Intuitionistic Fuzzy Ideals with Thresholds \((\alpha, \beta)\) in LA-rings

Nasreen Kausar\textsuperscript{1,}\textsuperscript{*}, Badar ul Islam\textsuperscript{2}, Syed Amjad Ahmad\textsuperscript{3}, Muhammad Azam Waqar\textsuperscript{4}

\textsuperscript{1} Department of Mathematics, University of Agriculture, FSD, Pakistan
\textsuperscript{2} Department of Electrical Engineering, NFC IEFR FSD, Pakistan
\textsuperscript{3} Department of Mechanical Engineering, NFC IEFR FSD, Pakistan
\textsuperscript{4} Department of School of Business Management, NFC IEFR FSD, Pakistan

Abstract. In this paper, we give characterizations of regular (intra-regular, both regular and intra-regular) LA-rings by the properties of intuitionistic fuzzy (left, right, quasi-, bi-, generalized bi-) ideals with thresholds \((\alpha, \beta)\).

2010 Mathematics Subject Classifications: 03F55, 08A72, 20N25

Key Words and Phrases: Intuitionistic fuzzy left (right, interior, quasi-, bi-, generalized bi-) ideals with thresholds \((\alpha, \beta)\), regular (intra-regular) LA-rings.

1. Introduction

In ternary operations, the commutative law is given by \(abc = cba\). Kazim et al \[18\], have generalized this notion by introducing the parenthesis on the left side of this equation to get a new pseudo associative law, that is \((ab)c = (cb)a\). This law \((ab)c = (cb)a\) is called the left invertive law. A groupoid \(S\) is called a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law. An LA-semigroup is a midway structure between a commutative semigroup and a groupoid. Ideals in LA-semigroups have been investigated by Protic et al \[24\].

In \[12\] (resp. \[8\]), a groupoid \(S\) is said to be medial (resp. paramedial) if \((ab)(cd) = (ac)(bd)\) (resp. \((ab)(cd) = (db)(ca)\)). In \[18\], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial by Protic et al \[24\] and also satisfies \(a(bc) = b(ac), (ab)(cd) = (dc)(ba)\).

Kamran \[14\], extended the notion of LA-semigroup to the left almost group (LA-group). An LA-semigroup \(G\) is called a left almost group, if there exists a left identity

\textsuperscript{*}Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v12i3.3441

Email addresses: kauasr.nasreen@gmail.com (K. Nasreen), badar.utp@gmail.com (I. Badar)
samjadahmad67@yahoo.com (S. A. Ahmad ), azamwaqar4@gmail.com (W. Azam)
\[ e \in G \text{ such that } ea = a \text{ for all } a \in G \text{ and for every } a \in G \text{ there exists } b \in G \text{ such that } ba = e. \]

Shah et al. [25], discussed the left almost ring (abbreviated as LA-ring) of finitely nonzero functions which is a generalization of commutative semigroup ring. By a left almost ring, we mean a non-empty set \( R \) with at least two elements such that \((R, +)\) is an LA-group, \((R, \cdot)\) is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring \((R, +, \cdot)\), we can always obtain an LA-ring \((R, \oplus, \cdot)\) by defining for all \( a, b \in R \), \( a \oplus b = b - a \) and \( a \cdot b \) is same as in the ring. Although the structure is non-associative and non-commutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

A non-empty subset \( A \) of \( R \) is called an LA-subring of \( R \) if \( a - b \) and \( ab \in A \) for all \( a, b \in A \). \( A \) is called a left (resp. right) ideal of \( R \) if \((A, +)\) is an LA-group and \( RA \subseteq A \) (resp. \( AR \subseteq A \)). \( A \) is called an ideal of \( R \) if it is both a left ideal and a right ideal of \( R \). A non-empty subset \( A \) of \( R \) is called an interior ideal of \( R \) if \((A, +)\) is an LA-group and \((RA)R \subseteq A \). A non-empty subset \( A \) of \( R \) is called a quasi-ideal of \( R \) if \((A, +)\) is an LA-group and \( AR \cap RA \subseteq A \). An LA-subring \( A \) of \( R \) is called a bi-ideal of \( R \) if \((AR)A \subseteq A \).

We will introduce the concept intuitionistic fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals with thresholds \((\alpha, \beta)\) of an LA-ring \( R \). We will establish a study by describing the different properties in terms of such ideals, which will be very useful for the characterizations of regular (intra-regular, both regular and intra-regular) LA-rings in terms of intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds \((\alpha, \beta)\).

2. Intuitionistic Fuzzy Ideals with Thresholds \((\alpha, \beta)\)

After the introduction of fuzzy set by Zadeh [31], several researchers explored the generalization of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [1, 2], as a generalization of the notion of fuzzy set.

Liu [20], introduced the concept of fuzzy subrings and fuzzy ideals of a ring. Many authors have explored the theory of fuzzy rings (for example [11, 19, 21, 22, 29]). Gupta et al. [11], gave the idea of intrinsic product of fuzzy subsets of a ring. Kuroki [19], characterized regular (intra-regular, both regular and intra-regular) rings in terms of fuzzy left (right, quasi-, bi-) ideals.

An intuitionistic fuzzy set (briefly, IFS) \( A \) in a non-empty set \( X \) is an object having the form \( A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\} \), where the functions \( \mu_A : X \to [0, 1] \) and \( \gamma_A : X \to [0, 1] \) denote the degree of membership and the degree of nonmembership, respectively and \( 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \) for all \( x \in X \) [1, 2].

An intuitionistic fuzzy set \( A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\} \) in \( X \) can be identified to be an ordered pair \((\mu_A, \gamma_A)\) in \( I^X \times I^X \), where \( I^X \) is the set of all functions from \( X \) to \([0, 1]\). For the sake of simplicity, we shall use the symbol \( A = (\mu_A, \gamma_A) \) for the IFS.
A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}.

Banerjee et al [3] and Hur et al [10], initiated the notion of intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring. Subsequently many authors studied the intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring by describing the different properties (see [9]). Shah et al [26], have initiated the concept of intuitionistic fuzzy normal LA-subrings of an LA-ring.

Bhakat et al [4–6], introduced the notion of \((\alpha, \beta)\)-fuzzy subgroups. It is a generalization of Rosenfeld fuzzy subgroups as \((\epsilon, \in \forall q)\)-fuzzy subgroups. Then many authors studied the algebraic structures by employing the idea of \((\alpha, \beta)\)-fuzzy subsets (for example [7, 13, 23]). Yuan et al [30], initiated the concept of fuzzy subgroups with thresholds. Shabir et al [28], gave the idea of fuzzy ideals with thresholds in semigroups.

Now we initiate the concept of intuitionistic fuzzy LA-subrings with thresholds \((\alpha, \beta)\) and intuitionistic fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals with thresholds \((\alpha, \beta)\) of an LA-ring \(R\).

An IFS \(A = (\mu_A, \gamma_A)\) of an LA-ring \(R\) is called an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \(R\) if

1. \(\max\{\mu_A(x-y), \alpha\} \geq \min\{\mu_A(x), \mu_A(y), \beta\}\),
2. \(\min\{\gamma_A(x-y), (1-\alpha)\} \leq \max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}\),
3. \(\max\{\mu_A(xy), \alpha\} \geq \min\{\mu_A(x), \mu_A(y), \beta\}\),
4. \(\min\{\gamma_A(xy), (1-\alpha)\} \leq \max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}\) for all \(x, y \in R\) and \(\alpha, \beta \in (0, 1]\) such that \(\alpha < \beta\).

An IFS \(A = (\mu_A, \gamma_A)\) of an LA-ring \(R\) is called an intuitionistic fuzzy left ideal with thresholds \((\alpha, \beta)\) of \(R\) if

1. \(\max\{\mu_A(x-y), \alpha\} \geq \min\{\mu_A(x), \mu_A(y), \beta\}\),
2. \(\min\{\gamma_A(x-y), (1-\alpha)\} \leq \max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}\),
3. \(\max\{\mu_A(xy), \alpha\} \geq \min\{\mu_A(y), \beta\}\),
4. \(\min\{\gamma_A(xy), (1-\alpha)\} \leq \max\{\gamma_A(y), (1-\beta)\}\) for all \(x, y \in R\) and \(\alpha, \beta \in (0, 1]\) such that \(\alpha < \beta\).

An IFS \(A = (\mu_A, \gamma_A)\) of an LA-ring \(R\) is called an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \(R\) if

1. \(\max\{\mu_A(x-y), \alpha\} \geq \min\{\mu_A(x), \mu_A(y), \beta\}\),
2. \(\min\{\gamma_A(x-y), (1-\alpha)\} \leq \max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}\),
3. \(\max\{\mu_A(xy), \alpha\} \geq \min\{\mu_A(x), \beta\}\),
4. \(\min\{\gamma_A(xy), (1-\alpha)\} \leq \max\{\gamma_A(x), (1-\beta)\}\) for all \(x, y \in R\) and \(\alpha, \beta \in (0, 1]\) such that \(\alpha < \beta\).

An IFS \(A = (\mu_A, \gamma_A)\) of an LA-ring \(R\) is called an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \(R\) if it is both an intuitionistic fuzzy left ideal with thresholds \((\alpha, \beta)\) and an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \(R\).

Every intuitionistic fuzzy left (resp. right, two-sided) ideal with thresholds \((\alpha, \beta)\) of \(R\) is an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \(R\), but converse is not true in general.
Example 1. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Define $+$ and $\cdot$ in $R$ as follows:

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Then $R$ is an LA-ring and $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring $R$. We define $(\alpha = 0.1, \beta = 0.7)$ $\mu_A(0) = \mu_A(4) = 0.7$, $\mu_A(1) = \mu_A(2) = \mu_A(3) = \mu_A(5) = \mu_A(6) = \mu_A(7) = 0.1$ and $\gamma_A(0) = \gamma_A(4) = 0.1$, $\gamma_A(1) = \gamma_A(2) = \gamma_A(3) = \gamma_A(5) = \gamma_A(6) = \gamma_A(7) = 0.7$. Since

$$
\max\{\mu_A(41), \alpha\} = \max\{\mu_A(3), \alpha\} = \max\{0.1, 0.1\} = 0.1.
$$

$$
\min\{\mu_A(4), \beta\} = \min\{0.7, 0.7\} = 0.7.
$$

Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta)$ of $R$, but not an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta)$ of $R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring $R$ is called an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta)$ of $R$ if

1. $\max\{\mu_A(x - y), \alpha\} \geq \min\{\mu_A(x), \mu_A(y), \beta\}$,
2. $\min\{\gamma_A(x - y), (1 - \alpha)\} \leq \max\{\gamma_A(x), \gamma_A(y), (1 - \beta)\}$,
3. $\max\{\mu_A((xy)z), \alpha\} \geq \min\{\mu_A(y), \beta\}$,
4. $\min\{\gamma_A((xy)z), (1 - \alpha)\} \leq \max\{\gamma_A(y), (1 - \beta)\}$ for all $x, y, z \in R$ and $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$.

An intuitionistic fuzzy LA-subring $A = (\mu_A, \gamma_A)$ with thresholds $(\alpha, \beta)$ of an LA-ring $R$ is called an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta)$ of $R$ if

1. $\max\{\mu_A(x - y), \alpha\} \geq \min\{\mu_A(x), \mu_A(y), \beta\}$,
2. $\min\{\gamma_A(x - y), (1 - \alpha)\} \leq \max\{\gamma_A(x), \gamma_A(y), (1 - \beta)\}$,
3. $\max\{\mu_A(x), \alpha\} \geq \min\{(\mu_A \circ R)(x), (R \circ \mu_A)(x), \beta\}$,
4. $\min\{\gamma_A(x), (1 - \alpha)\} \leq \max\{(\gamma_A \circ R)(x), (R \circ \gamma_A)(x), (1 - \beta)\}$ for all $x, y \in R$ and $\alpha, \beta \in [0, 1]$ such that $\alpha < \beta$. 

An intuitionistic fuzzy LA-subring $A = (\mu_A, \gamma_A)$ with thresholds $(\alpha, \beta)$ of an LA-ring $R$ is called an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta)$ of $R$ if
We define intuitionistic fuzzy sets $A$ with thresholds $(\alpha, \beta) \in (0, 1]$ such that $\alpha < \beta$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring $R$ is called an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta)$ of $R$ if

1. $\max\{\mu_A ((xy)z), \alpha\} \geq \min\{\mu_A (x), \mu_A (z), \beta\}$,
2. $\min\{\gamma_A ((xy)z), (1 - \alpha)\} \leq \max\{\gamma_A (x), \gamma_A (z), (1 - \beta)\}$ for all $x, y, z \in R$ and $\alpha, \beta \in (0, 1]$ such that $\alpha < \beta$.

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy sets of an LA-ring $R$, then the product of $A$ and $B$ is denoted by $A \circ B = (\mu_A \circ \mu_B, \gamma_A \circ \gamma_B)$ and defined by:

$$(\mu_A \circ \mu_B)(x) = \begin{cases} \vee \sum_{i=1}^{n} a_i b_i & \text{if } x = \sum_{i=1}^{n} a_i b_i, \ a_i, b_i \in R \\ 0 & \text{if } x \neq \sum_{i=1}^{n} a_i b_i \end{cases}$$

and

$$(\gamma_A \circ \gamma_B)(x) = \begin{cases} \wedge \sum_{i=1}^{n} a_i b_i & \text{if } x = \sum_{i=1}^{n} a_i b_i, \ a_i, b_i \in R \\ 1 & \text{if } x \neq \sum_{i=1}^{n} a_i b_i \end{cases}$$

Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring $R$ and $\alpha, \beta \in (0, 1]$ such that $\alpha < \beta$. We define an intuitionistic fuzzy set $A^\alpha_\beta$ of $R$ as follows: $(\mu_A)^\alpha_\beta(x) = (\mu_A(x) \wedge \beta) \lor \alpha$ and $(\gamma_A)^\alpha_\beta(x) = (\gamma_A(x) \lor (1 - \beta)) \land (1 - \alpha)$ for all $x \in R$.

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy sets of an LA-ring $R$. We define intuitionistic fuzzy sets $A \land^\alpha_\beta B$, $A \lor^\alpha_\beta B$, $A \circ^\alpha_\beta B$ and $A -^\alpha_\beta B$ of $R$ as follows:

$$(\mu_A \land^\alpha_\beta \mu_B)(x) = \{(\mu_A \land \mu_B)(x) \land \beta\} \lor \alpha$$

and

$$(\gamma_A \lor^\alpha_\beta \gamma_B)(x) = \{(\gamma_A \lor \gamma_B)(x) \lor (1 - \beta)\} \land (1 - \alpha).$$

$$(\mu_A \lor^\alpha_\beta \mu_B)(x) = \{(\mu_A \lor \mu_B)(x) \lor \beta\} \land \alpha$$

and

$$(\gamma_A \land^\alpha_\beta \gamma_B)(x) = \{(\gamma_A \land \gamma_B)(x) \land (1 - \beta)\} \lor (1 - \alpha).$$

$$(\mu_A \circ^\alpha_\beta \mu_B)(x) = \{(\mu_A \circ \mu_B)(x) \circ \beta\} \lor \alpha$$

and

$$(\gamma_A -^\alpha_\beta \gamma_B)(x) = \{(\gamma_A - \gamma_B)(x) - (1 - \beta)\} \land (1 - \alpha).$$

$$(\mu_A -^\beta_\alpha \mu_B)(x) = \{(\mu_A - \mu_B)(x) - \beta\} \lor \alpha$$

$$(\gamma_A - \gamma_B)(x) \land (1 - \beta) \lor (1 - \alpha).$$
and \((\gamma_A - \alpha \gamma_B)(x) = \{(\gamma_A - \gamma_B)(x) \lor (1 - \beta)\} \land (1 - \alpha)\),

for all \(x \in R\).

Now we are giving the central properties of such ideals of an LA-ring \(R\), which will be very helpful for further sections.

**Lemma 1.** Let \(A\) and \(B\) be two intuitionistic fuzzy sets of an LA-ring \(R\). Then the following properties holds.

1. \(A \land_{\alpha} B = A_{\beta} \land B_{\alpha}\).
2. \(A \lor_{\alpha} B = A_{\beta} \lor B_{\alpha}\).
3. \(A \circ_{\alpha} B \geq A_{\beta} \circ B_{\alpha}\).

If every element \(x\) of \(R\) is expressible as \(x = \sum_{i=1}^{n} a_i b_i\), then \(A \circ_{\alpha} B = A_{\beta} \circ B_{\alpha}\).

If \(\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})\) is an intuitionistic characteristic function of \(A\), then \((\chi_A)_{\alpha}^{\beta}\) is defined as

\[
(\mu_{\chi_A})_{\alpha}^{\beta}(x) = \begin{cases} 
\beta & \text{if } x \in A \\
\alpha & \text{if } x \notin A 
\end{cases}
\]

and

\[
(\gamma_{\chi_A})_{\alpha}^{\beta}(x) = \begin{cases} 
\beta & \text{if } x \in A \\
\alpha & \text{if } x \notin A 
\end{cases}
\]

**Lemma 2.** Let \(R\) be an LA-ring. Then the following properties hold.

1. \((A \circ_{\alpha} B) \circ_{\alpha} C = (C \circ_{\alpha} B) \circ_{\alpha} A\),
2. \((A \circ_{\alpha} B) \circ_{\alpha} (C \circ_{\alpha} D) = (A \circ_{\alpha} C) \circ_{\alpha} (B \circ_{\alpha} D)\) for all intuitionistic fuzzy sets \(A, B, C\) and \(D\) of \(R\).

**Proof.** Let \(A = (\mu_A, \gamma_A)\), \(B = (\mu_B, \gamma_B)\) and \(C = (\mu_C, \gamma_C)\) be intuitionistic fuzzy sets of an LA-ring \(R\). We have to show that \((A \circ_{\alpha} B) \circ_{\alpha} C = (C \circ_{\alpha} B) \circ_{\alpha} A\). Now

\[
((A \circ_{\alpha} B) \circ_{\alpha} C)(x) = \{((A \circ B) \circ C)(x) \lor \beta\} \lor \alpha
\]

\[
= \{((A \circ B) \circ A)(x) \lor \beta\} \lor \alpha
\]

\[
= ((C \circ_{\alpha} B) \circ_{\alpha} A)(x).
\]

In same lines, we can prove (2).

**Proposition 1.** Let \(R\) be an LA-ring with left identity \(e\). Then the following assertions hold.

1. \(A \circ_{\alpha} (B \circ_{\alpha} C) = B \circ_{\alpha} (A \circ_{\alpha} C)\),
2. \((A \circ_{\alpha} B) \circ_{\alpha} (C \circ_{\alpha} D) = (D \circ_{\alpha} B) \circ_{\alpha} (C \circ_{\alpha} A)\),
3. \((A \circ_{\alpha} B) \circ_{\alpha} (C \circ_{\alpha} D) = (D \circ_{\alpha} C) \circ_{\alpha} (B \circ_{\alpha} A)\) for all intuitionistic fuzzy sets \(A, B, C\) and \(D\) of \(R\).

**Proof.** Same as Lemma 2.
Theorem 1. Let $A$ and $B$ be two non-empty subsets of an LA-ring $R$. Then the following conditions hold.

1. $\chi_A \cap \alpha \chi_B = (\chi_{AB})^\alpha$.
2. $\chi_A \vee \alpha \chi_B = (\chi_{A\cup B})^\beta$.
3. $\chi_A \wedge \beta \chi_B = (\chi_{A\cap B})^\beta$.

Proof. Straight forward.

Theorem 2. Let $A$ be a non-empty subset of an LA-ring $R$. Then the following properties hold.

1. $A$ is an LA-subring of $R$ if and only if $\chi_A$ is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta)$ of $R$.
2. $A$ is a left (resp. right, two-sided) ideal of $R$ if and only if $\chi_A$ is an intuitionistic fuzzy left (resp. right, two-sided) ideal with thresholds $(\alpha, \beta)$ of $R$.

Proof. (1) Let $A$ be an LA-subring of an LA-ring $R$ and $x, y \in R$. If $x, y \notin A$, then by definition of intuitionistic characteristic function $\mu_{\chi_A}(x) = 0 = \mu_{\chi_A}(y)$ and $\gamma_{\chi_A}(x) = 1 = \gamma_{\chi_A}(y)$. Thus

$$\mu_{\chi_A}(x - y) \geq \min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y)\} = \min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\}$$

$$\Rightarrow \mu_{\chi_A}(x - y) \geq \min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\}$$

and

$$\mu_{\chi_A}(xy) \geq \min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y)\} = \min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\}$$

$$\Rightarrow \mu_{\chi_A}(xy) \geq \min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\}$$

Similarly, we have

$$\min\{\gamma_{\chi_A}(x - y), (1 - \alpha)\} \leq \max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y), (1 - \beta)\}$$

and

$$\min\{\gamma_{\chi_A}(xy), (1 - \alpha)\} \leq \max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y), (1 - \beta)\}.$$
Remark 1. (i) $A$ is an additive LA-subgroup of $R$ if and only if $\chi_A$ is an intuitionistic fuzzy additive LA-subgroup with thresholds $(\alpha,\beta)$ of $R$.

(ii) $A$ is an LA-subsemigroup of $R$ if and only if $\chi_A$ is an intuitionistic fuzzy LA-subsemigroup with thresholds $(\alpha,\beta)$ of $R$. 

\[
\begin{align*}
\max\{\mu_{\chi_A}(xy),\alpha\} & \geq \min\{\mu_{\chi_A}(x),\mu_{\chi_A}(y),\beta\} = \beta, \\
\min\{\gamma_{\chi_A}(x-y),(1-\alpha)\} & \leq \max\{\gamma_{\chi_A}(x),\gamma_{\chi_A}(y),(1-\beta)\} = 1 - \beta, \\
\min\{\gamma_{\chi_A}(xy),(1-\alpha)\} & \leq \max\{\gamma_{\chi_A}(x),\gamma_{\chi_A}(y),(1-\beta)\} = 1 - \beta,
\end{align*}
\]

$\chi_A$ being an intuitionistic fuzzy LA-subring with thresholds $(\alpha,\beta)$ of $R$. Thus

\[
\begin{align*}
\max\{\mu_{\chi_A}(x-y),\alpha\} & \geq \beta \text{ and } \max\{\mu_{\chi_A}(xy),\alpha\} \geq \beta, \\
\min\{\gamma_{\chi_A}(x-y),(1-\alpha)\} & \leq 1 - \beta \text{ and } \min\{\gamma_{\chi_A}(xy),(1-\alpha)\} \leq 1 - \beta.
\end{align*}
\]

This implies that $\mu_{\chi_A}(x-y) = 1 = \mu_{\chi_A}(xy)$ and $\gamma_{\chi_A}(x-y) = 0 = \gamma_{\chi_A}(xy)$, i.e., $x - y$ and $xy \in A$. Hence $A$ is an LA-subring of $R$.

(2) Let $A$ be a left ideal of an LA-ring $R$ and $x, y \in R$. If $y \notin A$, then by definition of intuitionistic characteristic function $\mu_{\chi_A}(y) = 0$ and $\gamma_{\chi_A}(y) = 1$. Thus

\[
\mu_{\chi_A}(xy) \geq \mu_{\chi_A}(y) = \min\{\mu_{\chi_A}(y),\beta\}
\]

$\Rightarrow \mu_{\chi_A}(xy) \geq \min\{\mu_{\chi_A}(y),\beta\}$

\[
\Rightarrow \max\{\mu_{\chi_A}(xy),\alpha\} \geq \min\{\mu_{\chi_A}(y),\beta\}
\]

and $\gamma_{\chi_A}(xy) \leq \gamma_{\chi_A}(y) = \max\{\gamma_{\chi_A}(y),(1-\beta)\}$

$\Rightarrow \gamma_{\chi_A}(xy) \leq \max\{\gamma_{\chi_A}(y),(1-\beta)\}$

$\Rightarrow \min\{\gamma_{\chi_A}(xy),(1-\alpha)\} \leq \max\{\gamma_{\chi_A}(y),(1-\beta)\}$.

Similarly, we have

\[
\begin{align*}
\max\{\mu_{\chi_A}(xy),\alpha\} & \geq \min\{\mu_{\chi_A}(y),\beta\}, \\
\min\{\gamma_{\chi_A}(xy),(1-\alpha)\} & \leq \max\{\gamma_{\chi_A}(y),(1-\beta)\},
\end{align*}
\]

when $y \in A$. Therefore the intuitionistic characteristic function $\chi_A$ of $A$ is an intuitionistic fuzzy left ideal with thresholds $(\alpha,\beta)$ of $R$.

Conversely, assume that the intuitionistic characteristic function $\chi_A$ of $A$ is an intuitionistic fuzzy left ideal with thresholds $(\alpha,\beta)$ of an LA-ring $R$. Let $y \in A$ and $z \in R$, then by definition $\mu_{\chi_A}(y) = 1$ and $\gamma_{\chi_A}(y) = 0$. Since

\[
\begin{align*}
\max\{\mu_{\chi_A}(zy),\alpha\} & \geq \min\{\mu_{\chi_A}(y),\beta\} = \beta, \\
\min\{\gamma_{\chi_A}(zy),(1-\alpha)\} & \leq \max\{\gamma_{\chi_A}(y),(1-\beta)\} = 1 - \beta,
\end{align*}
\]

$\chi_A$ being an intuitionistic fuzzy left ideal with thresholds $(\alpha,\beta)$ of $R$. Thus

\[
\begin{align*}
\max\{\mu_{\chi_A}(zy),\alpha\} & \geq \beta \text{ and } \min\{\gamma_{\chi_A}(zy),(1-\alpha)\} \leq 1 - \beta.
\end{align*}
\]

This implies that $\mu_{\chi_A}(zy) = 1$ and $\gamma_{\chi_A}(zy) = 0$, i.e., $zy \in A$. Therefore $A$ is a left ideal of $R$. 

Remark 1. (i) $A$ is an additive LA-subgroup of $R$ if and only if $\chi_A$ is an intuitionistic fuzzy additive LA-subgroup with thresholds $(\alpha,\beta)$ of $R$.

(ii) $A$ is an LA-subsemigroup of $R$ if and only if $\chi_A$ is an intuitionistic fuzzy LA-subsemigroup with thresholds $(\alpha,\beta)$ of $R$. 

Theorem 3. Let $A$ be an IFS of an LA-ring $R$. Then the following assertions hold.

(1) $A$ is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta)$ of $R$ if and only if $A \cap^\beta A \subseteq A^\beta$ and $A^\beta \subseteq A_{\alpha}$. 

(2) $A$ is an intuitionistic fuzzy left (resp. right) ideal with thresholds $(\alpha, \beta)$ of $R$ if and only if $R \cap^\beta A \subseteq A_{\alpha}$ (resp. $A \cap^\beta R \subseteq A_{\alpha}$) and $A^\beta \subseteq A_{\alpha}$. 

Proof. (1) Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta)$ of an LA-ring $R$ and $x \in R$. For $A \cap^\beta A \subseteq A^\beta$. If $(A \cap^\beta A)(x) = 0$, then obvious $A \cap^\beta A \subseteq A^\beta$, otherwise we have 

$$
(\mu_A \cap^\beta A)(x) = \{(\mu_A \cap A)(x) \land \beta) \lor \alpha = \left\{ \left( \bigvee_{a=1}^n a_i \cap b_i \{ \land_{a=1}^n \{ \mu_A(a_i) \land \mu_A(b_i) \} \} \right) \land \beta \right\} \lor \alpha \leq \left\{ \left( \bigvee_{a=1}^n a_i \cap b_i \{ \land_{a=1}^n \mu_A(a_i \cap b_i) \} \right) \land \beta \right\} \lor \alpha = \{(\mu_A(x) \land \beta) \lor \alpha = (\mu_A)^\beta(x), \Rightarrow \mu_A \cap^\beta A \subseteq (\mu_A)^\beta.
$$

Similarly, we have $\gamma_A \cap^\beta A \supseteq (\gamma_A)^\beta$. Thus $A \cap^\beta A \subseteq A^\beta$.

Now for $A \cap^\beta A \subseteq A^\beta$. If $(A \cap^\beta A)(x) = 0$, then obvious $A \cap^\beta A \subseteq A^\beta$, otherwise we have 

$$
(\mu_A \cap^\beta A)(x) = \{(\mu_A \cap A)(x) \land \beta) \lor \alpha = \left\{ \left( \bigvee_{a=1}^n a_i \cap b_i \{ \land_{a=1}^n \{ \mu_A(a_i) \land \mu_A(b_i) \} \} \right) \land \beta \right\} \lor \alpha \leq \left\{ \left( \bigvee_{a=1}^n a_i \cap b_i \{ \land_{a=1}^n \mu_A(a_i \cap b_i) \} \right) \land \beta \right\} \lor \alpha = \{(\mu_A(x) \land \beta) \lor \alpha = (\mu_A)^\beta(x), \Rightarrow \mu_A \cap^\beta A \subseteq (\mu_A)^\beta.
$$

Similarly, we have $\gamma_A \cap^\beta A \supseteq (\gamma_A)^\beta$. Thus $A \cap^\beta A \subseteq A^\beta$.

Conversely, assume that $A \cap^\beta A \subseteq A^\beta$ and $A \cap^\beta A \subseteq A^\beta$. Let $x, y \in R$ such that $a = xy$. Now 

$$
\max \{\mu_A(xy), \alpha\} = \max \{\mu_A(a), \alpha\} = \max \{\min \{\mu_A(a), \beta\}, \alpha\} = (\mu_A)^\beta(a) \geq (\mu_A \cap A)(a) = \{(\mu_A \cap A)(a) \land \beta \lor \alpha = \{(\bigvee_{a=1}^n a_i \cap b_i \{ \land_{a=1}^n \{ \mu_A(a_i) \land \mu_A(b_i) \} \} \land \beta \lor \alpha \geq \{(\mu_A(x) \land \mu_A(y)) \land \beta \lor \alpha = (\mu_A(x) \land \mu_A(y)) \lor \beta = \min \{\mu_A(x), \mu_A(y), \beta\}, \Rightarrow \max \{\mu_A(xy), \alpha\} \geq \min \{\mu_A(x), \mu_A(y), \beta\}.
$$

Similarly, we have $\min \{\gamma_A(xy), (1 - \alpha)\} \leq \max \{\gamma_A(x), \gamma_A(y), (1 - \beta)\}$. Now set $a = x - y$ and

$$
\max \{\mu_A(x - y), \alpha\} = \max \{\mu_A(a), \alpha\} = \max \{\min \{\mu_A(a), \beta\}, \alpha\}
$$
Lemma 3. If \( A \) and \( B \) are two intuitionistic fuzzy LA-subrings (resp. (left, right, two-sided) ideals) with thresholds \((\alpha, \beta)\) of an LA-ring \( R \), then \( A \wedge_{\alpha}^\beta B \) is also an intuitionistic fuzzy LA-subring (resp. (left, right, two-sided) ideal) with thresholds \((\alpha, \beta)\) of \( R \).

\[
\mu_{A}(a) \geq (\mu_{A} - \varphi_{\alpha}) A = ((\mu_{A} - \mu_{A})(a) \land \beta) \lor \alpha
\]

Similarly, we have \( \min\{\gamma_{A}(x-y), (1-\alpha)\} \leq \max\{\gamma_{A}(x), \gamma_{A}(y), (1-\beta)\} \). Hence \( A \) is an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \( R \).

(2) Assume that \( A \) is an intuitionistic fuzzy left ideal with thresholds \((\alpha, \beta)\) of an LA-ring \( R \) and \( x \in R \). If \( (R \circ_{\alpha}^\beta A)(x) = 0 \), then obvious \( R \circ_{\alpha}^\beta A \subseteq A_{\alpha}^\beta \), otherwise we have

\[
(R \circ_{\alpha}^\beta \mu_{A})(x) = \{(R \circ \mu_{A})(x) \land \beta) \lor \alpha
\]

Similarly, we have \( R \circ_{\alpha}^\beta \gamma_{A} \subseteq (\gamma_{A})_{\alpha}^\beta \). Thus \( R \circ_{\alpha}^\beta A \subseteq A_{\alpha}^\beta \).

Conversely, suppose that \( R \circ_{\alpha}^\beta A \subseteq A_{\alpha}^\beta \). Let \( y, z \in R \) such that \( x = yz \). Now

\[
\max\{\mu_{A}(yz), \alpha\} = \max\{\mu_{A}(x), \alpha\} = \max\{\min\{\mu_{A}(x), \beta\}, \alpha\}
\]

\[
= (\mu_{A})_{\alpha}^\beta(x) \geq (R \circ_{\alpha}^\beta \mu_{A})(x) = \{(R \circ \mu_{A})(x) \land \beta) \lor \alpha
\]

\[
\geq ((R(y) \land \mu_{A}(z)) \land \beta) \lor \alpha
\]

\[
= (1 \land \mu_{A}(z)) \land \beta = \min\{\mu_{A}(z), \beta\}
\]

\[
\Rightarrow \max\{\mu_{A}(yz), \alpha\} \geq \min\{\mu_{A}(z), \beta\}
\]

Similarly, we have \( \min\{\gamma_{A}(yz), (1-\alpha)\} \leq \max\{\gamma_{A}(z), (1-\beta)\} \). Therefore \( A \) is an intuitionistic fuzzy left ideal with thresholds \((\alpha, \beta)\) of \( R \).

Lemma 3. If \( A \) and \( B \) are two intuitionistic fuzzy LA-subrings (resp. (left, right, two-sided) ideals) with thresholds \((\alpha, \beta)\) of an LA-ring \( R \), then \( A \wedge_{\alpha}^\beta B \) is also an intuitionistic fuzzy LA-subring (resp. (left, right, two-sided) ideal) with thresholds \((\alpha, \beta)\) of \( R \).

Proof. Let \( A = (\mu_{A}, \gamma_{A}) \) and \( B = (\mu_{B}, \gamma_{B}) \) be two intuitionistic fuzzy LA-subrings with thresholds \((\alpha, \beta)\) of an LA-ring \( R \). We have to show that \( A \wedge_{\alpha}^\beta B \) is also an intuitionistic
fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \(R\). Now

\[
\max\{(\mu_A \land^\beta \mu_B)(x-y), \alpha\} = \max\{\{(\mu_A \land \mu_B)(x-y) \land \beta\} \lor \alpha\}, \alpha\} = \{(\mu_A \land \mu_B)(x-y) \land \beta\} \lor \alpha \geq \{\mu_A(x) \land \mu_A(y) \land \mu_B(x) \land \mu_B(y) \land \beta\} \lor \alpha = \{(\mu_A \land \mu_B)(x) \land \mu_A(y) \land \mu_B(y) \land \beta\} \lor \alpha = \{(\mu_A \land \mu_B)(x)) \land \mu_A(y) \land \mu_B(y) \land \beta\} \lor \alpha = \{(\mu_A \land \mu_B)(x) \land \mu_A(y)) \land \mu_B(y) \land \beta\} \lor \alpha = \{(\mu_A \land \mu_B)(x) \land \mu_A(y)) \land \mu_B(y) \land \beta\} \lor \alpha = \min\{(\mu_A \land^\beta \mu_B)(x), (\mu_A \land^\beta \mu_B)(y), \beta\}.
\]

Thus \(\max\{(\mu_A \land^\beta \mu_B)(x-y), \alpha\} \geq \min\{(\mu_A \land^\beta \mu_B)(x), (\mu_A \land^\beta \mu_B)(y), \beta\}\). Similarly, we have \(\max\{(\mu_A \land^\beta \mu_B)(xy), \alpha\} \geq \min\{(\mu_A \land^\beta \mu_B)(x), (\mu_A \land^\beta \mu_B)(y), \beta\}\). In same lines we have \(\min\{(\gamma_A \lor^\alpha \gamma_B)(x-y), (1-\alpha)\} \leq \max\{(\gamma_A \lor^\alpha \gamma_B)(x), (\gamma_A \lor^\alpha \gamma_B)(y), (1-\beta)\}\) and \(\min\{(\gamma_A \lor^\alpha \gamma_B)(xy), (1-\alpha)\} \leq \max\{(\gamma_A \lor^\alpha \gamma_B)(x), (\gamma_A \lor^\alpha \gamma_B)(y), (1-\beta)\}\).

Hence \(A \land^\beta B\) is an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \(R\).

**Lemma 4.** If \(A\) and \(B\) are two intuitionistic fuzzy LA-subrings with thresholds \((\alpha, \beta)\) of an LA-ring \(R\), then \(A \circ_\alpha B\) is also an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \(R\).

**Proof.** Suppose that \(A = (\mu_A, \gamma_A)\) and \(B = (\mu_B, \gamma_B)\) are two intuitionistic fuzzy LA-subrings with thresholds \((\alpha, \beta)\) of an LA-ring \(R\). We have to show that \(A \circ_\alpha B\) is also an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \(R\). Now

\[
(\mu_A \circ_\alpha \mu_B)^2 = (\mu_A \circ_\alpha \mu_B) \circ_\alpha (\mu_A \circ_\alpha \mu_B) = (\mu_A \circ_\alpha \mu_B)^2 \leq (\mu_A)^2 \circ_\alpha (\mu_B)^2 = (\mu_A \circ_\alpha \mu_B)^2
\]

and

\[
(\gamma_A \circ_\alpha \gamma_B)^2 = (\gamma_A \circ_\alpha \gamma_B) \circ_\alpha (\gamma_A \circ_\alpha \gamma_B) = (\gamma_A \circ_\alpha \gamma_B)^2 \leq (\gamma_A)^2 \circ_\alpha (\gamma_B)^2 = (\gamma_A \circ_\alpha \gamma_B)^2.
\]

Since \(\mu_B \circ_\alpha \mu_B \leq (\mu_B)^2 \) and \(\gamma_B \circ_\alpha \gamma_B \geq (\gamma_B)^2\), \(B = (\mu_B, \gamma_B)\) being an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\). This implies that \(\mu_A \circ_\alpha \mu_B \) and \(\gamma_A \circ_\alpha \gamma_B \) are two intuitionistic fuzzy LA-subrings with thresholds \((\alpha, \beta)\). Therefore \(A \circ_\alpha B\) is an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \(R\).

**Remark 2.** If \(A\) is an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of an LA-ring \(R\), then \(A \circ_\alpha A\) is also an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \(R\).
Lemma 5. Let $R$ be an LA-ring with left identity $e$. Then every intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta)$ of $R$ is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta)$ of $R$.

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta)$ of an LA-ring $R$ and $x, y \in R$. Thus

$$max\{\mu_A(xy), \alpha\} = max\{\mu_A((ex)y), \alpha\} = max\{\mu_A((yx)e), \alpha\} \geq min\{\mu_A(yx), \beta\} \geq min\{\mu_A(y), \beta\}$$

and

$$min\{\gamma_A(xy), (1-\alpha)\} = min\{\gamma_A((ex)y), (1-\alpha)\} = min\{\gamma_A((yx)e), (1-\alpha)\} \leq max\{\gamma_A(xy), (1-\beta)\} \leq max\{\gamma_A(y), (1-\beta)\}.$$ 

Therefore $A$ is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta)$ of $R$.

Lemma 6. If $A$ and $B$ are two intuitionistic fuzzy left (resp. right) ideals with thresholds $(\alpha, \beta)$ of an LA-ring $R$ with left identity $e$, then $A \circ^\alpha B$ is also an intuitionistic fuzzy left (resp. right) ideal with thresholds $(\alpha, \beta)$ of $R$.

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_A, \gamma_A)$ be two intuitionistic fuzzy left ideals with thresholds $(\alpha, \beta)$ of an LA-ring $R$. We have to show that $A \circ^\alpha B$ is also an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta)$ of $R$. Since, $\mu_A \circ^\alpha \mu_B - \beta \mu_A \circ^\alpha \mu_B \subseteq \mu_A \circ^\alpha \mu_B$ and $\gamma_A \circ^\alpha \gamma_B \subseteq \gamma_A \circ^\alpha \gamma_B$ by the Lemma 4. Now

$$R \circ^\alpha (\mu_A \circ^\alpha \mu_B) = (R \circ^\alpha R) \circ^\alpha (\mu_A \circ^\alpha \mu_B) \subseteq \mu_A \circ^\alpha \mu_B$$

and

$$R \circ^\alpha (\gamma_A \circ^\alpha \gamma_B) = (R \circ^\alpha R) \circ^\alpha (\gamma_A \circ^\alpha \gamma_B) \subseteq \gamma_A \circ^\alpha \gamma_B.$$ 

Hence $A \circ^\alpha B$ is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta)$ of $R$. Similarly, we can prove for right ideals.

Remark 3. If $A$ is an intuitionistic fuzzy left (resp. right) ideal with thresholds $(\alpha, \beta)$ of an LA-ring $R$ with left identity $e$, then $A \circ^\alpha A$ is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta)$ of $R$.

Lemma 7. If $A$ and $B$ are two intuitionistic fuzzy ideals with thresholds $(\alpha, \beta)$ of an LA-ring $R$, then $A \circ^\alpha B \subseteq A \wedge^\alpha B$.

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy ideals with thresholds $(\alpha, \beta)$ of an LA-ring $R$ and $x \in R$. If $(A \circ^\alpha B)(x) = 0$, then obvious $A \circ^\alpha B \subseteq A \wedge^\alpha B$, otherwise we have

$$(\mu_A \circ^\alpha \mu_B)(x) = \{(\mu_A \circ \mu_B)(x) \wedge \beta\} \vee \alpha$$
\[
\begin{align*}
&= \left\{ \left( \bigvee_{i=1}^{n} a_i b_i \left\{ \bigwedge_{i=1}^{n} \{ \mu_A (a_i) \wedge \mu_B (b_i) \} \right\} \right) \wedge \beta \right\} \vee \alpha \\
&\leq \left\{ \left( \bigvee_{i=1}^{n} a_i b_i \left\{ \bigwedge_{i=1}^{n} \{ \mu_A (a_i) \wedge \mu_B (a_i b_i) \} \right\} \right) \wedge \beta \right\} \vee \alpha \\
&= \left\{ \left( \bigvee_{i=1}^{n} a_i b_i \left\{ \bigwedge_{i=1}^{n} \{ (\mu_A \wedge \mu_B) (a_i b_i) \} \right\} \right) \wedge \beta \right\} \vee \alpha \\
&= \left\{ (\mu_A \wedge \mu_B) (x) \wedge \beta \right\} \vee \alpha = (\mu_A \wedge \mu_B) (x).
\end{align*}
\]

Similarly, we have \( \gamma_A \circ^\beta_A \gamma_B \supseteq \gamma_A \cup^\beta_A \gamma_B \). Hence \( A \circ^\beta_A B \subseteq A \wedge^\beta_A B \).

**Remark 4.** If \( A \) is an intuitionistic fuzzy ideal with thresholds \( (\alpha, \beta) \) of an LA-ring \( R \), then \( A \circ^\beta_A A \subseteq A \wedge^\beta_A A \).

**Lemma 8.** Let \( R \) be an LA-ring. Then \( A \circ^\beta_A B \subseteq A \wedge^\beta_A B \) for every intuitionistic fuzzy right ideal \( A \) with thresholds \( (\alpha, \beta) \) and every intuitionistic fuzzy left ideal \( B \) with thresholds \( (\alpha, \beta) \) of \( R \).

**Proof.** Same as Lemma 7.

**Theorem 4.** Let \( A \) be a non-empty subset of an LA-ring \( R \). Then the following conditions are true.

1. \( A \) is an interior ideal of \( R \) if and only if \( \chi_A \) is an intuitionistic fuzzy interior ideal with thresholds \( (\alpha, \beta) \) of \( R \).
2. \( A \) is a quasi-ideal of \( R \) if and only if \( \chi_A \) is an intuitionistic fuzzy quasi-ideal with thresholds \( (\alpha, \beta) \) of \( R \).
3. \( A \) is a bi-ideal of \( R \) if and only if \( \chi_A \) is an intuitionistic fuzzy bi-ideal with thresholds \( (\alpha, \beta) \) of \( R \).
4. \( A \) is a generalized bi-ideal of \( R \) if and only if \( \chi_A \) is an intuitionistic fuzzy generalized bi-ideal with thresholds \( (\alpha, \beta) \) of \( R \).

**Proof.** Let \( A \) be an interior ideal of an LA-ring \( R \), this implies that \( A \) is an additive LA-subgroup. Then \( \chi_A \) is an intuitionistic fuzzy additive LA-subgroup with thresholds \( (\alpha, \beta) \) of \( R \) by the Remark 1. Let \( x, y, a \in R \). If \( a \notin A \), then by definition of intuitionistic characteristic function \( \mu_{\chi_A} (a) = 0 \) and \( \gamma_{\chi_A} (a) = 1 \). Thus

\[
\mu_{\chi_A} ((xa)y) \geq \mu_{\chi_A} (a) = \min \{ \mu_{\chi_A} (a), \beta \} \\
\Rightarrow \mu_{\chi_A} ((xa)y) \geq \min \{ \mu_{\chi_A} (a), \beta \} \\
\Rightarrow \max \{ \mu_{\chi_A} ((xa)y), \alpha \} \geq \min \{ \mu_{\chi_A} (a), \beta \}.
\]

Similarly, we have \( \min \{ \gamma_{\chi_A} ((xa)y), (1 - \alpha) \} \leq \max \{ \gamma_{\chi_A} (a), (1 - \beta) \} \). In same lines, we have

\[
\max \{ \mu_{\chi_A} ((xa)y), \alpha \} \geq \min \{ \mu_{\chi_A} (a), \beta \} \\
\text{and } \min \{ \gamma_{\chi_A} ((xa)y), (1 - \alpha) \} \leq \max \{ \gamma_{\chi_A} (a), (1 - \beta) \}.
\]
when \( a \in A \). Hence the intuitionistic characteristic function \( \chi_A \) of \( A \) is an intuitionistic fuzzy interior ideal with thresholds \((\alpha, \beta)\) of \( R \).

Conversely, suppose that the intuitionistic characteristic function \( \chi_A \) of \( A \) is an intuitionistic fuzzy interior ideal with thresholds \((\alpha, \beta)\) of \( R \), this means that \( \chi_A \) is an intuitionistic fuzzy additive LA-subgroup with thresholds \((\alpha, \beta)\) of \( R \). Then \( A \) is an additive LA-subgroup of \( R \) by the Remark 1. Let \( t \in (RA)R \), so \( t = (xa)y \), where \( a \in A \) and \( x, y \in R \). Then by definition \( \mu_{\chi_A}(a) = 1 \) and \( \gamma_{\chi_A}(a) = 0 \). Since

\[
\max\{\mu_{\chi_A}(xa)y, \alpha\} \geq \min\{\mu_{\chi_A}(a), \beta\} = \beta
\]

and
\[
\min\{\gamma_{\chi_A}(xa)y, (1 - \alpha)\} \leq \max\{\gamma_{\chi_A}(a), (1 - \beta)\} = 1 - \beta,
\]

\( \chi_A \) being an intuitionistic fuzzy interior ideal with thresholds \((\alpha, \beta)\) of \( R \). This implies that \( \mu_{\chi_A}(xa)y \geq \beta \) and \( \gamma_{\chi_A}(xa)y \leq 1 - \beta \), thus \( \mu_{\chi_A}(xa)y = 1 \) and \( \mu_{\chi_A}(xa)y = 0 \), i.e., \((xa)y \in A \). Hence \( A \) is an interior ideal of \( R \).

(2) Let \( A \) be a quasi-ideal of \( R \), this implies that \( A \) is an additive LA-subgroup. Then \( \chi_A \) is an intuitionistic fuzzy additive LA-subgroup with thresholds \((\alpha, \beta)\) of \( R \) by the Remark 1. Let \( x \in R \) and \( x \notin A \), then \( x \notin RA \) or \( x \notin AR \). If \( x \notin RA \), then definition of intuitionistic characteristic function \((R \circ \mu_{\chi_A})(x) = 0 \) and \((R \circ \gamma_{\chi_A})(x) = 1 \). Thus

\[
\max\{\mu_{\chi_A}(x), \alpha\} \geq 0 = \min\{\mu_{\chi_A}(R)(x), (R \circ \mu_{\chi_A})(x), \beta\}
\]

and
\[
\min\{\gamma_{\chi_A}(x), (1 - \alpha)\} \leq 1 = \max\{\gamma_{\chi_A}(R)(x), (R \circ \gamma_{\chi_A})(x), (1 - \beta)\}.
\]

If \( x \in A \), then

\[
\max\{\mu_{\chi_A}(x), \alpha\} = 1 \geq \min\{\mu_{\chi_A}(R)(x), R \circ \mu_{\chi_A}(x), \beta\}
\]

and
\[
\min\{\gamma_{\chi_A}(x), (1 - \alpha)\} = 0 \leq \max\{\gamma_{\chi_A}(R)(x), R \circ \gamma_{\chi_A}(x), (1 - \beta)\}.
\]

Therefore the intuitionistic characteristic function \( \chi_A \) of \( A \) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \( R \).

Conversely, assume that the intuitionistic characteristic function \( \chi_A \) of \( A \) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \( R \), this means that \( \chi_A \) is an intuitionistic fuzzy additive LA-subgroup with thresholds \((\alpha, \beta)\) of \( R \). Then \( A \) is an additive LA-subgroup of \( R \) by the Remark 1. Let \( x \) be an element of \( AR \cap RA \), this means that \( x \in AR \) and \( RA \). Since

\[
\max\{\mu_{\chi_A}(x), \alpha\} \geq \min\{\mu_{\chi_A}(R)(x), (R \circ \mu_{\chi_A})(x), \beta\} = \min\{\mu_{\chi_A}(\mu_{RA})(x), \mu_{\chi_A}(\mu_{AR})(x), \beta\} = \min\{\mu_{\chi_{AR}}(x), \mu_{\chi_{RA}}(x), \beta\} = \beta.
\]

\( \Rightarrow \) \( \max\{\mu_{\chi_A}(x), \alpha\} \geq \beta \).

Similarly, we have \( \min\{\gamma_{\chi_A}(x), (1 - \alpha)\} \leq 1 - \beta \), thus \( \mu_{\chi_A}(x) = 1 \) and \( \gamma_{\chi_A}(x) = 0 \), i.e., \( x \in A \). Therefore \( A \) is a quasi-ideal of \( R \).

(3) Let \( A \) be a bi-ideal of \( R \), this implies that \( A \) is an LA-subring of \( R \). Then \( \chi_A \) is an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \( R \) by the Remark 1. Let
Let \( \beta (\max A \\setminus \text{olds} (x, a)), \alpha \), \( x, y \in A \), then by definition of intuitionistic characteristic function \( \mu_{\chi A}(x) = \mu_{\chi A}(y) = 0 \) and \( \gamma_{\chi A}(x) = \gamma_{\chi A}(y) = 1 \). Thus

\[
\mu_{\chi A}((xa)y) \geq \mu_{\chi A}(x) \land \mu_{\chi A}(y) = \min\{\mu_{\chi A}(x), \mu_{\chi A}(y), \beta\}
\]

\[
\Rightarrow \mu_{\chi A}((xa)y) \geq \min\{\mu_{\chi A}(x), \mu_{\chi A}(y), \beta\}
\]

\[
\Rightarrow \max\{\mu_{\chi A}((xa)y), \alpha\} \geq \min\{\mu_{\chi A}(x), \mu_{\chi A}(y), \beta\}.
\]

Similarly, we have \( \min\{\gamma_{\chi A}((xa)y), (1 - \alpha)\} \leq \max\{\gamma_{\chi A}(x), \mu_{\chi A}(y), (1 - \beta)\} \). In same lines we have

\[
\max\{\mu_{\chi A}((xa)y), \alpha\} \geq \min\{\mu_{\chi A}(x), \mu_{\chi A}(y), \beta\}
\]

and \( \min\{\gamma_{\chi A}((xa)y), (1 - \alpha)\} \leq \max\{\gamma_{\chi A}(x), \gamma_{\chi A}(y), (1 - \beta)\} \),

when \( x, y \in A \). since the intuitionistic characteristic function \( \chi_A \) of \( A \) is an intuitionistic fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of \( R \).

Conversely, suppose that the intuitionistic characteristic function \( \chi_A \) of \( A \) is an intuitionistic fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of \( R \), this means that \( \chi_A \) is an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \( R \). Then \( A \) is an LA-subring of \( R \) by the Remark 1. Let \( t \in (AR)A \), so \( t = (xa)y \), where \( x, y \in A \) and \( a \in R \). Then the definition \( \mu_{\chi A}(x) = \mu_{\chi A}(y) = 1 \) and \( \gamma_{\chi A}(x) = \gamma_{\chi A}(y) = 0 \). As

\[
\max\{\mu_{\chi A}((xa)y), \alpha\} \geq \min\{\mu_{\chi A}(x), \mu_{\chi A}(y), \beta\} = \beta
\]

and \( \min\{\gamma_{\chi A}((xa)y), (1 - \alpha)\} \leq \max\{\gamma_{\chi A}(x), \gamma_{\chi A}(y), (1 - \beta)\} = 1 - \beta, \)

\( \chi_A \) being an intuitionistic fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of \( R \). This implies that \( \mu_{\chi A}((xa)y) \geq \beta \) and \( \gamma_{\chi A}((xa)y) \leq 1 - \beta \), thus \( \mu_{\chi A}((xa)y) = 1 \) and \( \mu_{\chi A}((xa)y) = 0 \), i.e., \( (xa)y \in A \). Hence \( A \) is bi-ideal of \( R \). Similarly, we can prove (4).

**Theorem 5.** Let \( A = (\mu_A, \gamma_A) \) be an IFS of an LA-ring \( R \). Then \( A \) is an intuitionistic fuzzy interior ideal with thresholds \((\alpha, \beta)\) of \( R \) if and only if \( (R \circ_\alpha^\beta A) \circ_\alpha^\beta R \subseteq A_\alpha \) and \( A - \alpha A \subseteq A_\alpha \).

**Proof.** Suppose that \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy interior ideal with thresholds \((\alpha, \beta)\) of an LA-ring \( R \) and \( x \in R \). If \( (R \circ_\alpha^\beta A) \circ_\alpha^\beta R(x) = 0 \), then obvious \( (R \circ_\alpha^\beta A) \circ_\alpha^\beta R \subseteq A_\alpha \). Otherwise there exist \( a_i, b_i, c_i, d_i \in R \) such that \( x = \sum_{i=1}^n a_i b_i \) and \( a_i = \sum_{i=1}^n c_i d_i \).

Since \( A \) is an intuitionistic fuzzy interior ideal with thresholds \((\alpha, \beta)\) of \( R \), this implies that \( \max\{\mu_A((c_i d_i) b_i), \alpha\} \geq \min\{\mu_A(d_i), \beta\} \) and \( \min\{\gamma_A((c_i d_i) b_i), (1 - \alpha)\} \leq \max\{\gamma_A(d_i), (1 - \beta)\} \).

Now

\[
((R \circ_\alpha^\beta \mu_A) \circ_\alpha^\beta R)(x) = \{((R \circ_\alpha^\beta \mu_A) \circ_\alpha^\beta R)(x) \land \beta\} \lor \alpha
\]

\[
= \{(\vee_{x=\sum_{i=1}^n a_i b_i} \{\land_{i=1}^n \{((R \circ_\alpha^\beta \mu_A) \circ_\alpha^\beta R)(a_i) \land \beta\}\}) \land \beta\} \lor \alpha
\]

\[
= \{(\vee_{x=\sum_{i=1}^n a_i b_i} \{\land_{i=1}^n \{(R \circ_\alpha^\beta \mu_A)(a_i) \land 1\}\}) \land \beta\} \lor \alpha
\]
Theorem 6. Let \( A = (\mu_A, \gamma_A) \) be an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of an LA-ring \( R \). Then \( A \) is an intuitionistic fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of \( R \) if and only if \((A \circ_\alpha^\beta R) \circ_\alpha^\beta A \subseteq A_\alpha^\beta\).

Proof. Same as Theorem 5.

Theorem 7. Let \( A = (\mu_A, \gamma_A) \) be an IFS of an LA-ring \( R \). Then \( A \) is an intuitionistic fuzzy generalized bi-ideal with thresholds \((\alpha, \beta)\) of \( R \) if and only if \((A \circ_\alpha^\beta R) \circ_\alpha^\beta A \subseteq A_\alpha^\beta\) and \( A −_\alpha A \subseteq A_\alpha^\beta\).

Proof. Same as Theorem 5.
Lemma 9. If $A$ and $B$ are two intuitionistic fuzzy bi- (resp. generalized bi-, quasi-, interior) ideals with thresholds $(\alpha, \beta)$ of an LA-ring $R$, then $A \wedge^\alpha B$ is also an intuitionistic fuzzy bi- (resp. generalized bi-, quasi-, interior) ideal with thresholds $(\alpha, \beta)$ of $R$.

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy bi-ideals with thresholds $(\alpha, \beta)$ of an LA-ring $R$. We have to show that $A \wedge^\alpha B$ is also an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta)$ of $R$. Since $A$ and $B$ are intuitionistic fuzzy LA-subrings with thresholds $(\alpha, \beta)$ of $R$, then $A \wedge^\alpha B$ is also an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta)$ of $R$ by the Lemma 3. We have to show that $\max\{\mu_A(x) \wedge^\alpha \mu_B(y), \alpha\} \geq \min\{\mu_A \wedge^\alpha \mu_B(x), \mu_A \wedge^\alpha \mu_B(y), \beta\}$ and $\min\{\gamma_A \vee^\beta \gamma_B(x), \gamma_A \vee^\beta \gamma_B(y), (1 - \alpha)\} \leq \max\{\gamma_A \vee^\beta \gamma_B(x), (\gamma_A \vee^\beta \gamma_B)(y), (1 - \beta)\}$. Now

$$\max\{\mu_A \wedge^\alpha \mu_B((xa)y), \alpha\} = \max\{(\mu_A \wedge \mu_B)((xa)y) \wedge \beta\} \vee \alpha = \{\mu_A((xa)y) \wedge \mu_B((xa)y) \wedge \beta\} \vee \alpha \geq \mu_A(x) \wedge \mu_B(x) \wedge \mu_B(y) \wedge \beta \vee \alpha = \{\mu_A(x) \wedge \mu_B(x) \wedge \mu_B(y) \wedge \beta\} \vee \alpha = \{\gamma_A \wedge \gamma_B(x, \gamma_A \wedge \gamma_B)(y) \wedge \beta\} \vee \alpha = \{\gamma_A \wedge \gamma_B(x) \wedge \gamma_A \wedge \gamma_B(y) \wedge \beta\} \vee \alpha = \min\{\mu_A \wedge^\alpha \mu_B(x), (\mu_A \wedge^\alpha \mu_B)(y) \wedge \beta\}.\]$$

Thus $\max\{\mu_A \wedge^\alpha \mu_B((xa)y), \alpha\} \geq \min\{\mu_A \wedge^\alpha \mu_B(x), (\mu_A \wedge^\alpha \mu_B)(y), \beta\}$. Similarly, we have $\min\{\gamma_A \vee^\beta \gamma_B(x), \gamma_A \vee^\beta \gamma_B(y), (1 - \alpha)\} \leq \max\{\gamma_A \vee^\beta \gamma_B(x), (\gamma_A \vee^\beta \gamma_B)(y), (1 - \beta)\}$. Hence $A \wedge^\alpha B$ is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta)$ of $R$.

Lemma 10. If $A$ and $B$ are two intuitionistic fuzzy bi- (resp. generalized bi-, interior) ideals with thresholds $(\alpha, \beta)$ of an LA-ring $R$ with left identity $e$, then $A \circ^\beta B$ is also an intuitionistic fuzzy bi- (resp. generalized bi-, interior) ideal with thresholds $(\alpha, \beta)$ of $R$.

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy bi-ideals with thresholds $(\alpha, \beta)$ of an LA-ring $R$. We have to show that $A \circ^\alpha B$ is also an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta)$ of $R$. Since $A$ and $B$ are intuitionistic fuzzy LA-subrings with thresholds $(\alpha, \beta)$ of $R$, then $A \circ^\alpha B$ is also an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta)$ of $R$ by the Lemma 4. Now

$$(\mu_A \circ^\alpha \mu_B) \circ^\beta (\mu_A \circ^\beta \mu_B) = (\mu_A \circ^\alpha \mu_B) \circ^\beta (R \circ^\alpha \mu_B) \circ^\alpha (\mu_A \circ^\beta \mu_B) = (\mu_A \circ^\alpha R) \circ^\beta (\mu_B \circ^\alpha \mu_B) \circ^\alpha (\mu_A \circ^\beta \mu_B) = (\mu_A \circ^\alpha R) \circ^\beta (\mu_B \circ^\alpha (\mu_B \circ^\alpha \mu_B) \circ^\alpha \mu_B)$$
Let \((\mu_A, \gamma_A)\) be an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of an LA-ring \(R\) and \(x, y, z \in R\). Thus

\[
\max\{\mu_A((xy)z), \alpha\} \geq \min\{\mu_A(xy), \beta\} \geq \min\{\mu_A(y), \beta\},
\]

and

\[
\min\{\gamma_A((xy)z), (1-\alpha)\} \leq \max\{\gamma_A(xy), (1-\beta)\} \leq \max\{\gamma_A(y), (1-\beta)\}.
\]

Hence \(A\) is an intuitionistic fuzzy interior ideal with thresholds \((\alpha, \beta)\) of \(R\).

**Proposition 2.** Let \((\mu_A, \gamma_A)\) be an IFS of an LA-ring \(R\) with left identity \(e\). Then \(A\) is an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \(R\) if and only if \(A\) is an intuitionistic fuzzy interior ideal with thresholds \((\alpha, \beta)\) of \(R\).

**Proof.** Suppose that \(A = (\mu_A, \gamma_A)\) is an intuitionistic fuzzy interior ideal with thresholds \((\alpha, \beta)\) of an LA-ring \(R\) and \(x, y \in R\). Thus

\[
\max\{\mu_A(xy), \alpha\} = \max\{\mu_A((ex)y), \alpha\} = \min\{\mu_A(x), \beta\}
\]

and

\[
\min\{\gamma_A(xy), (1-\alpha)\} = \min\{\gamma_A((ex)y), (1-\alpha)\} \leq \max\{\gamma_A(x), (1-\beta)\}.
\]

So \(A\) is an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \(R\). Therefore \(A\) is an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \(R\) by the Lemma 5. Converse is true by the Lemma 11.

**Lemma 12.** Every intuitionistic fuzzy left (resp. right, two-sided) ideal with thresholds \((\alpha, \beta)\) of an LA-ring \(R\) is an intuitionistic fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of \(R\). The converse is not true in general.

**Proof.** Assume that \(A = (\mu_A, \gamma_A)\) is an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of an LA-ring \(R\) and \(x, y, z \in R\). Thus

\[
\max\{\mu_A((xy)z), \alpha\} \geq \min\{\mu_A(xy), \beta\} \geq \min\{\mu_A(x), \beta\}
\]

and

\[
\min\{\gamma_A((xy)z), (1-\alpha)\} \leq \max\{\gamma_A(xy), (1-\beta)\} \leq \max\{\gamma_A(x), (1-\beta)\}.
\]

This implies that \(\max\{\mu_A((xy)z), \alpha\} \geq \min\{\mu_A(x), \mu_A(z), \beta\}\). Similarly, we have

\[
\min\{\gamma_A((xy)z), (1-\alpha)\} \leq \max\{\gamma_A(x), (1-\beta)\}.
\]

So \(A\) is an intuitionistic fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of \(R\).
Lemma 13. Every intuitionistic fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of an LA-ring \(R\) is an intuitionistic fuzzy generalized bi-ideal with thresholds \((\alpha, \beta)\) of \(R\). The converse is not true in general.

**Proof.** Obvious.

Lemma 14. Every intuitionistic fuzzy left (resp. right, two-sided) ideal with thresholds \((\alpha, \beta)\) of an LA-ring \(R\) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \(R\). The converse is not true in general.

**Proof.** Let \(A = (\mu_A, \gamma_A)\) be an intuitionistic fuzzy left ideal with thresholds \((\alpha, \beta)\) of an LA-ring \(R\). Thus

\[
\begin{align*}
\max\{\mu_A(x), \alpha\} & \geq \min\{(R \circ \mu_A)(x), \beta\} \\
& \geq \min\{(\mu_A \circ R)(x), (R \circ \mu_A)(x), \beta\}
\end{align*}
\]

and

\[
\min\{\gamma_A(x), (1 - \alpha)\} \leq \max\{(R \circ \gamma_A)(x), (1 - \beta)\}
\]

\[
\leq \max\{(\gamma_A \circ R)(x), (R \circ \gamma_A)(x), (1 - \beta)\}.
\]

Hence \(A\) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \(R\).

**Proposition 3.** Every intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of an LA-ring \(R\) is an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \(R\).

**Proof.** Suppose that \(A = (\mu_A, \gamma_A)\) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of an LA-ring \(R\). Since \(\mu_A \circ_\alpha \mu_A \subseteq \mu_A \circ_\alpha R\) and \(\mu_A \circ_\alpha \mu_A \subseteq R \circ_\alpha \mu_A\), this implies that \(\mu_A \circ_\alpha \mu_A \subseteq \mu_A \circ_\alpha R \land R \circ_\alpha \mu_A \subseteq (\mu_A \circ_\alpha \mu_A)\). Similarly we have, \(\gamma_A \circ_\alpha \gamma_A \supseteq \gamma_A \circ_\alpha R \lor R \circ_\alpha \gamma_A \supseteq (\gamma_A \circ_\alpha \gamma_A)\). Therefore \(A\) is an intuitionistic fuzzy LA-subring with thresholds \((\alpha, \beta)\) of \(R\).

**Proposition 4.** Let \(A = (\mu_A, \gamma_A)\) be an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) and \(B = (\mu_B, \gamma_B)\) be an intuitionistic fuzzy left ideal with thresholds \((\alpha, \beta)\) of an LA-ring \(R\), respectively. Then \(A \circ_\alpha B\) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \(R\).

**Proof.** We have to show that \(A \circ_\alpha B\) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of an LA-ring \(R\). Since

\[
\begin{align*}
\max\{(\mu_A \circ_\alpha \mu_B)(x - y), \alpha\} & \geq \min\{(\mu_A \circ_\alpha \mu_B)(x), (\mu_A \circ_\alpha \mu_B)(y), \beta\} \\
& \geq \min\{(\gamma_A \lor_\alpha \gamma_B)(x - y), (1 - \alpha)\}
\end{align*}
\]

and

\[
\min\{(\gamma_A \lor_\alpha \gamma_B)(x), (\gamma_A \lor_\alpha \gamma_B)(y), (1 - \beta)\},
\]

by the Lemma 3 and

\[
((\mu_A \circ_\alpha \mu_B) \circ_\alpha R) \land (R \circ_\alpha (\mu_A \circ_\alpha \mu_B))
\]
\( A \beta \subseteq (\mu_A \circ_\alpha^\beta R) \wedge (R \circ_\alpha^\beta \mu_B) \subseteq (\mu_A)^\beta \wedge (\mu_B)^\beta = \mu_A \wedge^\beta \mu_B \\
\text{and } ((\gamma_A \lor_\alpha^\beta \gamma_B) \circ_\alpha^\beta R) \lor (R \circ_\alpha^\beta (\gamma_A \lor_\alpha^\beta \gamma_B)) \\
\supseteq (\gamma_A \circ_\alpha^\beta R) \lor (R \circ_\alpha^\beta \gamma_B) \supseteq (\gamma_A)^\beta \lor (\gamma_B)^\beta = \gamma_A \lor^\beta \gamma_B.
\]

Thus \( A \wedge^\beta B \) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \( R \).

**Lemma 15.** Let \( R \) be an LA-ring with left identity \( e \), such that \((xe)R = xR \) for all \( x \in R \). Then every intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \( R \) is an intuitionistic fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of \( R \).

**Proof.** Assume that \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of an LA-ring \( R \). This implies that \( A \) is an intuitionistic fuzzy LA-subring of thresholds \((\alpha, \beta)\) of \( R \). We have to show that \((\mu_A \circ_\alpha^\beta R) \circ_\alpha^\beta \mu_A \subseteq (\mu_A)^\alpha \) and \((\gamma_A \circ_\alpha^\beta R) \circ_\alpha^\beta \gamma_A \supseteq (\gamma_A)^\alpha \). Now

\[
(\mu_A \circ_\alpha^\beta R) \circ_\alpha^\beta \mu_A \subseteq (R \circ_\alpha^\beta \mu_A) \circ_\alpha^\beta \mu_A \subseteq R \circ_\alpha^\beta \mu_A,
\]

and

\[
(\mu_A \circ_\alpha^\beta R) \circ_\alpha^\beta \mu_A \subseteq (\mu_A \circ_\alpha^\beta e) \circ_\alpha^\beta (R \circ_\alpha^\beta R) \subseteq (\mu_A \circ_\alpha^\beta e) \circ_\alpha^\beta R^\beta = (\mu_A)^\alpha \circ_\alpha^\beta R.
\]

\[
\Rightarrow (\mu_A \circ_\alpha^\beta R) \circ_\alpha^\beta \mu_A \subseteq \mu_A \circ_\alpha^\beta R \land R \circ_\alpha^\beta \mu_A \subseteq (\mu_A)^\beta.
\]

Similarly, we have \((\gamma_A \circ_\alpha^\beta R) \circ_\alpha^\beta \gamma_A \supseteq (\gamma_A)^\beta \). So \( A \) is an intuitionistic fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of \( R \).

**Proposition 5.** If \( A \) and \( B \) are two intuitionistic fuzzy quasi-ideals with thresholds \((\alpha, \beta)\) of an LA-ring \( R \) with left identity \( e \), such that \((xe)R = xR \) for all \( x \in R \), then \( A \circ_\alpha^\beta B \) is an intuitionistic fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of \( R \).

**Proof.** Let \( A \) and \( B \) be two intuitionistic fuzzy quasi-ideals with thresholds \((\alpha, \beta)\) of an LA-ring \( R \), this implies that \( A \) and \( B \) be two intuitionistic fuzzy bi-ideals with thresholds \((\alpha, \beta)\) of \( R \), by the Lemma 15. Then \( A \circ_\alpha^\beta B \) is also an intuitionistic fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of \( R \) by the Lemma 10.

### 3. Regular LA-rings

In this section, we characterize regular LA-rings by the properties of intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds \((\alpha, \beta)\). An intuitionistic fuzzy ideal \( A = (\mu_A, \gamma_A) \) with thresholds \((\alpha, \beta)\) of an LA-ring \( R \) is an intuitionistic fuzzy idempotent with thresholds \((\alpha, \beta)\) of \( R \) if \( A \circ_\alpha^\beta A = A^\beta \).

**Lemma 16.** Every intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of a regular LA-ring \( R \) is an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \( R \).
Proof. Suppose that \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \( R \). Let \( x, y \in R \), this implies that there exists \( a \in R \), such that \( x = (xa)x \). Thus

\[
\max\{\mu_A(xy), \alpha\} = \max\{\mu_A(((xa)x)y), \alpha\} = \max\{\mu_A((yx)(xa)), \alpha\} \\
\geq \min\{\mu_A(yx), \beta\} \geq \min\{\mu_A(y), \beta\}
\]

and \( \min\{\gamma_A(xy), (1 - \alpha)\} = \min\{\gamma_A(((xa)x)y), (1 - \alpha)\} \leq \max\{\gamma_A(yx), (1 - \beta)\} \leq \max\{\gamma_A(y), (1 - \beta)\}. \)

Hence \( A \) is an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \( R \).

Lemma 17. Every intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of a regular LA-ring \( R \) is an intuitionistic fuzzy idempotent with thresholds \((\alpha, \beta)\).

Proof. Assume that \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \( R \) and \( A \circ_\alpha^\beta A \subseteq A \circ_\alpha^\beta A \). We have to show that \( A \circ_\alpha^\beta A \subseteq A \circ_\alpha^\beta A \). Let \( x \in R \), this means that there exists \( a \in R \) such that \( x = (xa)x \). Thus

\[
(\mu_A \circ_\alpha^\beta \mu_A)(x) = \{(\mu_A \circ \mu_A)(x) \land \beta\} \lor \alpha \\
= \{\sum_{i=1}^n \mu_A(a_i) \land \mu_A(b_i)\} \land \beta \lor \alpha \\
\geq \{\mu_A(xa) \land \mu_A(x)\} \land \beta \lor \alpha \\
= (\mu_A(xa) \lor \alpha) \land (\mu_A(x) \lor \alpha) \land (\beta \lor \alpha) \\
\geq (\mu_A(x) \lor \beta) \land (\mu_A(x) \lor \beta) = \mu_A(x) \lor \beta \\
= (\mu_A(x) \lor \beta) \lor \alpha = (\mu_A^\beta(x)) \lor \alpha \\
\Rightarrow (\mu_A^\beta \circ_\alpha^\beta \mu_A) \subseteq (\mu_A^\beta \circ_\alpha^\beta \mu_A).
\]

Similarly, we have \((\gamma_A^\beta \circ_\alpha^\beta \gamma_A)\). Therefore \( A \circ_\alpha^\beta A = A \circ_\alpha^\beta A \).

Remark 5. Every intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of a regular LA-ring \( R \) is an intuitionistic fuzzy idempotent with thresholds \((\alpha, \beta)\).

Proposition 6. Let \( A = (\mu_A, \gamma_A) \) be an IFS of a regular LA-ring \( R \). Then \( A \) is an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \( R \) if and only if \( A \) is an intuitionistic fuzzy interior ideal with thresholds \((\alpha, \beta)\) of \( R \).

Proof. Consider that \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy interior ideal with thresholds \((\alpha, \beta)\) of \( R \). Let \( x, y \in R \), then there exists an element \( a \in R \), such that \( x = (xa)x \). Thus

\[
\max\{\mu_A(xy), \alpha\} = \max\{\mu_A(((xa)x)y), \alpha\} \\
\geq \min\{\mu_A((yx)(xa)), \alpha\} \geq \min\{\mu_A(x), \beta\}
\]

and \( \min\{\gamma_A(xy), (1 - \alpha)\} = \min\{\gamma_A(((xa)x)y), (1 - \alpha)\} \)
Lemma 18. Let we have \((A, \mu)\) be a regular LA-ring. Then \((A \circ R) \land (R \circ A) = A\) for every intuitionistic fuzzy right ideal \(A\) with thresholds \((\alpha, \beta)\) of \(R\). Consequently \(A\) is an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \(R\). So \(A\) is an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \(R\) by the Lemma 16. Converse is true by the Lemma 11.

Remark 6. The concept of intuitionistic fuzzy (interior, two-sided) ideals coincides in regular LA-rings.

Proposition 7. Let \(R\) be a regular LA-ring. Then \((A \circ R) \land (R \circ A) = A\) for every intuitionistic fuzzy right ideal \(A\) with thresholds \((\alpha, \beta)\) of \(R\).

Proof. Suppose that \(A = (\mu, \gamma)\) is an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \(R\). This implies that \((A \circ R) \land (R \circ A) \subseteq A\), because every intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \(R\) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \(R\) by the Lemma 14. Let \(x \in R\), this implies that there exists \(a \in R\), such that \(x = (xa)x\). Thus
\[
(\mu_A \circ_R \gamma)(x) = \{(\mu_A \circ_R \gamma)(xa) \land \beta\} \lor \alpha \\
= \{(\sum_{i=1}^{n} a_i \land \beta) \lor \alpha \}
\]

Similarly, we have \((\mu_A)_A \subseteq R \circ \mu_A\), i.e., \((\mu_A)_A \subseteq (\mu_B)(R \circ \mu_A)\). In same lines, we have \((\gamma_A)_A \supseteq (\gamma_B)(R \circ \mu_A)\). Hence \((A \circ R) \land (R \circ A) = A\).

Lemma 18. Let \(R\) be a regular LA-ring. Then \(A \circ B = A \land B\) for every intuitionistic fuzzy right ideal \(A = (\mu, \gamma)\) with thresholds \((\alpha, \beta)\) and every intuitionistic fuzzy left ideal \(B = (\mu, \gamma)\) with thresholds \((\alpha, \beta)\) of \(R\).

Proof. Since \(A \circ B \subseteq A \land B\), for every intuitionistic fuzzy right ideal \(A = (\mu, \gamma)\) with thresholds \((\alpha, \beta)\) and every intuitionistic fuzzy left ideal \(B = (\mu, \gamma)\) with thresholds \((\alpha, \beta)\) of \(R\) by the Lemma 8. Let \(x \in R\), this means that there exists \(a \in R\) such that \(x = (xa)x\). Thus
\[
(\mu_A \circ_B \mu_B)(x) = \{(\mu_A \circ_B \mu_B)(xa) \land \beta\} \lor \alpha \\
= \{(\sum_{i} a_i \land \beta) \lor \alpha \}
\]
\[ \geq (\mu_A(x) \wedge \beta) \wedge \mu_B(x) \wedge \beta \]
\[ = \mu_A(x) \wedge \mu_B(x) \wedge \beta = (\mu_A \wedge \mu_B)(x) \wedge \beta \]
\[ = \{(\mu_A \wedge \mu_B)(x) \wedge \beta \} \vee \alpha = (\mu_A \wedge \mu_B)(x). \]
\[ \Rightarrow \mu_A \wedge \mu_B \subseteq \mu_A \cap_\alpha \mu_B. \]

Similarly, we have \( \gamma_A \cap_\alpha \gamma_B \geq \gamma_A \cap_\alpha \gamma_B, \) i.e., \( A \cap_\alpha B \subseteq A \cap_\alpha B. \) Therefore \( A \cap_\alpha B = A \cap_\alpha B. \)

**Lemma 19.** Let \( R \) be an LA-ring with left identity \( e. \) Then \( Ra \) is the smallest left ideal of \( R \) containing \( a. \)

**Proof.** Let \( x, y \in Ra \) and \( r \in R. \) This implies that \( x = r_1a \) and \( y = r_2a, \) where \( r_1, r_2 \in R. \) Now

\[ x - y = r_1a - r_2a = (r_1 - r_2)a \in Ra \]
and
\[ rx = r(r_1a) = (er)(r_1a) = (rr_1a)e = ((r_1a)(er))e \]
\[ = ((r_1)e)(ar)e = (e(ar))(r_1e) = (ar)(r_1e) \]
\[ = ((r_1e)r)a \in Ra. \]

Since \( a = ea \in Ra. \) Thus \( Ra \) is a left ideal of \( R \) containing \( a. \) Let \( I \) be another left ideal of \( R \) containing \( a. \) Since \( ra \in I, \) where \( ra \in Ra, \) i.e., \( Ra \subseteq I. \) Hence \( Ra \) is the smallest left ideal of \( R \) containing \( a. \)

**Lemma 20.** Let \( R \) be an LA-ring with left identity \( e. \) Then \( aR \) is a left ideal of \( R. \)

**Proof.** Straight forward.

**Proposition 8.** Let \( R \) be an LA-ring with left identity \( e. \) Then \( aR \cup Ra \) is the smallest right ideal of \( R \) containing \( a. \)

**Proof.** Let \( x, y \in aR \cup Ra, \) this means that \( x, y \in aR \) or \( Ra. \) Since \( aR \) and \( Ra \) both are left ideals of \( R, \) so \( x - y \in aR \) and \( Ra, \) i.e., \( x - y \in aR \cup Ra. \) We have to show that

\[ (aR \cup Ra)R \subseteq (aR \cup Ra). \]

Now
\[ (aR \cup Ra)R = (aR)R \cup (Ra)R = (RR)a \cup (Ra)(eR) \]
\[ \subseteq Ra \cup (Re)(aR) = Ra \cup R(aR) \]
\[ = Ra \cup a(RR) \subseteq Ra \cup aR = aR \cup Ra. \]
\[ \Rightarrow (aR \cup Ra)R \subseteq aR \cup Ra. \]

Since \( a \in Ra, \) i.e., \( a \in aR \cup Ra. \) Let \( I \) be another right ideal of \( R \) containing \( a. \) Since \( aR \in IR \subseteq I \) and \( Ra = (RR)a = (aR)R \in (IR)R \subseteq IR \subseteq I, \) i.e., \( aR \cup Ra \subseteq I. \) Therefore \( aR \cup Ra \) is the smallest right ideal of \( R \) containing \( a. \)
Theorem 8. Let $R$ be an LA-ring with left identity $e$, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

(1) $R$ is a regular.

(2) $A \wedge^\beta B = A \alpha^\beta R$ for every intuitionistic fuzzy right ideal $A$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy left ideal $B$ with thresholds $(\alpha, \beta)$ of $R$.

(3) $C^\alpha = (C \alpha^\beta R) \alpha^\beta C$ for every intuitionistic fuzzy quasi-ideal $C$ with thresholds $(\alpha, \beta)$ of $R$.

Proof. Consider that (1) holds and $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta)$ of $R$. This implies that $(C \alpha^\beta R) \alpha^\beta C \subseteq C^\alpha$, because every intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta)$ of $R$ is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta)$ of $R$ by the Lemma 15. Let $x \in R$, then there exists an element $a \in R$ such that $(x = (xa)a)$. Thus

\[
((\mu_C \alpha^\beta R) \alpha^\beta \mu_C)(x) = \{(\mu_C \alpha^\beta R) \alpha^\beta \mu_C(x) \wedge \beta \} \vee \alpha
\]

\[
\geq \{(\mu_C \alpha^\beta R)\alpha^\beta \mu_C(x) \wedge \beta \} \vee \alpha
\]

\[
= (\mu_C \alpha^\beta R)\alpha^\beta \mu_C(x) \wedge \beta
\]

\[
= (\mu_C \alpha^\beta R)\alpha^\beta \mu_C(x) \wedge \beta
\]

\[
= (\mu_C \alpha^\beta R)\alpha^\beta \mu_C(x) \wedge \beta
\]

\[
\Rightarrow (\mu_C \alpha^\beta R)\alpha^\beta \mu_C \subseteq (\mu_C \alpha^\beta R) \alpha^\beta \mu_C.
\]

Similarly, we have $(C \gamma^\beta C) \alpha^\beta C \wedge^\beta R \alpha^\beta C$. So $C^\alpha = (C \alpha^\beta R) \alpha^\beta C$, i.e., (1) implies (3). Suppose that (3) holds. Let $A$ be an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta)$ and $B$ be an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta)$ of $R$. This implies that $A$ and $B$ be intuitionistic fuzzy quasi-ideals with thresholds $(\alpha, \beta)$ of $R$ by the Lemma 14, so $A \wedge^\beta B$ be also an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta)$ of $R$. Then by our supposition, $A \wedge^\beta B = (A \wedge^\alpha B) \alpha^\beta R \alpha^\beta C \subseteq (A \wedge^\alpha B) \alpha^\beta B \subseteq A \wedge^\beta B$, i.e., $A \wedge^\beta B \subseteq A \wedge^\alpha B \subseteq A \wedge^\beta B$. Since $A \wedge^\alpha B \subseteq A \wedge^\beta B$, so $A \wedge^\alpha B = A \wedge^\alpha B$, i.e., $(3) \Rightarrow (2)$. Assume that (2) is true and $a \in R$. Then $Ra$ is a left ideal of $R$ containing $a$ by the Lemma 10 and $aR \cup Ra$ is a right ideal of $R$ containing $a$ by the Proposition 8. This means that $\chi_{Ra}$ is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta)$ and $\chi_{aRa}$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta)$ of $R$, by the Theorem 2. Then by our assumption $\chi_{Ra} \cap_{\alpha^\beta} R = (\chi_{Ra} \cap_{\alpha^\beta} R) \wedge_{\alpha^\beta} R = (\chi_{Ra} \cap_{\alpha^\beta} R) \wedge_{\alpha^\beta} R$ and $(\chi_{Ra} \cap_{\alpha^\beta} R) \cap_{\alpha^\beta} R = (\chi_{Ra} \cap_{\alpha^\beta} R) \cap_{\alpha^\beta} R$ by the Theorem 1. Thus $(aR \cup Ra) \cap Ra = (aR \cup Ra)Ra$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in (aR \cup Ra)Ra$, so $a \in$
(aR)(Ra) ∪ (Ra)(Ra). This implies that a ∈ (aR)(Ra) or a ∈ (Ra)(Ra). If a ∈ (aR)(Ra), then a = (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ey)a)x) = a((xy)y))a for any x, y ∈ R. If a ∈ (Ra)(Ra), then (Ra)(Ra) = (((R)Ra)Ra) = (aR)Ra, i.e., a ∈ (aR)(Ra). Therefore a is a regular, i.e., R is a regular. So (2) ⇒ (1).

**Theorem 9.** Let R be an LA-ring with left identity e, such that (xe)R = xR for all x ∈ R. Then the following conditions are equivalent.

(1) R is a regular.
(2) $A^\alpha_{\beta} = (A \circ^\beta_{\alpha} R) \circ^\beta_{\alpha} A$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta)$ of R.
(3) $B^\alpha_{\beta} = (B \circ^\beta_{\alpha} R) \circ^\beta_{\alpha} B$ for every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta)$ of R.
(4) $C^\alpha_{\beta} = (C \circ^\beta_{\alpha} R) \circ^\beta_{\alpha} C$ for every intuitionistic fuzzy generalized bi-ideal C with thresholds $(\alpha, \beta)$ of R.

**Proof.** (1) ⇒ (4), is obvious. (4) ⇒ (3), since every intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta)$ of R is an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta)$ of R by the Lemma 13. (3) ⇒ (2), since every intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta)$ of R is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta)$ of R by the Lemma 15. (2) ⇒ (1), by the Theorem 8.

**Theorem 10.** Let R be an LA-ring with left identity e, such that (xe)R = xR for all x ∈ R. Then the following conditions are equivalent.

(1) R is a regular.
(2) $A \wedge^\alpha_{\beta} I = (A \circ^\beta_{\alpha} I) \circ^\beta_{\alpha} A$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy ideal I with thresholds $(\alpha, \beta)$ of R.
(3) $B \wedge^\alpha_{\beta} I = (B \circ^\beta_{\alpha} I) \circ^\beta_{\alpha} B$ for every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy ideal I with thresholds $(\alpha, \beta)$ of R.
(4) $C \wedge^\alpha_{\beta} I = (C \circ^\beta_{\alpha} I) \circ^\beta_{\alpha} C$ for every intuitionistic fuzzy generalized bi-ideal C with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy ideal I with thresholds $(\alpha, \beta)$ of R.

**Proof.** Suppose that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta)$ and I = $(\mu_I, \gamma_I)$ be an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta)$ of R. Now $(C \circ^\beta_{\alpha} I) \circ^\beta_{\alpha} C ⊆ (R \circ^\beta_{\alpha} I) \circ^\beta_{\alpha} R ⊆ I \circ^\beta_{\alpha} R ⊆ I^\beta$ and $(C \circ^\beta_{\alpha} I) \circ^\beta_{\alpha} C ⊆ (C \circ^\beta_{\alpha} R) \circ^\beta_{\alpha} C ⊆ C^\beta$, i.e., $(C \circ^\beta_{\alpha} I) \circ^\beta_{\alpha} C ⊆ C^\beta$ and $I \wedge^\beta = C \wedge^\beta I$. Let $x ∈ R$, this implies that there exists $a ∈ R$ such that $x = (xa)x$. Now $xa = ((xa)x)a = (ax)(xa) = x((ax)a)$. Thus

$$\begin{align*}
((\mu_C \circ^\beta_{\alpha} \mu_I) \circ^\beta_{\alpha} \mu_C)(x) \\
= \{((\mu_C \circ \mu_I) \circ \mu_C)(x) \wedge \beta\} \vee \alpha \\
= \{\bigvee_{x=1}^{\infty} \sum_{i=1}^{\infty} \{((\mu_C \circ \mu_I) \circ \mu_C)(q_i)\} \wedge \beta\} \vee \alpha \\
\geq \{((\mu_C \circ \mu_I) \circ \mu_C)(x) \wedge \beta\} \vee \alpha \\
= ((\mu_C \circ \mu_I) \circ \mu_C)(x) \wedge (\mu_C \circ \mu_I) \circ \mu_C(x) \wedge (\beta \vee \alpha)
\end{align*}$$
Theorem 11. Let $R$ be an LA-ring with left identity $e$, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is a regular.
2. $A \wedge^\beta_\alpha D \subseteq D \circ^\beta_\alpha A$ for every intuitionistic fuzzy quasi-ideal $A$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy right ideal $D$ with thresholds $(\alpha, \beta)$ of $R$.
3. $B \wedge^\beta_\alpha D \subseteq D \circ^\beta_\alpha B$ for every intuitionistic fuzzy bi-ideal $B$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy right ideal $D$ with thresholds $(\alpha, \beta)$ of $R$.
4. $C \wedge^\beta_\alpha D \subseteq D \circ^\beta_\alpha C$ for every intuitionistic fuzzy generalized bi-ideal $C$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy right ideal $D$ with thresholds $(\alpha, \beta)$ of $R$.

Proof. $(1) \Rightarrow (4)$, is obvious. It is clear that $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$. Assume that $(2)$ holds. Then $A \wedge^\beta_\alpha D = (A \circ^\beta_\alpha R) \circ^\beta_\alpha A$, where $R$ itself is an intuitionistic fuzzy two-sided ideal with thresholds $(\alpha, \beta)$ of $R$, i.e., $A^\beta_\alpha = (A \circ^\beta_\alpha R) \circ^\beta_\alpha A$. Hence $R$ is a regular by the Theorem 8, i.e., $(2) \Rightarrow (1)$.

Theorem 12. Let $R$ be an LA-ring with left identity $e$, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is a regular.
2. $A \wedge^\beta_\alpha D \wedge^\beta_\alpha L \subseteq (A \circ^\beta_\alpha D) \circ^\beta_\alpha L$ for every intuitionistic fuzzy quasi-ideal $A$ with thresholds $(\alpha, \beta)$, every intuitionistic fuzzy right ideal $D$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy left ideal $L$ with thresholds $(\alpha, \beta)$ of $R$.
3. $B \wedge^\beta_\alpha D \wedge^\beta_\alpha L \subseteq (B \circ^\beta_\alpha D) \circ^\beta_\alpha L$ for every intuitionistic fuzzy bi-ideal $B$ with thresholds $(\alpha, \beta)$, every intuitionistic fuzzy right ideal $D$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy left ideal $L$ with thresholds $(\alpha, \beta)$ of $R$.
4. $C \wedge^\beta_\alpha D \wedge^\beta_\alpha L \subseteq (C \circ^\beta_\alpha D) \circ^\beta_\alpha L$ for every intuitionistic fuzzy generalized bi-ideal $C$ with thresholds $(\alpha, \beta)$, every intuitionistic fuzzy right ideal $D$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy left ideal $L$ with thresholds $(\alpha, \beta)$ of $R$. 

\[ = ((\mu_C \circ \mu_I) (xa) \vee \alpha) \wedge \mu_C (x) \wedge \beta \]
\[ = (\left\{ \begin{array}{l}
\mu_C (x) \wedge \mu_I ((ax) a) \vee \alpha \wedge \mu_C (x) \wedge \beta \\
\mu_C (x) \wedge (\mu_I ((ax) a) \vee \alpha) \wedge \mu_C (x) \wedge \beta
\end{array} \right.) \vee \alpha \wedge \mu_C (x) \wedge \beta
\]
\[ \geq (\{ \mu_C (x) \wedge \mu_I ((ax) a) \vee \alpha \} \wedge \mu_C (x) \wedge \beta
\]
\[ = (\mu_C (x) \wedge (\mu_I ((ax) a) \vee \alpha) \wedge \mu_C (x) \wedge \beta
\]
\[ \geq (\mu_C (x) \wedge (\mu_I (x) \wedge \beta) \wedge \mu_C (x) \wedge \beta
\]
\[ = (\mu_C (x) \wedge (\mu_I (x) \wedge \beta) \wedge \mu_C (x) \wedge \beta
\]
\[ = (\{ \mu_C \wedge \mu_I (x) \wedge \beta \} \vee \alpha) = (\mu_C \wedge \mu_I (x) \wedge \beta
\]
\[ \Rightarrow \mu_C \wedge \mu_I \mu \leq (\mu_C \circ \mu_I) \circ \mu
\]
Proof. Consider that (1) holds. Let \( C = (\mu_C, \gamma_C) \) be an intuitionistic fuzzy generalized bi-ideal with thresholds \((\alpha, \beta)\), \( L = (\mu_L, \gamma_L) \) be an intuitionistic fuzzy left ideal with thresholds \((\alpha, \beta)\) and \( D = (\mu_D, \gamma_D) \) be an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \( R \). Let \( x \in R \), then there exists an element \( a \in R \) such that \( x = (xa)x \). Now

\[
x = (xa)x, \\
xa = ((xa)x)a = (ax)(xa) = x((ax)a), \\
(ax)a = (a((xa)x))a = (a(ax))(xa)
\]

Therefore,\( x((xa)(ae))a = ((xa)(ae))xa = x(((xa)(ae))a) \) \( x(((nx)(ea))a) = x((nx)(ea))a = x((ae)(xn)) \)

\[
x = x((ae)(xn)) = x(xm). \\
\Rightarrow xa = x((xa)a) = x(x(xm)) = (ex)(x(xm)) = ((xm)x)(xe).
\]

Thus

\[
((\mu_C \circ_\alpha^\beta \mu_D \circ_\alpha^\beta \mu_L)(x) \\
= \{((\mu_C \circ_\alpha^\beta \mu_D \circ_\alpha^\beta \mu_L)(x) \land \beta \} \lor \alpha \}
\]

\[
= \{ \bigvee_{x=\sum_{i=1}^{n} p_i, \bigwedge_{i=1}^{n} \{ (\mu_C \circ_\alpha^\beta \mu_D (p_i) \land \mu_L (q_i) ) \} \land \beta \} \} \lor \alpha \}
\]

\[
\geq \{ (\mu_C \circ_\alpha^\beta \mu_D (xa) \land \mu_L (x) \land \beta \} \lor \alpha \}
\]

\[
= (\mu_C \circ_\alpha^\beta \mu_D (xa) \lor \alpha) \land (\mu_L (x) \lor \alpha) \land (\beta \lor \alpha)
\]

\[
= (\mu_C \circ_\alpha^\beta \mu_D (xa) \lor \alpha) \land (\mu_L (x) \land \beta
\]

\[
= ((\mu_C ((xm)x) \land \mu_D (xe)) \lor \alpha) \land (\mu_L (x) \land \beta
\]

\[
= (\mu_C ((xm)x) \lor \alpha) \land (\mu_D (xe) \lor \alpha) \land (\mu_L (x) \land \beta
\]

\[
= (\mu_C (x) \land \mu_C (x) \land (\mu_D (x) \land \beta) \land (\mu_L (x) \land \beta
\]

\[
= \mu_C (x) \land (\mu_D (x) \land (\mu_L (x) \lor \alpha \land (\mu_C \circ_\alpha^\beta \mu_D \circ_\alpha^\beta \mu_L))(x).
\]

Hence, \( C \land_\alpha^\beta D \land_\alpha^\beta L \subseteq (C \circ_\alpha^\beta D \circ_\alpha^\beta L) \), i.e., \( (1) \Rightarrow (4) \). It is clear that \( (4) \Rightarrow (3) \) and \( (3) \Rightarrow (2) \). Assume that \( (2) \) holds. Then \( A \land_\alpha^\beta R \land_\alpha^\beta L \subseteq (A \circ_\alpha^\beta R \circ_\alpha^\beta L), \) where \( A \) is an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \( R \), i.e., \( A \land_\alpha^\beta L \subseteq A \circ_\alpha^\beta L \). Since \( A \circ_\alpha^\beta L \subseteq A \land_\alpha^\beta L \), thus \( A \circ_\alpha^\beta L = A \land_\alpha^\beta L \). So \( R \) is regular by the Theorem 8, i.e., \( (2) \Rightarrow (1) \).

4. Intra-regular LA-rings

In this section, we characterize intra-regular LA-rings in terms of intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds \((\alpha, \beta)\).
Lemma 21. Every intuitionistic fuzzy left (right) ideal with thresholds \((\alpha, \beta)\) of an intra-regular LA-ring \(R\) is an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \(R\).

Proof. Suppose that \(A = (\mu_A, \gamma_A)\) is an intuitionistic fuzzy left ideal with thresholds \((\alpha, \beta)\) of \(R\). Let \(x, y \in R\), this implies that there exist \(a_i, b_i \in R\), such that \(x = \sum_{i=1}^{n}(a_i x^2)b_i\). Thus

\[
\begin{align*}
\max \{\mu_A(xy), \alpha\} &= \max \{\mu_A(((a_i x^2)b_i)y), \alpha\} \\
&= \max \{\mu_A((y b_i)(a_i x^2)), \alpha\} \\
&\geq \min \{\mu_A(a_i(xx)), \beta\} \geq \min \{\mu_A(xx), \beta\} \\
&\geq \min \{\mu_A(x), \beta\}
\end{align*}
\]

and
\[
\begin{align*}
\min \{\gamma_A(xy), (1-\alpha)\} &= \min \{\gamma_A(((a_i x^2)b_i)y), (1-\alpha)\} \\
&= \min \{\gamma_A((y b_i)(a_i x^2)), (1-\alpha)\} \\
&\leq \max \{\gamma_A(a_i(xx)), (1-\beta)\} \\
&\leq \max \{\gamma_A(xx), (1-\beta)\} = \max \{\gamma_A(x), (1-\beta)\}.
\end{align*}
\]

Hence \(A\) is an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \(R\).

Lemma 22. Let \(R\) be an intra-regular LA-ring with left identity \(e\). Then every intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \(R\) is an intuitionistic fuzzy idempotent with thresholds \((\alpha, \beta)\).

Proof. Assume that \(A = (\mu_A, \gamma_A)\) is an intuitionistic fuzzy ideal with thresholds \((\alpha, \beta)\) of \(R\) and \(A \circ^\beta_A \subseteq A^\beta_A\). Let \(x \in R\), this means that there exist \(a_i, b_i \in R\), such that \(x = \sum_{i=1}^{n}(a_i x^2)b_i\). Now

\[
x = (a_i x^2)b_i = (a_i(xx))b_i = (x(a_i)x)b_i = (x(a_i))(eb_i) = (x(e))((ae)b_i) = (x)(x)e b_i.
\]

Thus

\[
\begin{align*}
(\mu_A \circ^\beta_A \mu_A)(x) &= \{\mu_A \circ \mu_A\}(x) \wedge \beta \vee \alpha \\
&= \{(\mu_A \circ \mu_A)(x) \wedge \beta \} \vee \alpha \\
&\geq \{\mu_A(a_i(x) \wedge \mu_A((xe)b_i)) \wedge \beta \} \vee \alpha \\
&= (\mu_A(a_i(x) \vee \alpha) \wedge (\mu_A ((xe)b_i) \vee \alpha) \wedge (\beta \vee \alpha) \\
&\geq (\mu_A(x) \wedge \beta) \wedge (\mu_A(x) \wedge \beta) \wedge \beta \\
&= (\mu_A(x) \wedge (\mu_A(x) \wedge \beta)) \vee \alpha = (\mu_A(x))_\beta \alpha \\
&\Rightarrow (\mu_A(x))_\beta \alpha \subseteq (\mu_A \circ^\beta_A \mu_A).
\end{align*}
\]

Similarly, we have \((\gamma_A)(x))_\beta \alpha \subseteq (\gamma_A \circ^\beta_A \gamma_A). Therefore \(A^\beta_A = A \circ^\beta_A A\).
Proposition 9. Let $A$ be an IFS of an intra-regular LA-ring $R$ with left identity $e$. Then $A$ is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta)$ of $R$ if and only if $A$ is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta)$ of $R$.

Proof. Consider that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta)$ of $R$. Let $x, y \in R$, then there exist elements $a_i, b_i \in R$, such that $x = \sum_{i=1}^{n} (a_i x^2) b_i$. Thus

$$
\max\{\mu_A(xy), \alpha\} = \max\{\mu_A((a_i x^2)b_i)y), \alpha\} = \max\{\mu_A((yb_i)(a_i x^2)), \alpha\}
$$

$$
= \max\{\mu_A((yb_i)(a_i(xx))), \alpha\} = \max\{\mu_A((yb_i)(x(a_i x))), \alpha\}
$$

$$
= \max\{\mu_A((yx)(b_i(a_i x))), \alpha\} \geq \min\{\mu_A(x), \beta\}.
$$

Similarly, we have $\min\{\gamma_A(xy), (1-\alpha)\} \leq \max\{\gamma_A(x), (1-\beta)\}$, i.e., $A$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta)$ of $R$. Therefore $A$ is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta)$ of $R$ by the Lemma 21. Converse is true by the Lemma 11.

Remark 7. The concept of intuitionistic fuzzy (interior, two-sided) ideals with thresholds $(\alpha, \beta)$ coincides in intra-regular LA-rings with left identity.

Lemma 23. Let $R$ be an intra-regular LA-ring with left identity $e$. Then $B \wedge_{\alpha} A \subseteq A \circ_{\alpha} B$ for every intuitionistic fuzzy left ideal $A = (\mu_A, \gamma_A)$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy right ideal $B = (\mu_B, \gamma_B)$ with thresholds $(\alpha, \beta)$ of $R$.

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta)$ and $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta)$ of $R$. Let $x \in R$, this implies that there exist $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n} (a_i x^2) b_i$. Now

$$
x = (a_i x^2) b_i = (a_i(xx)) b_i = (x(a_i x)) b_i
$$

$$
= (x(a_i x))(eb_i) = (xe)((a_i x)b_i) = (a_i x)((xe)b_i).
$$

Thus

$$
(\mu_A \circ_{\alpha} \mu_B)(x) = \{(\mu_A \circ \mu_B)(x) \wedge \beta \} \lor \alpha
$$

$$
= \{\{\forall x = \sum_{i=1}^{n} p_i \{\wedge_{\alpha} \mu_A (p_i) \wedge \mu_B (q_i)\}\} \wedge \beta \} \lor \alpha
$$

$$
\geq \{\mu_A(a_i x) \wedge \mu_B ((xe)b_i) \} \wedge \beta \lor \alpha
$$

$$
= (\mu_A (a_i x) \lor \alpha) \wedge (\mu_B ((xe)b_i) \lor \alpha) \land (\beta \lor \alpha)
$$

$$
\geq (\mu_A (x) \lor \beta) \wedge (\mu_B (x) \land \beta) \land \beta
$$

$$
= \mu_A (x) \wedge \mu_B (x) \land \beta = \mu_B (x) \wedge \mu_A (x) \land \beta
$$

$$
= \{(\mu_B \wedge \mu_A)(x) \wedge \beta
$$

$$
= \{(\mu_B \wedge \mu_A)(x) \wedge \beta \} \lor \alpha = (\mu_B \wedge_{\alpha} \mu_A)(x).
$$

$$
\Rightarrow \mu_B \wedge_{\alpha} \mu_A \subseteq A \circ_{\alpha} B.
$$

Similarly, we have $\gamma_B \wedge_{\alpha} \gamma_A \geq \gamma_A \circ_{\alpha} \gamma_B$. Hence $B \wedge_{\alpha} A \subseteq A \circ_{\alpha} B$. 
Theorem 13. Let $R$ be an LA-ring with left identity $e$, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

(1) $R$ is an intra-regular.

(2) $B \land_{\alpha} A \subseteq A \lor_{\alpha} B$ for every intuitionistic fuzzy left ideal $A$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy right ideal $B$ with thresholds $(\alpha, \beta)$ of $R$.

**Proof.** (1) $\Rightarrow$ (2), is true by the Lemma 23. Assume that (2) holds and $a \in R$. Then $Ra$ is a left ideal of $R$ containing $a$ by the Lemma 19 and $aR \cup Ra$ is a right ideal of $R$ containing $a$ by the Proposition 8. This means that $\chi_{Ra}$ is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta)$ and $\chi_{aR \cup Ra}$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta)$ of $R$, by the Theorem 2. By our assumption $(\chi_{aR \cup Ra} \land_{\alpha} \chi_{Ra}) \subseteq (\chi_{Ra}) \land_{\alpha} (\chi_{aR \cup Ra})$, i.e., $(\chi_{aR \cup Ra}) \land_{\alpha} \chi_{Ra} \subseteq (\chi_{Ra}) \land_{\alpha} (\chi_{aR \cup Ra})$ by the Theorem 1. Thus $(aR \cup Ra) \land_{\alpha} Ra \subseteq (aR \cup Ra)$.

Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in Ra(aR \cup Ra) = (Ra)(aR) \cup (Ra)(Ra)$. This implies that $a \in (Ra)(aR)$ or $a \in (Ra)(Ra)$. If $a \in (Ra)(aR)$, then

\[
(Ra)(aR) = (Ra)((ea)(RR)) = (Ra)(((RR)(ae)) = (Ra)(((ae)(R))R) = (Ra)(((RR)a)R) = ((Ra)(aR) = ((Ra)a)R = ((Ra)(ea))R = ((Re)(aa))R = (Ra^2)R.
\]

Thus $a \in (Ra^2)R$. If $a \in (Ra)(Ra)$, then obvious $a \in (Ra^2)R$. So $a$ is an intra regular. Therefore $R$ is an intra-regular, i.e., (2) $\Rightarrow$ (1).

Theorem 14. Let $R$ be an LA-ring with left identity $e$, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

(1) $R$ is an intra-regular.

(2) $A \land_{\alpha} I = (A \lor_{\alpha} I) \land_{\alpha} A$ for every intuitionistic fuzzy quasi-ideal $A$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy ideal $I$ with thresholds $(\alpha, \beta)$ of $R$.

(3) $B \land_{\alpha} I = (B \lor_{\alpha} I) \land_{\alpha} B$ for every intuitionistic fuzzy bi-ideal $B$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy ideal $I$ with thresholds $(\alpha, \beta)$ of $R$.

(4) $C \land_{\alpha} I = (C \lor_{\alpha} I) \land_{\alpha} C$ for every intuitionistic fuzzy generalized bi-ideal $C$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy ideal $I$ with thresholds $(\alpha, \beta)$ of $R$.

**Proof.** Consider that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta)$ and $I = (\mu_I, \gamma_I)$ be an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta)$ of $R$. Now $(C \lor_{\alpha} I) \land_{\alpha} C \subseteq (R \land_{\alpha} I) \lor_{\alpha} R \subseteq I \land_{\alpha} R \subseteq I_{\alpha}^\beta$ and $(C \lor_{\alpha} I) \land_{\alpha} C \subseteq (C \lor_{\alpha} I) \land_{\alpha} C \subseteq C_{\alpha}^\beta$, thus $(C \lor_{\alpha} I) \land_{\alpha} C \subseteq C_{\alpha} \land_{\alpha} I_{\alpha}^\beta = C \land_{\alpha} I$. Let $x \in R$, then there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n} (a_i x^2) b_i$. Now

\[
x = (a_i x^2) b_i = (a_i (x x)) b_i = (x (a_i x)) b_i = (b_i (a_i x)) x.
\]

\[
b_i(a_i x) = b_i(a_i ((a_i x^2) b_i)) = b_i((a_i x^2) (a_i b_i)) = b_i((a_i x^2) c_i)
\]

\[
= (a_i x^2) (b_i c_i) = (a_i x^2) d_i = (a_i x^2) (ed_i) = (d_i e) (x^2 a_i)
\]

\[
= m_i(x^2 a_i) = x^2 (m_i a_i) = (xx) l_i = (l_i x) x = (l_i x) (ex)
\]
Thus
\[(\mu C \circ_\alpha^\beta \mu I) \circ_\alpha^\beta \mu C)(x) = \{(\mu C \circ \mu I) \circ_\alpha^\beta \mu C(x) \land \beta \land \alpha \}\lor \alpha \]
\[= \left\{ \left( \bigvee_{x=\sum_{i=1}^n p_i q_i} \left\{ \land_{i=1}^n \{ \mu C \circ \mu I(p_i) \land \mu C(q_i) \} \right\} \right) \land \beta \land \alpha \right\} \lor \alpha \]
\[\geq \left\{ (\mu C \circ \mu I) (b_i(a,x)) \land \mu C(x) \land \beta \lor \alpha \right\} \lor \alpha \]
\[= (\mu C \circ \mu I) (b_i(a,x)) \lor \alpha \land (\mu C(x) \lor \alpha) \land (\beta \lor \alpha) \]
\[= (\mu C \circ \mu I) (b_i(a,x)) \lor \alpha \land \mu C(x) \land \beta \]
\[= \left( \bigvee_{b_i(a,x)=\sum_{i=1}^n m_i} \land_{i=1}^n \{ \mu C(m_i) \land \mu I(m_i) \} \right) \lor \alpha \lor \mu C(x) \land \beta \]
\[\geq \left\{ (\mu C(x) \lor \alpha) \land (\mu I((x)e)i) \lor \alpha \land (\mu C(x) \lor \alpha) \land (\mu C(x) \lor \alpha) \land \mu C(x) \land \beta \lor \alpha \right\} \lor \alpha \]
\[= \{ (\mu C \circ \mu I)(x) \land \beta \lor \alpha \} \lor \alpha \]
\[= (\mu C \circ_\alpha^\beta \mu I)(x) \lor \alpha \]
\[\Rightarrow \mu C \circ_\alpha^\beta \mu I \subseteq (\mu C \circ_\alpha^\beta \mu I) \circ_\alpha^\beta \mu C. \]

Similarly, we have \(\gamma C \lor_\alpha^\beta \gamma I \subseteq (\gamma C \circ_\alpha^\beta \gamma I) \circ_\alpha^\beta \gamma C\). Hence \(C \land_\alpha^\beta I = (C \circ_\alpha^\beta I) \circ_\alpha^\beta C\), i.e., (1) implies (4). It is clear that (4) \(\Rightarrow\) (3) and (3) \(\Rightarrow\) (2). Suppose that (2) holds. Let \(A\) be an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) and \(I\) be an intuitionistic fuzzy two-sided ideal with thresholds \((\alpha, \beta)\) of \(R\). Since every intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \(R\) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \(R\) by the Lemma 14, This implies that \(A\) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \(R\). By our supposition \(A \land_\alpha^\beta I = (A \circ_\alpha^\beta I) \circ_\alpha^\beta A \subseteq (R \circ_\alpha^\beta I) \circ_\alpha^\beta A \subseteq I \circ_\alpha^\beta A\), i.e., \(A \land_\alpha^\beta I \subseteq I \circ_\alpha^\beta A\). So \(R\) is an intra-regular by the Theorem 13, i.e., (2) \(\Rightarrow\) (1).

**Theorem 15.** Let \(R\) be an LA-ring with left identity \(e\), such that \((xe)R = xR\) for all \(x \in R\). Then the following conditions are equivalent.

1. \(R\) is an intra-regular.
2. \(A \land_\alpha^\beta L \subseteq L \circ_\alpha^\beta A\) for every intuitionistic fuzzy quasi-ideal \(A\) with thresholds \((\alpha, \beta)\) and every intuitionistic fuzzy quasi-left ideal \(L\) with thresholds \((\alpha, \beta)\) of \(R\).
3. \(B \land_\alpha^\beta L \subseteq L \circ_\alpha^\beta B\) for every intuitionistic fuzzy bi-ideal \(B\) with thresholds \((\alpha, \beta)\) and every intuitionistic fuzzy left ideal \(L\) with thresholds \((\alpha, \beta)\) of \(R\).
4. \(C \land_\alpha^\beta L \subseteq L \circ_\alpha^\beta C\) for every intuitionistic fuzzy generalized bi-ideal \(C\) with thresholds \((\alpha, \beta)\) and every intuitionistic fuzzy left ideal \(L\) with thresholds \((\alpha, \beta)\) of \(R\).

**Proof.** Suppose that (1) holds. Let \(C = (\mu C, \gamma C)\) be an intuitionistic fuzzy generalized bi-ideal with thresholds \((\alpha, \beta)\) and \(L = (\mu L, \gamma L)\) be an intuitionistic fuzzy left ideal with
thresholds \((\alpha, \beta)\) of \(R\). Let \(x \in R\), this implies that there exist \(a_i, b_i \in R\) such that 
\[ x = \sum_{i=1}^{n} (a_i x^2) b_i. \]
Now \(x = (a_i(x)) b_i = (x(a_i)) b_i = (b_i(a_i)) x\). Thus
\[
(\mu_L \circ_\alpha \mu_C)(x) = \{(\mu_L \circ_\alpha \mu_C)(x) \land \beta \} \lor \alpha
\]
\[
= \{\left(\bigvee_{x=\sum_{i=1}^{n} p_i q_i} \left\{\land_{i=1}^{n} \{\mu_L(p_i) \land \mu_C(q_i)\}\right\} \land \beta \} \lor \alpha
\]
\[
\geq (\mu_L(b_i(a_i)) \land \mu_C(x)) \land \beta \lor \alpha
\]
\[
= (\mu_L(b_i(a_i)) \lor \alpha) \land (\mu_C(x) \lor \alpha) \land (\beta \lor \alpha)
\]
\[
\geq (\mu_L(x) \land \beta) \land \mu_C(x) \land \beta
\]
\[
= (\mu_L(x) \land \beta) \land (\mu_C(x) \lor \beta)
\]
\[
= (\mu_C \land \mu_L)(x) \land \beta
\]
\[
\Rightarrow (\mu_C \land \mu_L)(x) \land \beta \lor \alpha = (\mu_C \land_\alpha \mu_L)(x).
\]

Similarly, we have \(\gamma_C \land_\beta \gamma_L \supseteq \gamma_L \circ_\beta \gamma_C\). Hence \(C \land_\beta \land_\alpha L \subseteq L \circ_\beta \circ_\alpha C\), i.e., \((1)\) implies \((4)\). It is clear that \((4) \Rightarrow (3)\) and \((3) \Rightarrow (2)\). Assume that \((2)\) holds. Let \(A\) be an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) and \(L\) be an intuitionistic fuzzy left ideal with thresholds \((\alpha, \beta)\) of \(R\). Since every intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \(R\) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \(R\), this means that \(A\) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \(R\). By our assumption, \(A \land_\alpha L \subseteq L \land_\beta A\). Hence \(R\) is an intra-regular by the Theorem 13, i.e., \((2) \Rightarrow (1)\).

**Theorem 16.** Let \(R\) be an LA-ring with left identity \(e\), such that \((xe)R = xR\) for all \(x \in R\). Then the following conditions are equivalent.

1. \(R\) is an intra-regular.
2. \(A \land_\beta L \land_\alpha D \subseteq (L \circ_\alpha A) \circ_\beta D\) for every intuitionistic fuzzy quasi-ideal \(A\) with thresholds \((\alpha, \beta)\), every intuitionistic fuzzy left ideal \(L\) with thresholds \((\alpha, \beta)\) and every intuitionistic fuzzy right ideal \(D\) with thresholds \((\alpha, \beta)\) of \(R\).
3. \(B \land_\alpha L \land_\beta D \subseteq (L \circ_\beta B) \circ_\alpha D\) for every intuitionistic fuzzy bi-ideal \(B\) with thresholds \((\alpha, \beta)\), every intuitionistic fuzzy left ideal \(L\) with thresholds \((\alpha, \beta)\) and every intuitionistic fuzzy right ideal \(D\) with thresholds \((\alpha, \beta)\) of \(R\).
4. \(C \land_\beta L \land_\alpha D \subseteq (L \circ_\alpha C) \circ_\beta D\) for every intuitionistic fuzzy generalized bi-ideal \(C\) with thresholds \((\alpha, \beta)\), every intuitionistic fuzzy left ideal \(L\) with thresholds \((\alpha, \beta)\) and every intuitionistic fuzzy right ideal \(D\) with thresholds \((\alpha, \beta)\) of \(R\).

**Proof.** Assume that \((1)\) holds. Let \(C = (\mu_C, \gamma_C)\) be an intuitionistic fuzzy generalized bi-ideal with thresholds \((\alpha, \beta)\), \(L = (\mu_L, \gamma_L)\) be an intuitionistic fuzzy left ideal with thresholds \((\alpha, \beta)\) and \(D = (\mu_D, \gamma_D)\) be an intuitionistic fuzzy right ideal with thresholds \((\alpha, \beta)\) of \(R\). Let \(x \in R\), this means that there exist \(a_i, b_i \in R\) such that 
\[ x = \sum_{i=1}^{n} (a_i x^2) b_i. \]
Now
\[ x = (a_i(x)) b_i = (x(a_i)) b_i = (b_i(a_i)) x \]
and
\[ b_i(a_i) = b_i((a_i x^2) b_i) = b_i((a_i x^2)(a_i b_i)) \]
α,β intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds (1)

Let

Theorem 17.

i.e., (1) implies (4)

α,β fuzzy bi-ideal with thresholds (2)

R

Then the following conditions are equivalent.

Suppose that (2) holds. Then

Thus

\begin{align*}
\left( (\mu_L \circ^\beta \mu_C) \circ^\beta \mu_D \right)(x) &= \left( \left( \left( \mu_L \circ^\beta \mu_C \right) \circ^\beta \mu_D \right)(x) \right) \land \beta \right) \lor \alpha \\
&= \left( \left( \left( \mu_L \circ^\beta \mu_C \right) \circ^\beta \mu_D \right)(x) \right) \lor \alpha \\
&\geq \left( \left( \mu_L \circ^\beta \mu_C \right) \circ^\beta \mu_D \right)(x) \lor \beta \right) \lor \alpha \\
&= \left( \left( \mu_L \circ^\beta \mu_C \right) \circ^\beta \mu_D \right)(x) \lor \beta \right) \lor \alpha \\
&\geq \left( \left( \mu_L \circ^\beta \mu_C \right) \circ^\beta \mu_D \right)(x) \lor \beta \right) \lor \alpha \\
&= \left( \left( \mu_L \circ^\beta \mu_C \right) \circ^\beta \mu_D \right)(x) \lor \beta \right) \lor \alpha \\
&\Rightarrow \mu_C \land^\alpha \mu_L \land^\alpha \mu_D \subseteq \left( \mu_L \circ^\beta \mu_C \right) \circ^\beta \mu_D.
\end{align*}

Similarly, we have\( \mu_C \land^\alpha \mu_L \land^\alpha \mu_D \subseteq \left( \mu_L \circ^\beta \mu_C \right) \circ^\beta \mu_D \). Hence C \land^\alpha \mu D \subseteq \left( \mu_L \circ^\beta \mu C \right) \circ^\beta \mu D
given thresholds (1). Since (4) \Rightarrow (3) and (3) \Rightarrow (2). Suppose that (2)
holds. Then

A \land^\alpha \mu R \land^\alpha \mu D \subseteq \left( \mu_L \circ^\beta \mu A \right) \circ^\beta \mu D, \text{ where } A = \text{ an intuitionistic fuzzy left ideal with thresholds } 

\( (\alpha, \beta) \) of \( R \), i.e., \( A \land^\alpha \mu D \subseteq \left( \mu_L \circ^\beta \mu A \right) \circ^\beta \mu D \). Therefore \( R \) is an intra-regular, i.e., (2) \Rightarrow (1).

5. Regular and Intra-regular LA-rings

In this section, we characterize both regular and intra-regular LA-rings in terms of 

intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds (1, 2).

Theorem 17. Let \( R \) be an LA-ring with left identity \( e \), such that \((xe)R = xR \) for all \( x \in R \). Then the following conditions are equivalent.

1. \( R \) is both a regular and an intra-regular.

2. Every intuitionistic fuzzy quasi-ideal with thresholds (1, 2) of \( R \) is an intuitionistic fuzzy idempotent with thresholds (1, 2).

Proof. Suppose that \( R \) is both a regular and an intra-regular. Let \( A = (\mu_A, \gamma_A) \) be

an intuitionistic fuzzy quasi-ideal with thresholds (1, 2) of \( R \). Then \( A \) be an intuitionistic fuzzy bi-ideal with thresholds (1, 2) of \( R \) and \( A \circ^\beta \mu A \subseteq A \circ^\beta \mu A \). Let \( x \in R \), this implies
that there exists \( a \in R \) such that \( x = (xa)x \), and also there exist \( a_i, b_i \in R \) such that
\[
x = \sum_{i=1}^{n} (a_i x^2) b_i.
\]
Now
\[
x = (xa)x
\]
\[
xa = ((a, x^2) b_i) a = (ab_i) (a, x^2) = c_i (a, xx)
\]
\[
= c_i (x(a, x)) = x(c_i (a, x)) = x((ec_i) (a, x))
\]
\[
x((xa_i)(c_i e)) = x((xa_i)d_i) = x((d_i a)i) x
\]
\[
x(l_i x) = l_i (xx) = (el_i)(xx) = (xx)(l_i e)
\]
\[
= (xx)m_i = (m_i x)x.
\]
\[
m_i x = m_i ((a, x^2) b_i) = (a, x^2)(m_i b_i) = (a, (xx)) n_i
\]
\[
= (x(a, x)) n_i = (x(a, x))(en_i) = (xe)((a, x)n_i)
\]
\[
= (xe)((a, x)(en_i)) = (xe)((a, e)(xn_i))
\]
\[
= (xe)(x((a, e)n_i)) = (xe)(xu_i) = (x((x)u_i) = x w_i.
\]
\[
\Rightarrow xa = (m_i x)x = (x w_i)x.
\]
Thus
\[
(\mu_A \circ_\alpha^\beta \mu_A)(x) = \{(\mu_A \circ_\alpha \mu_A)(x) \wedge \beta\} \vee \alpha
\]
\[
= \{(\sum_{i=1}^{n} p_i q_i \{\wedge_{\beta}^{n} \{\mu_A (p_i) \wedge \mu_A (q_i)\}\}) \wedge \beta\} \vee \alpha
\]
\[
\geq \{(\mu_A ((x w_i)x) \wedge \mu_A (x)) \wedge \beta\} \vee \alpha
\]
\[
= (\mu_A ((x w_i)x) \wedge \alpha) \wedge (\mu_A (x) \wedge \alpha) \wedge (\beta \vee \alpha)
\]
\[
\geq (\mu_A (x) \wedge \mu_A (x) \wedge \beta) \wedge \mu_A (x) \wedge \beta
\]
\[
= \mu_A (x) \wedge \beta = (\mu_A (x) \wedge \beta) \vee \alpha = (\mu_A \circ_\alpha^\beta \mu_A)(x).
\]
\[
\Rightarrow (\mu_A \circ_\alpha^\beta \mu_A)(x) \subseteq \mu_A \circ_\alpha^\beta \mu_A.
\]
Similarly, we have \((\gamma_A)\circ_\alpha^\beta \gamma_A\). Hence \(A \circ_\alpha^\beta A\). Conversely, assume that every intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \(R\) is an intuitionistic fuzzy idempotent with thresholds \((\alpha, \beta)\). Let \(a \in R\), then \(R_a\) is a left ideal of \(R\) containing \(a\) by the Lemma 19. This implies that \(R_a\) is a quasi-ideal of \(R\), so \(\chi_{R_a}\) is an intuitionistic fuzzy quasi-ideal with thresholds \((\alpha, \beta)\) of \(R\) by the Theorem 4. By our assumption \((\chi_{R_a})\circ_\alpha^\beta \chi_{R_a}\), i.e., \(R_a = (R_a)(R_a)\). Since \(a \in R_a\), i.e., \(a \in (R_a)(R_a)\). Thus \(a\) is both a regular and an intra-regular by the Theorems 8 and 13, respectively. Hence \(R\) is both a regular and an intra-regular, i.e., \((2) \Rightarrow (1)\).

**Theorem 18.** Let \(R\) be an LA-ring with left identity \(e\), such that \((xe)R = xR\) for all \(x \in R\). Then the following conditions are equivalent.

1. \(R\) is both a regular and an intra-regular.
2. \(A \circ_\alpha^\beta B \subseteq A \circ_\alpha^\beta B\) for all intuitionistic fuzzy quasi-ideals \(A\) and \(B\) with thresholds \((\alpha, \beta)\) of \(R\).
(3) $A^\alpha B \subseteq A^\alpha B$ for every intuitionistic fuzzy quasi-ideal $A$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy bi-ideal $B$ with thresholds $(\alpha, \beta)$ of $R$.

(4) $A \land^\alpha B \subseteq A \circ^\alpha B$ for every intuitionistic fuzzy bi-ideal $A$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy quasi-ideal $B$ with thresholds $(\alpha, \beta)$ of $R$.

(5) $A \land^\alpha B \subseteq A \circ^\alpha B$ for all intuitionistic fuzzy bi-ideals $A$ and $B$ with thresholds $(\alpha, \beta)$ of $R$.

(6) $A \land^\alpha B \subseteq A \circ^\alpha B$ for every intuitionistic fuzzy bi-ideal $A$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy generalized bi-ideal $B$ with thresholds $(\alpha, \beta)$ of $R$.

(7) $A \land^\alpha B \subseteq A \circ^\alpha B$ for every intuitionistic fuzzy generalized bi-ideal $A$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy quasi-ideal $B$ with thresholds $(\alpha, \beta)$ of $R$.

(8) $A \land^\alpha B \subseteq A \circ^\alpha B$ for every intuitionistic fuzzy generalized bi-ideal $A$ with thresholds $(\alpha, \beta)$ and every intuitionistic fuzzy bi-ideal $B$ with thresholds $(\alpha, \beta)$ of $R$.

(9) $A \land^\alpha B \subseteq A \circ^\alpha B$ for all intuitionistic fuzzy generalized bi-ideals $A$ and $B$ with thresholds $(\alpha, \beta)$ of $R$.

Proof. Assume that (1) holds. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy generalized bi-ideals with thresholds $(\alpha, \beta)$ of $R$. Let $x \in R$, then means that there exists an element $a \in R$ such that $x = (xa)x$, and also there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^n (a_i x^2) b_i$. Since $x = (xa)x = ((x a)x)x$ by the Theorem 17. Thus

$$(\mu_A \circ^\alpha B)(x) = \{(\mu_A \circ \mu_B)(x) \land \beta \} \lor \alpha$$

= \{(\mu_A ((x a)x) \land \mu_B (x)) \land \beta \} \lor \alpha

\geq \{(\mu_A ((x w_i)x) \land \mu_B (x)) \land \beta \} \lor \alpha

= (\mu_A ((x w_i)x) \lor \alpha) \land (\mu_B (x) \lor \alpha) \land (\beta \lor \alpha)

\geq (\mu_A \land \mu_B (x) \land \beta = (\mu_A \land \mu_B (x) \land \beta

= (\mu_A \land \mu_B (x) \land \beta \} \lor \alpha

= (\mu_A \land \mu_B (x) \land \beta

\Rightarrow \mu_A \land^\alpha B \subseteq A \circ^\alpha B.$$

Similarly, we have $\gamma_A \land^\alpha B \subseteq \gamma_A \circ^\alpha B$. Therefore $A \land^\alpha B \subseteq A \circ^\alpha B$, i.e., (1) implies (9). It is clear that (9) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (4) $\Rightarrow$ (2) and (9) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (3). Suppose that (2) holds. Let $A$ be an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta)$ and $B$ be an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta)$ of $R$. Since every intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta)$ and intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta)$ of $R$ is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta)$ of $R$ by the Lemma 14. By our supposition, $A \land^\alpha B \subseteq A \circ^\alpha B$. Since $A \circ^\alpha B \subseteq A \land^\alpha B$, so $A \land^\alpha B = A \circ^\alpha B$, i.e., $R$ is a regular. Again by our supposition, $A \land^\alpha B = B \land^\alpha A \subseteq B \circ^\alpha A$, i.e., $R$ is an intra-regular. Therefore $R$ is both a regular and an intra-regular, i.e., (2) $\Rightarrow$ (1). In similar way, we can prove that (3) $\Rightarrow$ (1).
Theorem 19. Let $R$ be an LA-ring with left identity $e$, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is both a regular and an intra-regular.
2. $A \wedge R^\beta B \subseteq (A \circ R^\beta B) \wedge (B \circ A^\beta)$ for every intuitionistic fuzzy right ideal $A$ with thresholds $(\alpha, \beta)$ and $B$ with thresholds $(\alpha, \beta)$ of $R$.
3. $A \wedge R^\beta B \subseteq (A \circ R^\beta B) \wedge (B \circ A^\beta)$ for every intuitionistic fuzzy left ideal $A$ with thresholds $(\alpha, \beta)$ and $B$ with thresholds $(\alpha, \beta)$ of $R$.
4. $A \wedge R^\beta B \subseteq (A \circ R^\beta B) \wedge (B \circ A^\beta)$ for every intuitionistic fuzzy quasi-ideal $A$ with thresholds $(\alpha, \beta)$ and $B$ with thresholds $(\alpha, \beta)$ of $R$.
5. $A \wedge R^\beta B \subseteq (A \circ R^\beta B) \wedge (B \circ A^\beta)$ for every intuitionistic fuzzy generalized bi-ideal $B$ with thresholds $(\alpha, \beta)$ of $R$.

Proof. Consider that (1) holds. Since $A \wedge R^\beta B \subseteq A \circ R^\beta B$ and $A \wedge R^\beta B \subseteq B \circ A^\beta$ for all intuitionistic fuzzy generalized bi-ideals $A$ and $B$ with thresholds $(\alpha, \beta)$ of $R$ by the Theorem 18. Hence $A \wedge R^\beta B \subseteq (A \circ R^\beta B) \wedge (B \circ A^\beta)$, i.e., (1) $\Rightarrow$ (14). It is clear that (14) $\Rightarrow$ (13) $\Rightarrow$ (12) $\Rightarrow$ (9) $\Rightarrow$ (6) $\Rightarrow$ (2), (14) $\Rightarrow$ (11) $\Rightarrow$ (10) $\Rightarrow$ (9), (14) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (6) and (14) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2). Suppose that (2) holds. Let $A$ be an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta)$ and $B$ be an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta)$ of $R$. By our supposition $A \wedge R^\beta B \subseteq (A \circ R^\beta B) \wedge (B \circ A^\beta)$, i.e., $R$ is an intra-regular. Again $A \wedge R^\beta B \subseteq (A \circ R^\beta B) \wedge (B \circ A^\beta) \subseteq A \circ R^\beta B$. Since $A \circ R^\beta B \subseteq A \wedge R^\beta B$, so $A \wedge R^\beta B = A \circ R^\beta B$, i.e., $R$ is a regular. Hence $R$ is both a regular and an intra-regular, i.e., (2) $\Rightarrow$ (1).
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