\gamma_{\chi_A}(a) = 1 = \gamma_{\chi_A}(b)$. By our supposition

$$\begin{aligned} \mu_{\chi_A}(a-b) &\leq \mu_{\chi_A}(a) \vee \mu_{\chi_A}(b) = 0 \vee 0 = 0, \\ \mu_{\chi_A}(ab) &\leq \mu_{\chi_A}(a) \vee \mu_{\chi_A}(b) = 0 \vee 0 = 0, \\ \gamma_{\chi_A}(a-b) &\geq \gamma_{\chi_A}(a) \wedge \gamma_{\chi_A}(b) = 1 \wedge 1 = 1, \\ \gamma_{\chi_A}(ab) &\geq \gamma_{\chi_A}(a) \wedge \gamma_{\chi_A}(b) = 1 \wedge 1 = 1. \end{aligned}$$

Thus $\mu_{\chi_A}(a-b) = 0 = \mu_{\chi_A}(ab)$ and $\gamma_{\chi_A}(a-b) = 1 = \gamma_{\chi_A}(ab)$, i.e., $a - b$ and $ab \in A$. Hence $A = A_1 \times A_2 \times ... \times A_n$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Lemma 7. *If $A = A_1 \times A_2 \times ... \times A_n$ and $B = B_1 \times B_2 \times ... \times B_n$ are two LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then their intersection $A \cap B$ is also an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proof. Straight forward.

Theorem 6. *Let $A = A_1 \times A_2 \times ... \times A_n$ and $B = B \times B_2 \times ... \times B_n$ be two LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Then $A \cap B$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ if and only if the intuitionistic anti characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = A \cap B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $Z = A \cap B$ be an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ and $a = (a_1, a_2, ..., a_n), b = (b_1, b_1, ..., b_n) \in R_1 \times R_2 \times ... \times R_n$. If $a, b \in Z = A \cap B$, then by definition of intuitionistic anti characteristic function $\mu_{\chi_Z}(a) = 0 = \mu_{\chi_Z}(b)$ and $\gamma_{\chi_Z}(a) = 1 = \gamma_{\chi_Z}(b)$. Since $a - b$ and $ab \in Z$, Z being an LA-subring. This means that

$$\begin{aligned} \mu_{\chi_Z}(a-b) &= 0 = 0 \vee 0 = \mu_{\chi_Z}(a) \vee \mu_{\chi_Z}(b), \\ \mu_{\chi_Z}(ab) &= 0 = 0 \vee 0 = \mu_{\chi_Z}(a) \vee \mu_{\chi_Z}(b), \\ \gamma_{\chi_Z}(a-b) &= 1 = 1 \wedge 1 = \gamma_{\chi_Z}(a) \wedge \gamma_{\chi_Z}(b), \\ \gamma_{\chi_Z}(ab) &= 1 = 1 \wedge 1 = \gamma_{\chi_Z}(a) \wedge \gamma_{\chi_Z}(b). \end{aligned}$$

Thus

$$\begin{aligned} \mu_{\chi_Z}(a-b) &\leq max\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\}, \\ \mu_{\chi_Z}(ab) &\leq max\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\}, \\ \gamma_{\chi_Z}(a-b) &\geq min\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\}, \\ \gamma_{\chi_Z}(ab) &\geq min\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\}. \end{aligned}$$

As ab and $ba \in Z$, this implies that $\mu_{\chi_Z}(ab) = 0 = \mu_{\chi_Z}(ba)$ and $\gamma_{\chi_Z}(ab) = 1 = \gamma_{\chi_Z}(ba)$, i.e., $\mu_{\chi_Z}(ab) = \mu_{\chi_Z}(ba)$ and $\gamma_{\chi_Z}(ab) = \gamma_{\chi_Z}(ba)$. Similarly, we have

$$\begin{aligned} \mu_{\chi_Z}(a-b) &\leq max\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\},\ \mu_{\chi_Z}(ab) \leq max\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\}, \\ \gamma_{\chi_Z}(a-b) &\geq min\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\},\ \gamma_{\chi_Z}(ab) \geq min\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\}, \\ \gamma_{\chi_Z}(ab) &= \gamma_{\chi_Z}(ba),\ \gamma_{\chi_Z}(ab) = \gamma_{\chi_Z}(ba), \end{aligned}$$

when $a, b \notin Z$. Hence the intuitionistic anti characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of Z is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Conversely, assume that the intuitionistic anti characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = A \cap B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Let $a, b \in Z = A \cap B$, by definition, we have $\mu_{\chi_Z}(a) = 0 = \mu_{\chi_Z}(b)$ and $\gamma_{\chi_Z}(a) = 1 = \gamma_{\chi_Z}(b)$. By our assumption

$$\begin{aligned} \mu_{\chi_Z}(a-b) &\leq \mu_{\chi_Z}(a) \vee \mu_{\chi_Z}(b) = 0 \vee 0 = 0, \\ \mu_{\chi_Z}(ab) &\leq \mu_{\chi_Z}(a) \vee \mu_{\chi_Z}(b) = 0 \vee 0 = 0, \\ \gamma_{\chi_Z}(a-b) &\geq \gamma_{\chi_Z}(a) \wedge \gamma_{\chi_Z}(b) = 1 \wedge 1 = 1, \\ \gamma_{\chi_Z}(ab) &\geq \gamma_{\chi_Z}(a) \wedge \gamma_{\chi_Z}(b) = 1 \wedge 1 = 1. \end{aligned}$$

Thus $\mu_{\chi_Z}(a-b) = 0 = \mu_{\chi_Z}(ab)$ and $\gamma_{\chi_Z}(a-b) = 1 = \gamma_{\chi_Z}(ab)$, i.e., $a - b$ and $ab \in Z$. Hence Z is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Corollary 4. *Let $\{B_i\}_{i\in I} = \{A_{i1} \times A_{i2} \times ... \times A_{in}\}_{i\in I}$ be a family of LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Then $B = \cap B_i$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ if and only if the intuitionistic anti characteristic function $\chi_B = \langle \mu_{\chi_B}, \gamma_{\chi_B}\rangle$ of $B = \cap B_i$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Theorem 7. *If $A = A_1 \times A_2 \times ... \times A_n$ and $B = B_1 \times B_2 \times ... \times B_n$ are two intuitionistic anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then their intersection $A \cap B$ is also an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $A = A_1 \times A_2 \times ... \times A_n = \{((a), \mu_{A_1\times A_2\times ...\times A_n}(a), \gamma_{A_1\times A_2\times ...\times A}(a)) \mid$ for all $a = (a_1, a_2, ..., a_n) \in R_1 \times R_2 \times ... \times R_n\}$ and $B = B_1 \times B_2 \times ... \times B_n = \{((b), \mu_{B_1\times B_2\times ...\times B_n}(b), \gamma_{B_1\times B_2\times ...\times B_n}(b)) \mid$ for all $b = (b_1, b_2, ..., b_n) \in R_1 \times R_2 \times ... \times R_n\}$ be two intuitionistic anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Let $Z = A \cap B$ and $Z = \{((z), \mu_Z(z), \gamma_Z(z)) \mid$ for all $z = (z_1, z_2, ..., z_n) \in R_1 \times R_2 \times ... \times R_n\}$, where

$$\begin{aligned}
\mu_Z(z_1, z_2, ..., z_n) &= \mu_{A\cap B}(z_1, z_2, ..., z_n) \\
&= max\{\mu_A(z_1, z_2, ..., z_n), \mu_B(z_1, z_2, ..., z_n)\} \\
\text{and } \gamma_Z(z_1, z_2, ..., z_n) &= \gamma_{A\cap B}(z_1, z_2, ..., z_n) \\
&= min\{\gamma_A(z_1, z_2, ..., z_n), \gamma_B(z_1, z_2, ..., z_n)\}.
\end{aligned}$$

Let $z = (z_1, z_n, ..., z_n), w = (w_1, w_2, ..., w_n) \in R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
\mu_Z(z - w) &= \mu_Z(z - w) = max\{\mu_A(z - w), \mu_B(z - w)\} \\
&\leq \{\mu_A(z) \vee \mu_A(w)\} \vee \{\mu_B(z) \vee \mu_B(w)\} \\
&= \mu_A(z) \vee \{\mu_A(w) \vee \mu_B(z)\} \vee \mu_B(w) \\
&= \mu_A(z) \vee \{\mu_B(z) \vee \mu_A(w)\} \vee \mu_B(w) \\
&= \{\mu_A(z) \vee \mu_B(z)\} \vee \{\mu_A(w) \vee \mu_B(w)\} \\
&= max\{\mu_{A\cap B}(z), \mu_{A\cap B}(w)\} \\
&= max\{\mu_Z(z), \mu_Z(w)\}
\end{aligned}$$

and

$$\begin{aligned}
\mu_Z(z \circ w) &= \mu_Z(z \circ w) = max\{\mu_A(z \circ w), \mu_B(z \circ w)\} \\
&\leq \{\mu_A(z) \vee \mu_A(w)\} \vee \{\mu_B(z) \vee \mu_B(w)\} \\
&= \mu_A(z) \vee \{\mu_A(w) \vee \mu_B(z)\} \vee \mu_B(w) \\
&= \mu_A(z) \vee \{\mu_B(z) \vee \mu_A(w)\} \vee \mu_B(w) \\
&= \{\mu_A(z) \vee \mu_B(z)\} \vee \{\mu_A(w) \vee \mu_B(w)\} \\
&= max\{\mu_{A\cap B}(z), \mu_{A\cap B}(w)\} \\
&= max\{\mu_Z(z), \mu_Z(w)\}.
\end{aligned}$$

Thus

$$\mu_Z((z_1, z_2, ..., z_n) - (w_1, w_2, ..., w_n))$$

$$\begin{aligned}
&\leq \ max\{\mu_Z(z_1, z_2, ..., z_n), \mu_Z(w_1, w_2, ..., w_n)\} \\
&\quad \text{and } \mu_Z((z_1, z_2, ..., z_n) \circ (w_1, w_2, ..., w_n)) \\
&\leq \ max\{\mu_Z(z_1, z_2, ..., z_n), \mu_Z(w_1, w_2, ..., w_n)\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\quad \gamma_Z((z_1, z_2, ..., z_n) - (w_1, w_2, ..., w_n)) \\
&\geq \ min\{\gamma_Z(z_1, z_2, ..., z_n), \gamma_Z(w_1, w_2, ..., w_n)\} \\
&\quad \text{and } \gamma_Z((z_1, z_2, ..., z_n) \circ (w_1, w_2, ..., w_n)) \\
&\geq \ min\{\gamma_Z(z_1, z_2, ..., z_n), \gamma_Z(w_1, w_2, ..., w_n)\}
\end{aligned}$$

Thus $Z = (\mu_Z, \gamma_Z)$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
&\quad \mu_Z((z_1, z_2, ..., z_n) \circ (w_1, w_2, ..., w_n)) \\
&= \ \mu_{A\cap B}(z_1w_1, z_2w_2, ..., z_nw_n) \\
&= \ max\{\mu_A(z_1w_1, z_2w_2, ..., z_nw_n), \mu_B(z_1w_1, z_2w_2, ..., z_nw_n)\} \\
&= \ max\{\mu_A(w_1z_1, w_2z_2, ..., w_nz_n), \mu_B(w_1z_1, w_2z_2, ..., w_nz_n)\} \\
&= \ \mu_{A\cap B}(w_1z_1, w_2z_2, ..., w_nz_n) \\
&= \ \mu_Z((w_1, w_2, ..., w_n) \circ (z_1, z_2, ..., z_n)).
\end{aligned}$$

Similarly

$$\gamma_Z((z_1, z_2, ..., z_n) \circ (w_1, w_2, ..., w_n)) = \gamma_Z((w_1, w_2, ..., w_n) \circ (z_1, z_2, ..., z_n)).$$

Hence $Z = A \cap B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Corollary 5. *If $\{B_i\}_{i\in I} = \{A_{i1} \times A_{i2} \times ... \times A_{in}\}_{i\in I}$ is a family of intuitionistic anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then $B = \cap B_i$ is also an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proposition 4. *If an IFS $A = A_1 \times A_2 \times ... \times A_n$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then $\Box A = (\mu_A, \overline{\mu}_A)$ (resp. $\Diamond A = (\overline{\gamma}_A, \gamma_A)$) is also an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $A_1 \times A_2 \times ... \times A_n$ be an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. We have to show that $\Box A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times ...\times A_n}, \overline{\mu}_{A_1\times A_2\times ...\times A_n})$ is also an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
&\quad \overline{\mu}_{A_1\times A_2\times ...\times A_n}((x_1, x_2, ..., x_n) - (y_1, y_2, ..., y_n)) \\
&= \ 1 - \mu_{A_1\times A_2\times ...\times A_n}((x_1, x_2, ..., x_n) - (y_1, y_2, ..., y_n))
\end{aligned}$$

$$
\begin{aligned}
&\geq 1 - max\,\{\mu_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n), \mu_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= min\,\{1-\mu_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n), 1-\mu_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= min\{\overline{\mu}_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n), \overline{\mu}_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&\quad \text{and } \overline{\mu}_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&= 1-\mu_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&\geq 1 - max\,\{\mu_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n), \mu_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= min\,\{1-\mu_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n), 1-\mu_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= min\{\overline{\mu}_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n), \overline{\mu}_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}.
\end{aligned}
$$

Thus $\Box A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times\ldots\times A_n}, \overline{\mu}_{A_1\times A_2\times\ldots\times A_n})$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$
\begin{aligned}
&= \overline{\mu}_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&= 1-\mu_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&= 1-\mu_{A_1\times A_2\times\ldots\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n))\\
&= \overline{\mu}_{A_1\times A_2\times\ldots\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)).
\end{aligned}
$$

Hence $\Box A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times\ldots\times A_n}, \overline{\mu}_{A_1\times A_2\times\ldots\times A_n})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Corollary 6. *An IFS $A = A_1 \times A_2 \times ... \times A_n$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2\times ...\times R_n$ if and only if $\Box A = (\mu_A, \overline{\mu}_A)$ (resp. $\Diamond A = (\overline{\gamma}_A, \gamma_A)$) is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Theorem 8. *An IFS $A_1\times A_2\times...\times A_n = (\mu_{A_1\times A_2\times\ldots\times A_n}, \gamma_{A_1\times A_2\times\ldots\times A_n})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ if and only if the fuzzy subsets $\mu_{A_1\times A_2\times\ldots\times A_n}$ and $\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}$ are anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times\ldots\times A_n}, \gamma_{A_1\times A_2\times\ldots\times A_n})$ be an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. This implies that $\mu_{A_1\times A_2\times\ldots\times A_n}$ is an anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. We have to show that $\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}$ is also an anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$
\begin{aligned}
&\quad \overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)-(y_1,y_2,...,y_n))\\
&= 1-\gamma_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)-(y_1,y_2,...,y_n))\\
&\leq 1-min\{\gamma_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n), \gamma_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= max\{1-\gamma_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n), 1-\gamma_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= max\{\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n), \overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}.\\
&\quad \text{and } \overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))
\end{aligned}
$$

$$
\begin{aligned}
&= 1-\gamma_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&\leq 1-min\{\gamma_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n),\gamma_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= max\{1-\gamma_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n),1-\gamma_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= max\{\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n),\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}.
\end{aligned}
$$

Thus $\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}$ is an anti fuzzy LA-subring of an LA-ring $R_1\times R_2\times\ldots\times R_n$. Now

$$
\begin{aligned}
&\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&= 1-\gamma_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&= 1-\gamma_{A_1\times A_2\times\ldots\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n))\\
&= \overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)).
\end{aligned}
$$

Hence $\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}$ is an anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2\times\ldots\times R_n$.

Conversely, suppose that $\mu_{A_1\times A_2\times\ldots\times A_n}$ and $\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}$ are anti fuzzy normal LA-subrings of an LA-ring $R_1\times R_2\times\ldots\times R_n$. We have to show that $A_1\times A_2\times\ldots\times A_n = (\mu_{A_1\times A_2\times\ldots\times A_n},\gamma_{A_1\times A_2\times\ldots\times A_n})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2\times\ldots\times R_n$. Now

$$
\begin{aligned}
&1-\gamma_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)-(y_1,y_2,...,y_n))\\
&= \overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)-(y_1,y_2,...,y_n))\\
&\leq max\{\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n),\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= max\{1-\gamma_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n),1-\gamma_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= 1-min\{\gamma_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n),\gamma_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&\text{and } 1-\gamma_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&= \overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&\leq max\{\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n),\overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= max\{1-\gamma_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n),1-\gamma_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}\\
&= 1-min\{\gamma_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,...,x_n),\gamma_{A_1\times A_2\times\ldots\times A_n}(y_1,y_2,...,y_n)\}.
\end{aligned}
$$

Thus $A_1\times A_2\times\ldots\times A_n = (\mu_{A_1\times A_2\times\ldots\times A_n},\gamma_{A_1\times A_2\times\ldots\times A_n})$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1\times R_2\times\ldots\times R_n$. Now

$$
\begin{aligned}
&1-\gamma_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&= \overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&= \overline{\gamma}_{A_1\times A_2\times\ldots\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n))\\
&= 1-\gamma_{A_1\times A_2\times\ldots\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)).
\end{aligned}
$$

Hence $A_1\times A_2\times\ldots\times A_n = (\mu_{A_1\times A_2\times\ldots\times A_n},\gamma_{A_1\times A_2\times\ldots\times A_n})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2\times\ldots\times R_n$.

Theorem 9. *An IFS* $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \gamma_{A_1\times A_2\times...\times A_n})$ *is an intuitionistic anti fuzzy normal LA-subring of an LA-ring* $R_1 \times R_2 \times ... \times R_n$ *if and only if the fuzzy subsets* $\overline{\mu}_{A_1\times A_2\times...\times A_n}$ *and* $\gamma_{A_1\times A_2\times...\times A_n}$ *are fuzzy normal LA-subrings of an LA-ring* $R_1 \times R_2 \times ... \times R_n$.

Proof. Let $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \gamma_{A_1\times A_2\times...\times A_n})$ be an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. This means that $\gamma_{A_1\times A_2\times...\times A_n}$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. We have to show that $\overline{\mu}_{A_1\times A_2\times...\times A_n}$ is also a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
& \overline{\mu}_{A_1\times A_2\times...\times A_n}((x_1, x_2, ..., x_n) - (y_1, y_2, ..., y_n)) \\
= \ & 1 - \mu_{A_1\times A_2\times...\times A_n}((x_1, x_2, ..., x_n) - (y_1, y_2, ..., y_n)) \\
\geq \ & 1 - max\{\mu_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n), \mu_{A_1\times A_2\times...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \ & min\{1 - \mu_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n), 1 - \mu_{A_1\times A_2\times...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \ & min\{\overline{\mu}_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n), \overline{\mu}_{A_1\times A_2\times...\times A_n}(y_1, y_2, ..., y_n)\} \\
& \text{and } \overline{\mu}_{A_1\times A_2\times...\times A_n}((x_1, x_2, ..., x_n) \circ (y_1, y_2, ..., y_n)) \\
= \ & 1 - \mu_{A_1\times A_2\times...\times A_n}((x_1, x_2, ..., x_n) \circ (y_1, y_2, ..., y_n)) \\
\geq \ & 1 - max\{\mu_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n), \mu_{A_1\times A_2\times...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \ & min\{1 - \mu_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n), 1 - \mu_{A_1\times A_2\times...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \ & min\{\overline{\mu}_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n), \overline{\mu}_{A_1\times A_2\times...\times A_n}(y_1, y_2, ..., y_n)\}.
\end{aligned}$$

Thus $\overline{\mu}_{A_1\times A_2\times...\times A_n}$ is a fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
& \overline{\mu}_{A_1\times A_2\times...\times A_n}((x_1, x_2, ..., x_n) \circ (y_1, y_2, ..., y_n)) \\
= \ & 1 - \mu_{A_1\times A_2\times...\times A_n}((x_1, x_2, ..., x_n) \circ (y_1, y_2, ..., y_n)) \\
= \ & 1 - \mu_{A_1\times A_2\times...\times A_n}((y_1, y_2, ..., y_n) \circ (x_1, x_2, ..., x_n)) \\
= \ & \overline{\mu}_{A_1\times A_2\times...\times A_n}((y_1, y_2, ..., y_n) \circ (x_1, x_2, ..., x_n)).
\end{aligned}$$

Hence $\overline{\mu}_{A_1\times A_2\times...\times A_n}$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Conversely, assume that $\overline{\mu}_{A_1\times A_2\times...\times A_n}$ and $\gamma_{A_1\times A_2\times...\times A_n}$ are fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. We have to show that $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \gamma_{A_1\times A_2\times...\times A_n})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
& 1 - \mu_{A_1\times A_2\times...\times A_n}((x_1, x_2, ..., x_n) - (y_1, y_2, ..., y_n)) \\
= \ & \overline{\mu}_{A_1\times A_2\times...\times A_n}((x_1, x_2, ..., x_n) - (y_1, y_2, ..., y_n)) \\
\geq \ & min\{\overline{\mu}_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n), \overline{\mu}_{A_1\times A_2\times...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \ & min\{1 - \mu_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n), 1 - \mu_{A_1\times A_2\times...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \ & 1 - max\{\mu_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n), \mu_{A_1\times A_2\times...\times A_n}(y_1, y_2, ..., y_n)\} \\
& \text{and } 1 - \mu_{A_1\times A_2\times...\times A_n}((x_1, x_2, ..., x_n) \circ (y_1, y_2, ..., y_n)) \\
= \ & \overline{\mu}_{A_1\times A_2\times...\times A_n}((x_1, x_2, ..., x_n) \circ (y_1, y_2, ..., y_n)) \\
\geq \ & min\{\overline{\mu}_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n), \overline{\mu}_{A_1\times A_2\times...\times A_n}(y_1, y_2, ..., y_n)\}
\end{aligned}$$

$$\begin{aligned}
&= min\{1-\mu_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n), 1-\mu_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
&= 1-max\{\mu_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n), \mu_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\}.
\end{aligned}$$

Thus $A_1\times A_2\times...\times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \gamma_{A_1\times A_2\times...\times A_n})$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R_1\times R_2\times...\times R_n$. Now

$$\begin{aligned}
& 1-\mu_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
=\ & \overline{\mu}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
=\ & \overline{\mu}_{A_1\times A_2\times...\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)) \\
=\ & 1-\mu_{A_1\times A_2\times...\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)).
\end{aligned}$$

Hence $A_1\times A_2\times...\times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \gamma_{A_1\times A_2\times...\times A_n})$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R_1\times R_2\times...\times R_n$.

Proposition 5. *Let $A = A_1\times A_2\times...\times A_n$ and $B = B_1\times B_2\times...\times B_n$ be intuitionistic fuzzy sets of LA-rings $R = R_1\times R_2\times...\times R_n$ and $R' = R_1'\times R_2'\times...\times R_n'$ with left identities $e = (e_1, e_2, ..., e_n)$ and $e' = (e_1\prime, e_2\prime, ..., e_n\prime)$, respectively. If $A\times B$ is an intuitionistic anti fuzzy LA-subring of an LA-ring $R\times R'$, then at least one of the following two statements must hold.*
1. *$\mu_A(x)\geq\mu_B(e')$ and $\gamma_A(x)\leq\gamma_B(e')$, for all $x\in R$.*
2. *$\mu_B(x)\geq\mu_A(e)$ and $\gamma_B(x)\leq\gamma_A(e)$, for all $x\in R'$.*

Proof. Let $A\times B$ be an intuitionistic anti fuzzy LA-subring of an LA-ring $R\times R\prime$. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a and b in R and R', respectively such that

$$\begin{aligned}
\mu_A(a) &\leq \mu_B(e') \text{ and } \gamma_A(a)\geq\gamma_B(e'). \\
\mu_B(b) &\leq \mu_A(e) \text{ and } \gamma_B(b)\geq\gamma_A(e).
\end{aligned}$$

Thus

$$\begin{aligned}
\mu_{A\times B}(a,b) &= max\{\mu_A(a),\mu_B(b)\} \\
&\leq max\{\mu_A(e),\mu_B(e')\} \\
&= \mu_{A\times B}(e,e') \\
\text{and } \gamma_{A\times B}(a,b) &= min\{\gamma_A(a),\gamma_B(b)\} \\
&\geq min\{\gamma_A(e),\gamma_B(e')\} \\
&= \gamma_{A\times B}(e,e').
\end{aligned}$$

Therefore $A\times B$ is not an intuitionistic anti fuzzy LA-subring of an LA-ring $R\times R'$. Hence either $\mu_A(x)\geq\mu_B(e')$ and $\gamma_A(x)\leq\gamma_B(e')$, for all $x\in R_1$ or $\mu_B(x)\geq\mu_A(e)$ and $\gamma_B(x)\leq\gamma_A(e)$, for all $x\in R_2$.

Theorem 10. *Let $A = A_1 \times A_2 \times ... \times A_n$ and $B = B_1 \times B_2 \times ... \times B_n$ be intuitionistic fuzzy sets of LA-rings $R = R_1 \times R_2 \times ... \times R_n$ and $R' = R'_1 \times R'_2 \times ... \times R'_n$ with left identities $e = (e_1, e_2, ..., e_n)$ and $e' = (e_1\prime, e_2\prime, ..., e_n\prime)$, respectively and $A \times B$ is an intuitionistic anti fuzzy normal LA-subring of an LA-ring $R \times R'$. Then the following conditions are true.*

1. If $\mu_A(x) \geq \mu_B(e')$ and $\gamma_A(x) \leq \gamma_B(e')$, for all $x \in R$, then A is an intuitionistic anti fuzzy normal LA-subring of R.

2. If $\mu_B(x') \geq \mu_A(e)$ and $\gamma_B(x') \leq \gamma_A(e)$, for all $x' \in R'$, then B is an intuitionistic anti fuzzy normal LA-subring of R'.

Proof. 1. Let $\mu_A(x) \geq \mu_B(e')$ and $\gamma_A(x) \leq \gamma_B(e')$ for all $x \in R$, and $y \in R$. We have to show that A is an intuitionistic anti fuzzy normal LA-subring of R. Now

$$\begin{aligned} \mu_A(x-y) &= \mu_A(x+(-y)) \\ &= max\{\mu_A(x+(-y)), \mu_B(e'+(-e'))\} \\ &= \mu_{A\times B}(x+(-y), e'+(-e')) \\ &= \mu_{A\times B}((x,e')+(-y,-e')) \\ &= \mu_{A\times B}((x,e')-(y,e')) \\ &\leq \mu_{A\times B}(x,e') \vee \mu_{A\times B}(y,e') \\ &= max\{max\{\mu_A(x), \mu_B(e')\}, max\{\mu_A(y), \mu_B(e')\}\} \\ &= \mu_A(x) \vee \mu_A(y) \end{aligned}$$

and

$$\begin{aligned} \mu_A(xy) &= max\{\mu_A(xy), \mu_B(e'e')\} \\ &= \mu_{A\times B}(xy, e'e') \\ &= \mu_{A\times B}((x,e') \circ (y,e')) \\ &\leq \mu_{A\times B}(x,e') \vee \mu_{A\times B}(y,e') \\ &= max\{max\{\mu_A(x), \mu_B(e')\}, max\{\mu_A(y), \mu_B(e')\}\} \\ &= \mu_A(x) \vee \mu_A(y). \end{aligned}$$

Similarly, we have

$$\gamma_A(x-y) \geq min\{\gamma_A(x), \gamma_A(y)\} \text{ and } \gamma_A(xy) \geq min\{\gamma_A(x), \gamma_A(y)\}.$$

Thus A is an intuitionistic anti fuzzy LA-subring of R. Now

$$\begin{aligned} \mu_A(xy) &= max\{\mu_A(xy), \mu_B(e'e')\} \\ &= \mu_{A\times B}\left(xy, e'e'\right) \\ &= \mu_{A\times B}\left(\left(x,e'\right) \circ \left(y,e'\right)\right) \\ &= \mu_{A\times B}\left(\left(y,e'\right) \circ \left(x,e'\right)\right) \\ &= \mu_{A\times B}(yx, e'e') \\ &= max\{\mu_A(yx), \mu_B(e'e')\} \end{aligned}$$

$$= \quad \mu_A(yx).$$

Similarly, $\gamma_B(xy) = \gamma_B(yx)$. Hence A is an intuitionistic anti fuzzy normal LA-subring of R. 2. is same as 1.

Conclusion 1. *Our aim is to encourage the research of associative algebraic structure by studying a class of non-associative and non-commutative algebraic structure means LA-ring and explored new methodological developments on LA-ring, which will be helpful in future. The objective of this paper is to initiate the notion of intuitionistic anti fuzzy normal subrings on LA-ring and established some imperative properties of such subrings. We hope that in future, this concept would be a useful contribution in the theory of non-associative algebraic structures.*

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