



## Forcing Independent Domination Number of a Graph

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**Abstract.** In this paper, we obtain the forcing independent domination number of some special graphs. Further, we determine the forcing independent domination number of graphs under some binary operations such join, corona and lexicographic product of two graphs.

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### 1. Introduction

Let  $G = (V(G), E(G))$  be a graph and  $v \in V(G)$ . The *open neighborhood* of  $v$  in  $G$  is the set  $N(v) = \{u \in V(G) : uv \in E(G)\}$  and the *closed neighborhood* of  $v$  is the set  $N[v] = N(v) \cup \{v\}$ . For  $X \subseteq V(G)$ , the *open neighborhood* of  $X$  is the set  $N(X) = \cup_{v \in X} N_G(v)$  and its *closed neighborhood* is the set  $N[X] = N(X) \cup X$ .

A set  $I \subseteq V(G)$  is an *independent set* of  $G$  if  $I \cap N(I) = \emptyset$ . A set  $D \subseteq V(G)$  is a *dominating set* of  $G$  if  $N[D] = V(G)$ . A set  $T \subseteq V(G)$  is an *independent dominating set* of  $G$  if  $T$  is both independent and dominating set. The *independent domination number*  $\gamma_i(G)$  of  $G$  is the minimum cardinality of an independent dominating set. If  $S$  is an independent dominating set with  $|S| = \gamma_i(G)$ , then we call  $S$  a  $\gamma_i$ -set of  $G$ . A maximum independent set ( $\alpha$ -set) is an independent set of largest possible size for a given graph  $G$ . This size, denoted by  $\alpha(G)$ , is called the independence number of  $G$ .

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Let  $I$  be a  $\gamma_i$ -set of a graph  $G$ . A subset  $D$  of  $I$  is said to be a *forcing subset for  $I$*  if  $I$  is the unique  $\gamma_i$ -set containing  $D$ . The *forcing independent domination number of  $I$*  is given by  $f\gamma_i(I) = \min\{|D| : D \text{ is a forcing subset for } I\}$ . The *forcing independent domination number of  $G$*  is given by

$$f\gamma_i(G) = \min\{f\gamma_i(I) : I \text{ is a } \gamma_i\text{-set of } G\}.$$

Let  $B$  be an  $\alpha$ -set of a graph  $G$ . A subset  $P$  of  $B$  is said to be a *forcing subset for  $B$*  if  $B$  is the unique  $\alpha$ -set containing  $P$ . The *forcing independence number of  $B$*  is given by  $f\alpha(B) = \min\{|P| : P \text{ is a forcing subset for } B\}$ . The *forcing independence number of  $G$*  is given by

$$f\alpha(G) = \min\{f\alpha(B) : B \text{ is an } \alpha\text{-set of } G\}.$$

Chartrand et. al [3] initiated the investigation on the relation between forcing and domination concepts in 1997 and used the term "forcing domination number". Independent domination under some binary operations such as corona and composition is studied by Canoy [2]. In 2013, Larson et. al [5] investigated the forcing independence number. In 2018, Canoy et. al [1] investigated the forcing domination number of graphs under some binary operations.

Let  $G$  and  $H$  be two graphs. The *join* of  $G$  and  $H$ , denoted by  $G + H$  is the graph with vertex set  $V(G + H) = V(G) \cup V(H)$  and edge set  $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$ . The *corona*  $G \circ H$  of  $G$  and  $H$  is the graph obtained by taking one copy of  $G$  and  $|V(G)|$  copies of  $H$ , and then forming the join  $\langle\{v}\rangle + H^v = v + H^v$ , where  $H^v$  is a copy of  $H$ , for each  $v \in V(G)$ . The *lexicographic product* (or *composition*)  $G[H]$  of  $G$  and  $H$  is the graph with  $V(G[H]) = V(G) \times V(H)$ , and  $(u, u')(v, v') \in E(G[H])$  if and only if either  $uv \in E(G)$  or  $u = v$  and  $u'v' \in E(H)$ .

Note that for each  $\emptyset \neq C \subseteq V(G) \times V(H)$ , the  $G$ -projection and  $H$ -projection of  $C$  are, respectively, the sets  $C_G = \{x \in V(G) : (x, a) \in C \text{ for some } a \in V(H)\}$  and  $C_H = \{a \in V(H) : (y, a) \in C \text{ for some } y \in V(G)\}$ . Observe that any non-empty subset  $C$  of  $V(G) \times V(H)$  can be written as  $C = \cup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ , where  $S = C_G \subseteq V(G)$  and  $T_x = \{a \in C_H : (x, a) \in C\}$  for each  $x \in S$ .

## 2. Known Results

**Theorem 2.1.** [4] For any graph  $G$ ,  $\lceil \frac{n}{1+\Delta} \rceil \leq \gamma_i(G) \leq n - \Delta$ .

**Theorem 2.2.** [4] The independent domination number of a cycle  $C_n$  with  $n \geq 3$ , a path  $P_n$  with  $n \geq 1$  and complete bipartite graph  $K_{r,s}$  are given by

i.  $\gamma_i(P_n) = \gamma_i(C_n) = \lceil \frac{n}{3} \rceil$ ,

ii.  $\gamma_i(K_{r,s}) = \min\{r, s\}$ .

**Theorem 2.3.** [2] Let  $G$  be a connected graph and  $H$  be any graph. Then  $C \subseteq V(G \circ H)$  is an independent dominating set in  $G \circ H$  if and only if  $C \cap V(G)$  is an independent set in  $G$  and  $C \cap V(v + H^v)$  is an independent dominating set in  $v + H^v$  for every  $v \in V(G)$ .

**Theorem 2.4.** [2] *Let  $G$  be a connected graph of order  $n$  and  $H$  any graph with  $\gamma_i(H) \neq 1$ . If  $C \subseteq V(G \circ H)$  is a minimum independent dominating set in  $G \circ H$ , then  $C \cap V(G)$  is a maximum independent set in  $G$ .*

**Theorem 2.5.** [2] *Let  $G$  and  $H$  be nontrivial connected graphs. A subset  $C = \cup_{x \in S} (\{x\} \times T_x)$  of  $V(G[H])$ , is an independent dominating set in  $G[H]$  if and only if  $S$  is an independent dominating set in  $G$  and  $T_x$  is an independent dominating set in  $H$  for every  $x \in S$ .*

**Corollary 2.6.** [2] *Let  $G$  and  $H$  be nontrivial connected graphs. Then  $\gamma_i(G[H]) = \gamma_i(G)\gamma_i(H)$ .*

**Corollary 2.7.** [2] *Let  $G$  be a connected graph and  $K_n$  the complete graph of order  $n \geq 1$ . Then  $\gamma_i(G[K_n]) = \gamma_i(G)$ .*

### 3. Main Results

The next two results follow directly from the definition of forcing independent domination and Theorem 2.1.

**Remark 3.1.** *Let  $G$  be a graph. Then*

- (i)  $f\gamma_i(G) = 0$  if and only if  $G$  has a unique  $\gamma_i$ -set.
- (ii)  $f\gamma_i(G) = 1$  if and only if  $G$  has at least two  $\gamma_i$ -sets and there exists a vertex which is contained in exactly one  $\gamma_i$ -set of  $G$ .

**Remark 3.2.** *Let  $G$  be any graph of order  $n$ . Then*

$$0 \leq f\gamma_i(G) \leq n - \Delta.$$

**Theorem 3.3.** *Let  $G$  be a connected graph. Then  $f\gamma_i(G) = \gamma_i(G)$  if and only if for every  $\gamma_i$ -set  $S$  of  $G$  and for each  $x \in S$ , there exists  $y_x \in V(G) \setminus S$  such that  $[S \setminus \{x\}] \cup \{y_x\}$  is a  $\gamma_i$ -set of  $G$ .*

*Proof.* Suppose that  $f\gamma_i(G) = \gamma_i(G)$ . Let  $S$  be a  $\gamma_i$ -set of  $G$ . Then, by assumption,  $\gamma_i(G) = |S| = f\gamma_i(G)$ , that is,  $S$  is the only forcing subset for itself. Let  $x \in S$ . Since  $S \setminus \{x\}$  is not a forcing subset for  $S$ , there exists a  $y_x \in V(G) \setminus S$  such that  $[S \setminus \{x\}] \cup \{y_x\}$  is a  $\gamma_i$ -set of  $G$ . Conversely, suppose that every  $\gamma_i$ -set  $S'$  of  $G$  satisfies the given condition. Let  $S$  be a  $\gamma_i$ -set of  $G$  such that  $f\gamma_i(G) = f\gamma_i(S)$ . Suppose further that  $S$  has a forcing subset  $P$  with  $|P| < |S|$ , that is,  $S = P \cup K$ , where  $K = \{x \in S : x \notin P\}$ . Pick  $x \in K$ . By assumption, there exists  $y_x \in V(G) \setminus S$  such that  $[S \setminus \{x\}] \cup \{y_x\} = T$  is a  $\gamma_i$ -set of  $G$ . Hence,  $T = P \cup R$ , where  $R = [K \setminus \{x\}] \cup \{y_x\}$ , is a  $\gamma_i$ -set containing  $P$ , a contradiction. Hence,  $S$  is the only forcing subset for  $S$ . Therefore,  $f\gamma_i(G) = |S| = \gamma_i(G)$ .  $\square$

**Theorem 3.4.** *For any complete graph  $K_n$  with  $n \geq 1$  vertices,*

$$f\gamma_i(K_n) = \begin{cases} 0, & n = 1, \\ 1, & n > 1. \end{cases}$$

*Proof.* Each vertex of  $K_n$  forms a  $\gamma_i$ -set of  $K_n$ , so by Remark 3.1, the result follows.  $\square$

**Theorem 3.5.** For any path  $P_n$  with  $n \geq 1$  vertices,

$$f\gamma_i(P_n) = \begin{cases} 0, & \text{if } n = 1 \text{ or } n \equiv 0(\text{mod } 3), \\ 1, & \text{otherwise.} \end{cases}$$

*Proof.* Suppose that  $P_n = [u_1, u_2, \dots, u_n]$ . By Theorem 2.2(i),  $\gamma_i(P_n) = \lceil \frac{n}{3} \rceil$ . Note that  $P_1 \cong K_1$ . Then  $f\gamma_i(P_1) = 0$  by Theorem 3.4. Next, let  $n \geq 2$  and consider the following cases:

Case 1:  $n \equiv 0(\text{mod } 3)$

Let  $S = \{u_2, u_5, u_8, \dots, u_{n-1}\} = \{u_{3k-1} : k = 1, 2, \dots, \frac{n}{3}\}$ . Clearly,  $S \cap N(S) = \emptyset$  and  $|S| = \lceil \frac{n}{3} \rceil$ . Since  $S$  is the only  $\gamma_i$ -set of  $P_n$ ,  $f\gamma_i(P_n) = 0$  by Remark 3.1(i).

Case 2:  $n \equiv 1(\text{mod } 3)$

Let  $S_1 = \{u_1\} \cup \{u_{3k} : k = 1, 2, \dots, \frac{n-1}{3}\}$ ,  $S_2 = \{u_1\} \cup \{u_{3k+1} : k = 1, 2, \dots, \frac{n-2}{3}\}$  and  $S_{k,p} = \{u_1\} \cup \{u_{3k+1} : k = 1, 2, \dots, \frac{n-4}{3}\} \cup \{u_{3p} : k < p \leq \frac{n-1}{3}\}$ . Then for all  $i \in \{1, 2\}$  and for all  $k, p$  with  $k \in \{1, 2, \dots, \frac{n-4}{3}\}$  and  $k < p \leq \frac{n-1}{3}$ ,  $S_i \cap N(S_i) = \emptyset$  and  $|S_i| = \lceil \frac{n}{3} \rceil$ , that is,  $S_i$  and  $S_{k,p}$  are  $\gamma_i$ -sets of  $P_n$  and  $u_3 \in S_1 \setminus (S_2 \cup S_{k,p})$ . Let  $S$  be a  $\gamma_i$ -set of  $P_n$  such that  $u_2 \in S$ . Then  $u_3 \in S_1 \setminus S$ . Since  $S_1$  is the only  $\gamma_i$ -set containing  $u_3$ , by Remark 3.1(ii),  $f\gamma_i(S_1) = 1 = f\gamma_i(P_n)$ .

Case 3:  $n \equiv 2(\text{mod } 3)$

Then the set  $S_1 = \{u_1, u_4, u_7, \dots, u_{n-1}\} = \{u_{3k+1} : k = 0, 1, \dots, \frac{n-2}{3}\}$  is the only  $\gamma_i$ -set of  $P_n$  that contains  $u_1$ . Since  $S_2 = \{u_2, u_4, u_7, \dots, u_{n-1}\}$  is also a  $\gamma_i$ -set of  $P_n$ , it follows from Remark 3.1(ii) that  $f\gamma_i(S_1) = 1 = f\gamma_i(P_n)$ .  $\square$

**Theorem 3.6.** For any cycle  $C_n$  with  $n \geq 3$  vertices,

$$f\gamma_i(C_n) = \begin{cases} 1, & \text{if } n = 4 \text{ or } n \equiv 0(\text{mod } 3), \\ 2, & \text{otherwise.} \end{cases}$$

*Proof.* Suppose that  $C_n = [u_1, u_2, \dots, u_n, u_1]$ . By Theorem 2.2(i),  $\gamma_i(C_n) = \lceil \frac{n}{3} \rceil$ . If  $n = 4$ , then  $S_1 = \{u_1, u_3\}$  and  $S_2 = \{u_2, u_4\}$  are the only  $\gamma_i$ -sets of  $C_4$ . Since  $S_1$  contains an element which is not in  $S_2$ ,  $f\gamma_i(S_1) = f\gamma_i(C_4) = 1$  by Remark 3.1(ii).

Next, let  $n \geq 3$ , where  $n \neq 4$ , and consider the following cases:

Case 1:  $n \equiv 0(\text{mod } 3)$

Let  $I_1 = \{u_{3k} : k = 1, 2, \dots, \frac{n}{3}\}$ ,  $I_2 = \{u_{3k+1} : k = 0, 1, \dots, \frac{n-3}{3}\}$ , and  $I_3 = \{u_{3k+2} : k = 0, 1, \dots, \frac{n-3}{3}\}$ . Then for all  $j \in \{1, 2, 3\}$ ,  $I_j \cap N(I_j) = \emptyset$  and  $|I_j| = \lceil \frac{n}{3} \rceil$ . Thus,  $I_1, I_2$  and  $I_3$  are the only  $\gamma_i$ -sets of  $C_n$ . Clearly,  $I_1$  contains an element which is not in  $I_2$  and  $I_3$ , by Remark 3.1(ii),  $f\gamma_i(I_1) = f\gamma_i(C_n) = 1$ .

Case 2:  $n \equiv 1 \pmod{3}$

Then clearly,  $S = \{u_1, u_3\} \cup \{u_{3k+2} : k = 1, 2, \dots, \frac{n-4}{3}\}$  is a  $\gamma_i$ -set of  $C_n$ . Clearly,  $S$  contains  $u_1$  and  $u_5$  where  $d(u_1, u_3) = d(u_3, u_5) = 2$ . Replacing any of the vertices  $u_{3k+2}$  ( $k \neq 1$ ) to form another  $\gamma_i$ -set is not possible since  $d(u_5, u_8) = d(u_1, u_{n-2}) = d(u_{3k+2}, u_{3k+5}) = 3$  for all  $k \in \{2, 3, \dots, \frac{n-7}{3}\}$ . Since  $u_1$  is also contained in the  $\gamma_i$ -set  $S' = \{u_1, u_{n-1}\} \cup \{u_{3k} : k = 1, 2, \dots, \frac{n-4}{3}\}$ , no vertex of  $C_n$  is contained in a unique  $\gamma_i$ -set. Thus,  $f\gamma_i(S) \geq 2$ . Clearly,  $\{u_1, u_5\}$  is uniquely contained in  $S$ . Therefore,  $f\gamma_i(S) = 2 = f\gamma_i(C_n)$ .

Case 3:  $n \equiv 2 \pmod{3}$

Suppose that  $n = 5$ . The  $\gamma_i$ -sets of  $C_5$  are  $S_1 = \{u_1, u_3\}$ ,  $S_2 = \{u_1, u_4\}$ ,  $S_3 = \{u_2, u_4\}$ ,  $S_4 = \{u_2, u_5\}$  and  $S_5 = \{u_3, u_5\}$ . Clearly, for each  $u_i \in S_j$  where  $i, j \in \{1, 2, 3, 4, 5\}$ , there exists  $u_k \in V(C_5) \setminus S_j$  such that  $[S_j \setminus \{u_i\}] \cup \{u_k\}$  is a  $\gamma_i$ -set of  $G$ . By Theorem 3.3,  $f\gamma_i(C_5) = 2$ . Now, suppose that  $n > 5$ . Then  $S = \{u_1\} \cup \{u_{3k} : k = 1, 2, \dots, \frac{n-2}{3}\}$  is a  $\gamma_i$ -set of  $C_n$ . Clearly,  $S$  contains  $u_1$  and  $u_3$  where  $d(u_1, u_3) = 2$ . Since  $u_1$  is also contained in the  $\gamma_i$ -set  $S' = \{u_1, u_4\} \cup \{u_{3k} : k = 2, 3, \dots, \frac{n-2}{3}\}$ , no vertex of  $C_n$  is contained in a unique  $\gamma_i$ -set. Thus,  $f\gamma_i(S) \geq 2$ . Since  $\{u_1, u_3\}$  is a forcing subset for  $S$ ,  $f\gamma_i(S) = 2 = f\gamma_i(C_n)$ .  $\square$

**Theorem 3.7.** *Let  $G$  and  $H$  be any graphs. Then  $S_0 \subseteq V(G + H)$  is a  $\gamma_i$ -set of  $G + H$  if and only if one of the following holds:*

- (i)  $S_0$  is a  $\gamma_i$ -set of  $G$  and  $\gamma_i(G) < \gamma_i(H)$
- (ii)  $S_0$  is a  $\gamma_i$ -set of  $H$  and  $\gamma_i(H) < \gamma_i(G)$
- (iii)  $S_0$  is either a  $\gamma_i$ -set of  $G$  or  $H$ , and  $\gamma_i(H) = \gamma_i(G)$ .

In particular,  $\gamma_i(G + H) = \min\{\gamma_i(G), \gamma_i(H)\}$ .

*Proof.* Clearly,  $S \subseteq V(G + H)$  is an independent dominating set of  $G + H$  if and only if either  $S$  is an independent dominating set of  $G$  or  $S$  is an independent dominating set of  $H$ . In particular,  $\gamma_i(G + H) = \min\{\gamma_i(G), \gamma_i(H)\}$ . Hence,  $S_0$  is a  $\gamma_i$ -set of  $G + H$  if and only if one of (i), (ii), and (iii) holds.  $\square$

**Theorem 3.8.** *For any graphs  $G$  and  $H$  with  $\gamma_i(G) = \gamma_i(H)$ ,*

$$f\gamma_i(G + H) = \begin{cases} 1, & \text{if either } G \text{ or } H \text{ has a unique } \gamma_i\text{-set,} \\ \min\{f\gamma_i(G), f\gamma_i(H)\}, & \text{otherwise.} \end{cases}$$

*Proof.* By Theorem 3.7,  $\gamma_i(G + H) = \gamma_i(G) = \gamma_i(H)$ . Suppose that either  $G$  or  $H$  has a unique  $\gamma_i$ -set. W.l.o.g., suppose that  $G$  has a unique  $\gamma_i$ -set, say  $S$ . Then by Corollary 3.7,  $S$  and the  $\gamma_i$ -sets of  $H$  are  $\gamma_i$ -sets of  $G + H$ . Clearly, for any  $x \in S$ ,  $\{x\}$  is uniquely contained in  $S$  and not in any  $\gamma_i$ -set of  $G + H$ . By Remark 3.1(ii),  $f\gamma_i(S) = 1 = f\gamma_i(G + H)$ .

Suppose that both  $G$  and  $H$  have no unique  $\gamma_i$ -sets. We may assume that  $f\gamma_i(G) \leq f\gamma_i(H)$ . Since every  $\gamma_i$ -set of  $G$  and  $H$  is a  $\gamma_i$ -set of  $G + H$ ,  $f\gamma_i(G) \geq f\gamma_i(G + H)$ . Now, let  $S_0$  be a  $\gamma_i$ -set of  $G + H$  such that  $f\gamma_i(G + H) = f\gamma_i(S_0)$ . If  $S_0 \subseteq V(G)$ , then  $S_0$  is a  $\gamma_i$ -set of  $G$ . Hence,  $f\gamma_i(G + H) = f\gamma_i(S_0) \geq f\gamma_i(G)$ . If  $S_0 \subseteq V(H)$ , then  $S_0$  is a  $\gamma_i$ -set of  $H$ . Hence,  $f\gamma_i(G + H) = f\gamma_i(S_0) \geq f\gamma_i(H) \geq f\gamma_i(G)$ . Hence, in any case,  $f\gamma_i(G + H) = f\gamma_i(G)$ .  $\square$

**Theorem 3.9.** For any graphs  $G$  and  $H$  with  $\gamma_i(G) \neq \gamma_i(H)$ ,

$$f\gamma_i(G + H) = \begin{cases} 0, & \text{if } \gamma_i(G) < \gamma_i(H) \text{ and } G \text{ has a unique } \gamma_i\text{-set or} \\ & \text{if } \gamma_i(H) < \gamma_i(G) \text{ and } H \text{ has a unique } \gamma_i\text{-set,} \\ f\gamma_i(G), & \text{if } \gamma_i(G) < \gamma_i(H) \text{ and } G \text{ has no unique } \gamma_i\text{-sets,} \\ f\gamma_i(H), & \text{if } \gamma_i(H) < \gamma_i(G) \text{ and } H \text{ has no unique } \gamma_i\text{-sets.} \end{cases}$$

*Proof.* Suppose that  $\gamma_i(G) < \gamma_i(H)$ . By Theorem 3.7,  $\gamma_i(G + H) = \gamma_i(G)$ . Suppose that  $G$  has a unique  $\gamma_i$ -set, say  $S$ . Then by Corollary 3.7,  $S$  is the only  $\gamma_i$ -set of  $G + H$ . By Remark 3.1(i),  $f\gamma_i(G + H) = 0$ . Now, suppose that  $G$  has no unique  $\gamma_i$ -sets. By Corollary 3.7, the  $\gamma_i$ -sets of  $G$  are also the  $\gamma_i$ -sets of  $G + H$ . Thus,  $f\gamma_i(G + H) = f\gamma_i(G)$ . Similarly, if  $\gamma_i(H) < \gamma_i(G)$ , then  $f\gamma_i(G + H) = 0$  whenever  $H$  has a unique  $\gamma_i$ -set, and  $f\gamma_i(G + H) = f\gamma_i(H)$  whenever  $H$  has no unique  $\gamma_i$ -sets.  $\square$

The next results are direct consequences of Theorem 3.8 and Theorem 3.9

**Corollary 3.10.** For any graph  $H$ ,

$$f\gamma(K_1 + H) = \begin{cases} 1, & \gamma_i(H) = 1, \\ 0, & \gamma_i(H) > 1. \end{cases}$$

**Corollary 3.11.** For the complete bipartite graph  $K_{n,m}$  such that  $n, m \geq 1$ ,

$$f\gamma_i(K_{n,m}) = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$$

**Corollary 3.12.** For the generalized fan  $F_{n,m} = \overline{K}_n + P_m$ , where  $n \geq 1$  and  $m \geq 2$ ,

$$f\gamma_i(F_{n,m}) = \begin{cases} 0, & \text{if either } n < \lceil \frac{m}{3} \rceil \text{ or } n > \lceil \frac{m}{3} \rceil \text{ with } m \equiv 0 \pmod{3}, \\ 1, & \text{if either } n = \lceil \frac{m}{3} \rceil \text{ or } n > \lceil \frac{m}{3} \rceil \text{ with } m \not\equiv 0 \pmod{3}. \end{cases}$$

**Corollary 3.13.** For the fan  $F_n = K_1 + P_n$ , where  $n \geq 2$ ,

$$f\gamma_i(F_n) = \begin{cases} 0, & n > 3, \\ 1, & n \leq 3. \end{cases}$$

**Corollary 3.14.** For the generalized wheel  $W_{n,m} = \overline{K}_n + C_m$ , where  $n \geq 1$  and  $m \geq 3$ ,

$$f\gamma_i(W_{n,m}) = \begin{cases} 0, & \text{if } n < \lceil \frac{m}{3} \rceil \\ 1, & \text{if either } n = \lceil \frac{m}{3} \rceil \text{ or } n > \lceil \frac{m}{3} \rceil \text{ with } m = 4 \text{ or } m \equiv 0 \pmod{3}, \\ 2, & \text{if } n > \lceil \frac{m}{3} \rceil \text{ with } m \neq 4 \text{ or } m \not\equiv 0 \pmod{3}. \end{cases}$$

**Corollary 3.15.** For the wheel  $W_n = K_1 + C_n$ , where  $n \geq 3$ ,

$$f\gamma_i(W_n) = \begin{cases} 0, & n > 3, \\ 1, & n = 3. \end{cases}$$

The following results are restatements of Theorems 2.3 and 2.4.

**Theorem 3.16.** Let  $G$  be a connected graph of order  $n$  and let  $H$  be any graph. Then  $C \subseteq V(G \circ H)$  is an independent dominating set in  $G \circ H$  if and only if  $C = A \cup (\bigcup_{v \in V(G) \setminus A} S_v)$ ,

where  $A$  is an independent set (may be empty) of  $G$  and  $S_v$  is an independent dominating set of  $H^v$  for all  $v \in V(G) \setminus A$ .

**Theorem 3.17.** Let  $G$  be a connected graph of order  $n$  and let  $H$  be any graph with  $\gamma_i(H) = 1$ . Then  $C$  is a  $\gamma_i$ -set of  $G \circ H$  if and only if  $C = A \cup (\bigcup_{v \in V(G) \setminus A} S_v)$  where

$A$  is an independent set of  $G$  and  $S_v$  is a  $\gamma_i$ -set of  $H^v$  for each  $v \in V(G) \setminus A$ . In particular,  $\gamma_i(G \circ H) = n$ .

**Theorem 3.18.** Let  $G$  be a connected graph of order  $n$  and let  $H$  be any graph with  $\gamma_i(H) \geq 2$ . Then  $C$  is a  $\gamma_i$ -set of  $G \circ H$  if and only if  $C = A \cup (\bigcup_{v \in V(G) \setminus A} S_v)$  where

$A$  is a maximum independent set of  $G$  and  $S_v$  is a  $\gamma_i$ -set of  $H^v$  for each  $v \in V(G) \setminus A$ . In particular,

$$\gamma_i(G \circ H) = \alpha(G) + [n - \alpha(G)]\gamma_i(H).$$

**Theorem 3.19.** Let  $G$  be a connected graph of order  $n$  and let  $H$  be any graph with  $\gamma_i(H) = 1$ . Then

$$f\gamma_i(G \circ H) = \begin{cases} \gamma_i(G), & \text{if } H \text{ has a unique } \gamma_i\text{-set,} \\ n, & \text{otherwise.} \end{cases}$$

*Proof.* Since  $\gamma_i(H) = 1$ , by Theorem 3.17,  $\gamma_i(G \circ H) = n$ . Suppose that  $H$  has a unique  $\gamma_i$ -set, say  $P = \{x\}$ . For each  $v \in V(G)$ , let  $P_v = \{x_v\} \subseteq V(H^v)$  be such that  $\langle P \rangle \cong \langle P_v \rangle$ , where  $\langle P \rangle$  is the subgraph induced by  $P$ . Let  $S = T \cup U$  where  $T$  is a  $\gamma_i$ -set of  $G$  and  $U = \{x_v \in V(H^v) : x_v \in P_v \forall v \in V(G) \setminus T\}$ . Clearly,  $S$  is a  $\gamma_i$ -set of  $G \circ H$ . Since  $H$  has a unique  $\gamma_i$ -set and any vertex  $v \in V(G) \setminus T$  is adjacent to a vertex in  $T$ , no element in  $U$  can be replaced by any vertex in  $v + V(H^v)$  for all  $v \in V(G) \setminus T$  to form another  $\gamma_i$ -set of  $G \circ H$ . Hence,  $T$  is uniquely contained in  $S$ , that is,  $T$  is a forcing subset for  $S$ .

Therefore,  $f\gamma_i(S) \leq |T|$ . Suppose that there exists  $B \subsetneq S$  such that  $f\gamma_i(S) = |B| < |T|$ . Suppose that  $B \cap U \neq \emptyset$ , say  $w \in B \cap U$ . Let  $v \in V(G) \setminus T$  such that  $w = x_v \in V(H^v)$ . Pick  $y \in T \cap N_G(v)$  and let  $T' = T \setminus \{y\}$  and  $U' = U \cup \{x_y\}$ . Then  $S' = T' \cup U'$  is a  $\gamma_i$ -set of  $G \circ H$  with  $S' \neq S$  and  $B \subseteq S'$ , a contradiction. Hence,  $B \subsetneq T$ . Let  $z \in T \setminus B$  and let  $S_z = (T \setminus \{z\}) \cup (U \cup \{x_z\})$  where  $\langle \{x_z\} \rangle \cong \langle x \rangle$  and  $x_z \in V(H^z)$ . Then  $S_z$  is a  $\gamma_i$ -set of  $G \circ H$ ,  $S_z \neq S$  and  $B \subseteq S_z$ . This is a contradiction since  $B$  is a forcing subset for  $S$ . Therefore,  $f\gamma_i(S) = |B| = |T| = \gamma_i(G)$ .

Now, suppose that  $H$  does not have a unique  $\gamma_i$ -set. Let  $C$  be a  $\gamma_i$ -set of  $G \circ H$  and let  $S$  be a forcing subset for  $C$ . By Theorem 3.17,  $C = A \cup (\bigcup_{v \in V \setminus A} S_v)$  where  $A$  is an independent set of  $G$  and  $S_v$  is a  $\gamma_i$ -set of  $H^v$  for each  $v \in V(G) \setminus A$ . Suppose that  $S \neq C$ , say  $w \in C \setminus S$ . Let  $z \in V(G)$  such that  $w \in V(z + H^z)$ . If  $w = z$ , then  $w \in A$ . Let  $A' = A \setminus \{w\}$  and let  $S_w = \{x_w\}$  be a  $\gamma_i$ -set of  $H^w$ . Then by Theorem 3.17,  $C' = A' \cup (\bigcup_{v \in V \setminus A'} S_v)$  is a  $\gamma_i$ -set of  $G \circ H$  with  $C' \neq C$  and  $S \subseteq C'$ . If  $w \neq z$ , then  $S_z = \{w\}$  is a  $\gamma_i$ -set of  $H^z$ . Let  $S_z^* = \{w'\}$  be a  $\gamma_i$ -set of  $H^z$  with  $w \neq w'$ . Then  $C^* = A \cup (\bigcup_{v \in V \setminus (A \cup \{z\})} S_v) \cup S_z^*$  is a  $\gamma_i$ -set of  $G \circ H$  with  $C^* \neq C$  and  $S \subseteq C^*$ . In either case, we get a contradiction. Thus,  $S = C$  and  $f\gamma_i(C) = |C| = n$ . Consequently,  $f\gamma_i(G \circ H) = n$ . □

**Theorem 3.20.** *Let  $G$  be a connected graph of order  $n$  and let  $H$  be any graph with  $\gamma_i(H) \neq 1$ . Then*

$$f\gamma_i(G \circ H) = \begin{cases} f\alpha(G), & \text{if } H \text{ has a unique } \gamma_i\text{-set} \\ [n - \alpha(G)][f\gamma_i(H)], & \text{if } H \text{ has no unique } \gamma_i\text{-sets.} \end{cases}$$

*In particular,  $f\gamma_i(G \circ H) = 0$  if  $G$  has a unique  $\alpha$ -set and  $H$  has a unique  $\gamma_i$ -set.*

*Proof.* Since  $\gamma_i(H) \geq 2$ , by Theorem 3.18,  $\gamma_i(G \circ H) = \alpha(G) + [n - \alpha(G)]\gamma_i(H)$ . Let  $T$  be a maximum independent set of  $G$ . Then  $|T| = \alpha(G)$  and  $|V(G) \setminus T| = n - \alpha(G)$ . Consider the following cases:

Case 1: Suppose that  $H$  has a unique  $\gamma_i$ -set, say  $R$ .

For each  $v \in V(G)$ , let  $R_v \subseteq V(H^v)$  such that  $\langle R_v \rangle \cong \langle R \rangle$ .

Suppose that  $G$  has a unique  $\alpha$ -set, say  $D$ . Then by Theorem 3.18,  $C = D \cup (\bigcup_{v \in V(G) \setminus D} R_v)$  is the unique  $\gamma_i$ -set of  $G \circ H$ . Thus, by Remark 3.1(i),  $f\gamma_i(G \circ H) = 0$ .

Suppose that  $G$  does not have a unique  $\alpha$ -set. Let  $A$  be an  $\alpha$ -set of  $G$  and let  $D_A$  be a forcing subset for  $A$  such that  $f\alpha(G) = f\alpha(A) = |D_A|$ . Let  $C = A \cup (\bigcup_{v \in V(G) \setminus A} R_v)$ . Then, by Theorem 3.18,  $C$  is a  $\gamma_i$ -set of  $G \circ H$ . Since each  $H^v$  has a unique  $\gamma_i$ -set  $R_v$ , it follows that  $D_A$  is a forcing subset for  $C$ . Thus,

$$f\gamma_i(G \circ H) \leq f\gamma_i(C) \leq |D_A| = f\alpha(G).$$

Next, let  $C_0$  be a  $\gamma_i$ -set of  $G \circ H$  such that  $f\gamma_i(G \circ H) = f\gamma_i(C_0)$ . Then  $C_0 = A_0 \cup (\bigcup_{v \in V(G) \setminus A_0} R_v)$ , where  $A_0$  is an  $\alpha$ -set of  $G$ . Let  $S$  be a forcing subset for  $C_0$  such that  $f\gamma_i(C_0) = |S|$ . Since each  $H^v$  has a unique  $\gamma_i$ -set  $R_v$ ,  $S \subseteq A_0$ . Since  $S$  is a forcing subset for  $C_0$ ,  $S$  must be a forcing subset for the  $\alpha$ -set  $A_0$ . Thus,

$$f\gamma_i(G \circ H) = f\gamma_i(C_0) = |S| \geq f\alpha(A_0) \geq f\alpha(G).$$

Therefore,  $f\gamma_i(G \circ H) = f\alpha(G)$ .

Case 2: Suppose that  $H$  does not have a unique  $\gamma_i$ -set.

Let  $Q$  be a  $\gamma_i$ -set of  $H$  with  $f\gamma_i(H) = f\gamma_i(Q)$  and let  $P_Q$  be a forcing subset for  $Q$  with  $f\gamma_i(Q) = |P_Q|$ . For each  $v \in V(G)$ , let  $Q_v \subseteq V(H^v)$  and  $P_{Q_v} \subseteq Q_v$  such that  $\langle Q_v \rangle \cong \langle Q \rangle$  and  $\langle P_{Q_v} \rangle \cong \langle P_Q \rangle$ . Let  $A_Q$  be an  $\alpha$ -set of  $G$ . Then by Theorem 3.18,  $C_Q = A_Q \cup (\bigcup_{v \in V(G) \setminus A_Q} Q_v)$  is a  $\gamma_i$ -set of  $G \circ H$ . Let  $S = \bigcup_{v \in V(G) \setminus A_Q} P_{Q_v}$ . Then  $S$  is a forcing subset for  $C_Q$ . Thus,

$$f\gamma_i(G \circ H) \leq f\gamma_i(C_Q) \leq |S| = [n - \alpha(G)]f\gamma_i(H).$$

Next, let  $C'$  be a  $\gamma_i$ -set of  $G \circ H$  such that  $f\gamma_i(G \circ H) = f\gamma_i(C')$ . Then by Theorem 3.18, let  $C' = A' \cup (\bigcup_{v \in V(G) \setminus A'} R_v)$ , where  $A'$  is an  $\alpha$ -set of  $G$  and  $R_v$  is a  $\gamma_i$ -set of  $H^v$  for

each  $v \in V(G) \setminus A'$ . Let  $S'$  be a forcing subset for  $C'$  such that  $f\gamma_i(C') = |S'|$ . Suppose that there exists  $w \in V(G) \setminus A'$  such that  $S' \cap R_w = S_w$  is not a forcing subset for  $R_w$ . Let  $R'_w$  be a  $\gamma_i$ -set of  $H^w$  with  $R'_w \neq R_w$ . Then  $C'' = A' \cup (\bigcup_{v \in V(G) \setminus (A' \cup \{w\})} R_v) \cup R'_w$  is a  $\gamma_i$ -set of  $G \circ H$  with  $C'' \neq C'$  and  $S' \subseteq C''$ , a contradiction. Thus,  $S_v = S' \cap R_v$  is a forcing subset for  $R_v$  for each  $v \in V(G) \setminus A'$ . Note that  $S_v \neq \emptyset$  for each  $v \in V(G) \setminus A'$ . Let  $S_0 = \bigcup_{v \in V(G) \setminus A'} S_v$ . Then

$$f\gamma_i(G \circ H) = |S'| \geq |S_0| = \sum_{v \in V(G) \setminus A'} |S_v| \geq \sum_{v \in V(G) \setminus A'} f\gamma_i(H) = [n - \alpha(G)]f\gamma_i(H).$$

Therefore,  $f\gamma_i(G \circ H) = [n - \alpha(G)]f\gamma_i(H)$ . □

**Theorem 3.21.** *Let  $G$  and  $H$  be connected graphs. Then*

$$f\gamma_i(G[H]) = \begin{cases} f\gamma_i(G), & \text{if } H \text{ has a unique } \gamma_i\text{-set,} \\ [\gamma_i(G)][f\gamma_i(H)], & \text{if } H \text{ has no unique } \gamma_i\text{-sets.} \end{cases}$$

*In particular,  $f\gamma_i(G[H]) = 0$  if  $G$  and  $H$  have unique  $\gamma_i$ -sets. Also,  $f\gamma_i(G[H]) = \gamma_i(G[H])$  if  $f\gamma_i(H) = \gamma_i(H)$ .*

*Proof.* By Corollary 2.6,  $\gamma_i(G[H]) = \gamma_i(G)\gamma_i(H)$ . Consider the following cases:

Case 1: Suppose that  $H$  has a unique  $\gamma_i$ -set, say  $R$ .

Let  $S$  be a  $\gamma_i$ -set of  $G$  and let  $U$  be a forcing subset for  $S$  such that  $f\gamma_i(G) = f\gamma_i(S) = |U|$ . By Theorem 2.5,  $C = S \times R$  is a  $\gamma_i$ -set of  $G[H]$ . Now, since  $U$  is a forcing subset for  $S$ ,  $U_c = U \times \{c\}$  is a forcing subset for  $C$  for each  $c \in R$ . Hence, for each  $c \in R$ ,

$$f\gamma_i(G[H]) \leq f\gamma_i(C) \leq |U_c| = |U| = f\gamma_i(G).$$

Let  $C_0 = S_0 \times R$  be a  $\gamma_i$ -set of  $G[H]$  such that  $f\gamma_i(G[H]) = f\gamma_i(C_0)$ . By Theorem 2.5,  $S_0$  is a  $\gamma_i$ -set of  $G$ . Let  $Q_0$  be a forcing subset for  $C_0$  with  $f\gamma_i(C_0) = |Q_0|$ . Let  $Q_0 = \cup_{x \in K} [\{x\} \times T_x]$ , where  $K \subseteq S_0$  and  $T_x \subseteq R$  for all  $x \in K$ . Since  $Q_0$  is a forcing subset for  $C_0$ , it follows that  $K$  is a forcing subset for  $S_0$ . Choose any  $x \in K$  and  $a \in T_x$ . Then  $Q_a = K \times \{a\} \subseteq Q_0$ . Thus,

$$f\gamma_i(G[H]) = f\gamma_i(C_0) = |Q_0| \geq |Q_a| = |K| \geq f\gamma_i(S_0) \geq f\gamma_i(G).$$

Therefore,  $f\gamma_i(G[H]) = f\gamma_i(G)$ . In particular, if  $G$  has a unique  $\gamma_i$ -set, then  $f\gamma_i(G[H]) = 0$ .

Case 2: Suppose that  $H$  does not have a unique  $\gamma_i$ -set.

Let  $R_0$  be a  $\gamma_i$ -set of  $H$  and  $T_0$  be a forcing subset for  $R_0$  such that  $f\gamma_i(H) = f\gamma_i(R_0) = |T_0|$ . Let  $S_0$  be a  $\gamma_i$ -set of  $G$ . For each  $x \in S_0$ , let  $T_x = T_0$  and  $R_x = R_0$ . By Theorem 2.5,  $C = \cup_{x \in S_0} [\{x\} \times R_x]$  is a  $\gamma_i$ -set of  $G[H]$ . Then  $C_0 = \cup_{x \in S_0} [\{x\} \times T_x] = S_0 \times T_0$  is a forcing subset for  $C$ . Hence,

$$f\gamma_i(G[H]) \leq f\gamma_i(C_0) \leq |C_0| = |S_0 \times T_0| = \gamma_i(G)f\gamma_i(H).$$

Next, let  $C = \cup_{x \in S} [\{x\} \times T_x]$  be a  $\gamma_i$ -set of  $G[H]$  and let  $D$  be a forcing subset for  $C$  such that  $f\gamma_i(G[H]) = f\gamma_i(C) = |D|$ . Then by Theorem 2.5,  $S$  is a  $\gamma_i$ -set of  $G$  and  $T_x$  is a  $\gamma_i$ -set of  $H$  for each  $x \in S$ . Let  $D = \cup_{x \in K} [\{x\} \times E_x]$  where  $K \subseteq S$  and  $E_x \subseteq T_x$  for each  $x \in S$ . Suppose that  $K \neq S$ , say  $v \in S \setminus K$ . Let  $T'_v$  be a  $\gamma_i$ -set of  $H$  with  $T'_v \neq T_v$ . Then  $C' = \cup_{x \in S \setminus \{v\}} [\{x\} \times T_x] \cup \{v\} \times T'_v$  is a  $\gamma_i$ -set of  $G[H]$  and  $D \subseteq C' \neq C$ , a contradiction. Thus,  $K = S$  and since  $D$  be a forcing subset for  $C$ ,  $E_x$  must be a forcing subset for  $T_x$  for each  $x \in S$ . Hence,

$$f\gamma_i(G[H]) = |D| = \sum_{x \in S} |E_x| \geq \gamma_i(G)f\gamma_i(H).$$

Therefore,  $f\gamma_i(G[H]) = \gamma_i(G)f\gamma_i(H)$ . In particular, if  $f\gamma_i(H) = \gamma_i(H)$ , then  $f\gamma_i(G[H]) = \gamma_i(G)\gamma_i(H) = \gamma_i(G[H])$ .  $\square$

Since the complete graph  $K_n$  has no unique  $\gamma_i$ -sets and  $f\gamma_i(K_n) = 1$  except when  $n = 1$ , the following result is immediate from Theorem 3.21.

**Corollary 3.22.** *Let  $G$  be a connected graph and  $K_n$  the complete graph of order  $n \geq 1$ . Then*

$$f\gamma_i(G[K_n]) = \begin{cases} f\gamma_i(G), & n = 1 \\ \gamma_i(G), & n > 1 \end{cases}$$

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