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On Companion B-algebras

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Abstract. This study introduces the concept of companion B-algebra and establishes some of its properties. Also, this paper introduces the notions of $\odot$-subalgebra and $\odot$-ideal of a companion B-algebra and investigates their relationship. Furthermore, this study establishes some homomorphic properties of $\odot$-subalgebra and $\odot$-ideal.

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1. Introduction

Y. Imai and K. Iséki [7] first initiated the study of BCK-algebras in 1966. In the same year, K. Iséki [6] introduced another class of algebras, called BCI-algebras, which are generalizations of BCK-algebras.

In 1999, J. Neggers and H. S. Kim [9], introduced the notion of d-algebra which is another generalization of BCK-algebra. In 2007, P. J. Allen, H. S. Kim and J. Neggers [3] developed the concept of companion d-algebra to demonstrate considerable parallelism with the theory of BCK-algebras.

In 2002, J. Neggers and H. S. Kim [11] introduced and investigated another class of algebras called B-algebras and described it to have nice properties without being complicated. P. J. Allen, J. Neggers and H. S. Kim [2] proved that every group, under some conditions, determines a B-algebra. Also, M. Kondo and Y. B. Jun [8] proved the converse.

This paper extends the study of B-algebras by defining the concept of companion operation and companion B-algebras and establishing some of its properties. This study also introduces the concepts of subalgebra and ideal of a companion B-algebra and determines some of its homomorphic properties.

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2. Preliminaries

Definition 2.1. [11] A *B-algebra* $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms: for all x, y, z in X,

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = x * (z * (0 * y))$.

Example 2.2. The set of integers together with the usual subtraction and the constant 0 is a B-algebra.

Theorem 2.3. [11] *If $(X, *, 0)$ is a B-algebra, then the following hold: for any $x, y, z \in X$,*

(a) $(x * y) * (0 * y) = x$

(b) $y * z = y * (0 * (0 * z))$

(c) $x * (y * z) = (x * (0 * z)) * y$

(d) $x * y = 0$ *implies* $x = y$

(e) $0 * x = 0 * y$ *implies* $x = y$

(f) $0 * (0 * x) = x$.

Theorem 2.4. [13] *If $(X, *, 0)$ is a B-algebra, then the following hold: for any $x, y, z \in X$, $0 * (x * y) = y * x$.*

Definition 2.5. [11] A B-algebra $(X, *, 0)$ is *commutative* if for any $x, y \in X$, $x * (0 * y) = y * (0 * x)$.

Theorem 2.6. [2] *Let $(X, *, 0)$ be a B-algebra. If $x \circ y = x * (0 * y)$ for all $x, y \in X$, then $(X, \circ)$ is a group.*

Theorem 2.7. [11] *Let $(G, \circ)$ be a group with identity e. If we define $x * y = x \circ y^{-1}$, then $(G, *, e)$ is a B-algebra.*

Definition 2.8. [12] Let $(X, *, 0)$ be a B-algebra. A nonempty subset H of X is called a *B-subalgebra* of X if $x * y \in H$ for any $x, y \in H$.

Definition 2.9. [5] Let $(X, *, 0)$ be a B-algebra. A nonempty subset I of X is called a *B-ideal* of X if $0 \in I$ and $x * y \in I$ and $y \in I$ imply $x \in I$.

Theorem 2.10. [1] *Every subalgebra of a B-algebra X is an ideal.*

Definition 2.11. [10] Let $(A, *_A, 0_A)$ and $(B, *_B, 0_B)$ be B-algebras. The mapping $\phi : A \to B$ is called a *B-homomorphism* if $\phi(x *_A y) = \phi(x) *_B \phi(y)$ for any $x, y \in A$. The *kernel of f* is defined as $Ker f = \{x \in A \ : \ \phi(x) = 0_B\}$.

3. Basic Properties of Companion B-algebra

Definition 3.1. Let $(X, *, 0)$ be a B-algebra. A binary operation $\odot$ on X is called a *subcompanion operation* of X if it satisfies for any $x, y \in X$,

$$((x \odot y) * x) * y = 0 \tag{SC}$$

A subcompanion operation $\odot$ is a *companion operation* of X if for any x, y, $z \in X$, $(z * x) * y = 0$ implies $z * (x \odot y) = 0$. (C)
A *companion B-algebra* $(X, *, \odot, 0)$ is a B-algebra $(X, *, 0)$ with companion operation $\odot$.

Example 3.2. Consider the B-algebra $(X, *, 0)$ with $*$ defined below [11]. Define an operation $\odot$ on X as follows:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

$\odot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	0	5	3	4
2	2	0	1	4	5	3
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

By routine calculations, $(X, *, \odot, 0)$ is a companion B-algebra.

Example 3.3. Consider the B-algebra $X = (\mathbb{Z}, -, 0)$. Then for all $x, y, z \in \mathbb{Z}$, $((x + y) - x) - y = 0$ and if $(z - x) - y = 0$, then $z - (x + y) = (z - x) - y = 0$. Hence, the binary operation "+" is a companion operation of $\mathbb{Z}$. Therefore, $(\mathbb{Z}, -, +, 0)$ is a companion B-algebra.

Theorem 3.4. *Let $(X, *, 0)$ be a B-algebra. If X has a companion operation $\odot$, then it is unique.*

Proof: Assume that the binary operations $\odot_1$ and $\odot_2$ are companion operations on X. Then by (SC) applied on $\odot_1$, for any $x, y \in X$, $((x \odot_1 y) * x) * y = 0$. By (C) applied on $\odot_2$, $(x \odot_1 y) * (x \odot_2 y) = 0$. Then by Theorem 2.3(d), $x \odot_1 y = x \odot_2 y$. Thus, $\odot_1 = \odot_2$ and the companion operation is unique. ∎

Theorem 3.5. *Let $(X, *, \odot, 0)$ be a companion B-algebra. Let $\star$ be a binary operation on X such that for all $x, y, z \in X$, $(x * y) * z = x * (y \star z)$. Then $(X, *, \star, 0)$ is a companion B-algebra and $\star$ is exactly the operation $\odot$.*

Proof: Suppose $x, y, z \in X$. By hypothesis and Definition 2.1(I), $((x \star y) * x) * y = (x \star y) * (x \star y) = 0$. Hence, $\star$ is a subcompanion operation. Now, let $(z * x) * y = 0$. Then by hypothesis, $z * (x \star y) = (z * x) * y = 0$. Thus, $\star$ is a companion operation, which is unique by Theorem 3.4. Therefore, $(X, *, \star, 0)$ is a companion B-algebra. ∎

Example 3.6. Let $X = \{0, 1, 2, 3\}$ be a set with the following table of operations:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

$\odot$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Then $(X, *, 0)$ is a B-algebra [2] and by routine calculations, $(X, *, \odot, 0)$ is a companion B-algebra. If $x = 1$ and $y = 3$, then $((1*3)*1)*3 = 2 \neq 0$. Hence, $*$ is not a subcompanion operation and so not a companion operation.

Remark 3.7. *If* $(X, *, 0)$ *is a* B-*algebra, then* $(X, *, *, 0)$ *is not always a companion* B-*algebra.*

In Example 3.6, the condition $x * y = y * (0 * x)$ does not hold.

Example 3.8. Consider the Klein B-algebra $K_4 = \{0, 1, 2, 3\}$ with the following table of operation [4]:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $x * y = y * (0 * x)$ for any $x, y \in K_4$ and $(K_4, *, *, 0)$ is a companion B-algebra.

The observation in Example 3.8 is generalized in the next theorem.

Theorem 3.9. *Let* $(X, *, 0)$ *be a B-algebra.* X *satisfies* $x * y = y * (0 * x)$ *for any* $x, y \in X$ *if and only if* $(X, *, *, 0)$ *is a companion B-algebra.*

Proof: Suppose $x * y = y * (0 * x)$. By Definition 2.1(III), assumption and Definition 2.1(I), $((x * y) * x) * y = (x * y) * (y * (0 * x)) = (x * y) * (x * y) = 0$. Suppose $(z * x) * y = 0$. By Definition 2.1(III), $z * (y * (0 * x)) = 0$ and by assumption, $z * (x * y) = 0$. Therefore, $(X, *, *, 0)$ is a companion B-algebra.

Conversely, suppose $(X, *, *, 0)$ is a companion B-algebra. By Definition 3.1, $(X, *, 0)$ is a B-algebra. Let $x, y \in X$. Then by (SC), $((x * y) * x) * y = 0$. By Definition 2.1(III), $(x * y) * (y * (0 * x)) = 0$. So, $x * y = y * (0 * x)$ by Theorem 2.3(d). ■

Lemma 3.10. *Let* $(X, *, \odot, 0)$ *be a companion B-algebra. Then for any* $x, y, z \in X$*, the following hold:*

(a) $0 \odot y = y$ *and* $y \odot 0 = y$;

(b) $x \odot y = y * (0 * x)$;

(c) *if* $x * z = y$*, then* $x = z \odot y$;

(d) $\odot$ *is associative in* X;

(e) $x = (x \odot y) \odot (0 * y)$;

(f) *if* $(X, *, 0)$ *is commutative, then* $x \odot y = x * (0 * y)$.

Proof: Let $(X, *, \odot, 0)$ be a companion B-algebra and $x, y, z \in X$.

(a) In (SC), take $x = 0$, that is, $0 = ((0 \odot y) * 0) * y = (0 \odot y) * y$. By Theorem 2.3(d), $0 \odot y = y$. Now, take $x = y$ and $y = 0$ in (SC). Then, $0 = ((y \odot 0) * y) * 0 = (y \odot 0) * y$. Hence, by Theorem 2.3(d), $y \odot 0 = y$.

(b) By (SC), $((x \odot y) * x) * y = 0$. So, by Definition 2.1(III), $(x \odot y) * (y * (0 * x)) = 0$. Thus, by Theorem 2.3(d), $x \odot y = y * (0 * x)$.

(c) If $x * z = y$, then $(x * z) * y = y * y = 0$. By (C), $x * (z \odot y) = 0$. Hence, by Theorem 2.3(d), $x = z \odot y$.

(d) By Lemma 3.10(b), Definition 2.1(III) and Theorem 2.3(c), we have

$$\begin{aligned}(x \odot y) \odot z &= z * (0 * (x \odot y)) \\ &= z * (0 * (y * (0 * x))) \\ &= z * ((0 * x) * y) \\ &= (z * (0 * y)) * (0 * x) \\ &= (y \odot z) * (0 * x) \\ &= x \odot (y \odot z).\end{aligned}$$

Thus, the companion operation $\odot$ is associative.

(e) Note that by Theorem 2.3(f), Definitions 2.1(I), 2.1(III), Theorems 2.3(b) and 2.4, and Lemma 3.10(b),

$$\begin{aligned}x &= 0 * (0 * x) \\ &= ((0 * y) * (0 * y)) * (0 * x) \\ &= (0 * y) * ((0 * x) * (0 * (0 * y))) \\ &= (0 * y) * ((0 * x) * y) \\ &= (0 * y) * (0 * (y * (0 * x))) \\ &= (0 * y) * (0 * (x \odot y)) \\ &= (x \odot y) \odot (0 * y).\end{aligned}$$

(f) Suppose $(X, *, 0)$ is commutative. By Lemma 3.10(b) and Definition 2.5, $x \odot y = y * (0 * x) = x * (0 * y)$. ■

Notice that in Example 3.6, X is commutative and $1 \odot 1 = 2 \neq 0$. Hence, we have found $x = 1 \in X$ such that $x \odot x \neq 0$. Also, $1 \neq 3 = 0 * 1$. Thus, we have the following remark.

Remark 3.11. *If $(X, *, \odot, 0)$ is a companion* B-*algebra, then $(X, \odot, 0)$ is not necessarily a* B-*algebra.*

Proposition 3.12. *Suppose $(X, *, \odot, 0)$ is a companion B-algebra. If $(X, *, 0)$ is a commutative B-algebra and $x = 0 * x$ for any $x \in X$, then $(X, \odot, 0)$ is a B-algebra.*

Proof: Suppose $(X, *, 0)$ is a commutative B-algebra and $x, y \in X$. By Lemma 3.10(b), Definition 2.5 and by assumption, $x \odot y = y * (0 * x) = x * (0 * y) = x * y$. Hence, $(X, \odot, 0) = (X, *, 0)$ is a B-algebra. ■

Example 3.13. Consider the companion B-algebra in Example 3.2 and consider the following table of operation:

$\otimes$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	0	4	5	3
2	2	0	1	5	3	4
3	3	4	4	0	2	1
4	4	3	5	1	0	2
5	5	4	3	2	1	0

Applying Theorem 2.6, we conclude that $(X, \otimes, 0)$ is the group where $x \otimes y = x * (0 * y)$. Note that $\odot \neq \otimes$ since $1 \odot 5 = 4 \neq 3 = 1 \otimes 5$. Thus, by definition of $\otimes$, $x \odot y \neq x \otimes y = x * (0 * y)$. Hence, we cannot apply Theorem 2.6 to immediately conclude that $(X, \odot, 0)$ is a group. However, the following theorem says so.

Theorem 3.14. *Let $(X, *, \odot, 0)$ be a companion B-algebra. Then $(X, \odot, 0)$ is a group.*

Proof: Note that $X \neq \varnothing$ since $0 \in X$. By Lemma 3.10(d) the companion operation $\odot$ is associative. Note that by Lemma 3.10(a), 0 acts as the $\odot$-identity element in $(X, \odot, 0)$. Find y such that $x \odot y = 0$ and $y \odot x = 0$. Suppose $x \odot y = 0$. Then by Lemma 3.10(b), $y * (0 * x) = 0$. So, by Theorem 2.3(d), $y = 0 * x$. Also, suppose $y \odot x = 0$. By Lemma 3.10(b), $x * (0 * y) = 0$. Then by Theorem 2.3(f), $(0 * (0 * x)) * (0 * y) = 0$ and by Theorem 2.3(d), $0 * (0 * x) = 0 * y$. Hence, by Theorem 2.3(e), $y = 0 * x$. Thus, we have found $x^{-1} = y = 0 * x$ in $(X, \odot, 0)$. Therefore, $(X, \odot, 0)$ is a group. ■

Remark 3.15. *For any $x \in X$, $x^{-1} = 0 * x$ is called the inverse of x in the group $(X, \odot, 0)$.*

Theorem 3.16. *Let $(G, \circ)$ be a group with identity e. Then G determines a companion B-algebra $(G, *, \otimes, e)$ where $x * y = x \circ y^{-1}$ and $x \otimes y = y * x^{-1}$.*

Proof: Let $(G, \circ)$ be a group with identity e and $x, y \in G$. Define two binary operations $*$ and $\otimes$ by $x * y = x \circ y^{-1}$ and $x \otimes y = y * x^{-1}$. By Theorem 2.7, $(G, *, e)$ is a B-algebra. Observe that

$$\begin{aligned}
((x \otimes y) * x) * y &= ((x \otimes y) \circ x^{-1}) * y \\
&= ((x \otimes y) \circ x^{-1}) \circ y^{-1} \\
&= (x \otimes y) \circ (x^{-1} \circ y^{-1}) \\
&= (x \otimes y) \circ (y \circ x)^{-1}
\end{aligned}$$

$$\begin{aligned} &= (y * x^{-1}) \circ (y \circ x)^{-1} \\ &= (y \circ (x^{-1})^{-1}) \circ (y \circ x)^{-1} \\ &= (y \circ x) \circ (y \circ x)^{-1} = e. \end{aligned}$$

Hence, $\otimes$ is a subcompanion operation on G. Suppose $(z * x) * y = e$. Then $z \circ (y \circ x)^{-1} = z \circ (x^{-1} \circ y^{-1}) = (z \circ x^{-1}) \circ y^{-1} = (z * x) \circ y^{-1} = (z * x) * y = e$. Observe that $z * (x \otimes y) = z * (y * x^{-1}) = z * (y \circ (x^{-1})^{-1}) = z * (y \circ x) = z \circ (y \circ x)^{-1} = e$. Hence, $\otimes$ is a companion operation on G. Thus, $(G, *, \otimes, e)$ is a companion B-algebra. ■

Consider the B-algebra given in Example 3.2. Note that X is not commutative since there exist $x = 3$ and $y = 4$ such that $3*(0*4) = 2 \neq 1 = 4*(0*3)$. Define $x \circ y = x*(0*y)$. If $x = 3$ and $y = 2$, then $((x \circ y)*x)*y = 2 \neq 0$. Hence, $\circ$ is not a subcompanion operation.

Remark 3.17. *If $(X, *, 0)$ is a* B*-algebra, then $(X, *, \circ, 0)$ is not necessarily a companion* B*-algebra where the operation $\circ$ is defined by $x \circ y = x * (0 * y)$.*

Example 3.18. Let $X = \{0, 1, 2\}$ be a set with the following table of operations, where $x \circ y = x * (0 * y)$:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

$\circ$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

By routine calculations, $(X, *, 0)$ is a commutative B-algebra and $(X, *, \circ, 0)$ is a companion B-algebra.

Theorem 3.19. *If $(X, *, 0)$ is a commutative B-algebra, then $(X, *, \circ, 0)$ is a companion B-algebra where $x \circ y = x * (0 * y)$.*

Proof: Let $(X, *, 0)$ be a commutative B-algebra and $x, y, z \in X$. Define the operation $\circ$ by $x \circ y = x * (0 * y)$. Note that by Definition 2.1(III), the definition of $\circ$, Definitions 2.5 and 2.1(I), we have

$$\begin{aligned} ((x \circ y) * x) * y &= (x \circ y) * (y * (0 * x)) \\ &= (x * (0 * y)) * (y * (0 * x)) \\ &= (y * (0 * x)) * (y * (0 * x)) = 0. \end{aligned}$$

Now, suppose $(z * x) * y = 0$. Then by the definition of $\circ$, Definition 2.5 and Definition 2.1(III), $z * (x \circ y) = z * (x * (0 * y)) = z * (y * (0 * x)) = (z * x) * y = 0$. Hence, $\circ$ is a companion operation. Therefore, $(X, *, \circ, 0)$ is a companion B-algebra. ■

4. On $\odot$-subalgebras

Definition 4.1. Let $(X, *, \odot, 0)$ be a companion B-algebra and I be a nonempty subset of X. Then I is called a *$\odot$-subalgebra* if $x \odot y \in I$ for any $x, y, \in I$.

Example 4.2. In Example 3.2, the set $I_1 = \{0, 1, 2\}$ is a $\odot$-subalgebra of X, while $I_2 = \{3, 4, 5\}$ is not a $\odot$-subalgebra since $3 \odot 4 = 1 \notin I_2$.

Theorem 4.3. *Let $(X, *, \odot, 0)$ be a companion B-algebra. If I is a B-ideal of X, then I is a $\odot$-subalgebra of X.*

Proof: Let $(X, *, \odot, 0)$ be a companion B-algebra and I be a B-ideal of X. Then $I \neq \varnothing$. Let $x, y \in I$. By (SC), $((x \odot y) * x) * y = 0 \in I$. Since I is a B-ideal of X and $y \in I$, $(x \odot y) * x \in I$ by Definition 2.9. Furthermore, since $x \in I$, $x \odot y \in I$. Therefore, I is a $\odot$-subalgebra of X. ■

The converse of Theorem 4.3 need not be true in general. In the companion B-algebra $(\mathbb{Z}, -, +, 0)$ in Example 3.3, $I = \mathbb{Z}^+$ is a $\odot$-subalgebra since for all $x, y \in I$, $x + y \in I$. However, $0 \notin I$, thus, I is not a B-ideal. Hence, we have the following remark.

Remark 4.4. *If I is a $\odot$-subalgebra of a companion* B*-algebra $(X, *, \odot, 0)$, then I is not necessarily a* B*-ideal.*

Let $(\mathbb{Z}, -, +, 0)$ be the companion B-algebra given in Example 3.3. Then $I = \mathbb{Z}^+$ is a $+$-subalgebra . Note that $I_1 = \mathbb{Z}^+ \cup \{0\}$ is a B-ideal since $0 \in I_1$. Now, let $x - y \in I_1$ and $y \in I_1$. Then $x - y \geq 0$ and $y \geq 0$. So $x \geq 0$ and $x \in I_1$.

Theorem 4.5. *Let $(X, *, \odot, 0)$ be a companion B-algebra. Suppose I is a $\odot$-subalgebra and $0 \in I$. Then I is a B-ideal.*

Proof: Suppose I is a $\odot$-subalgebra of X and $0 \in I$. Let $u * v \in I$ and $v \in I$. Then by Theorem 2.3(a) and Lemma 3.10(b), $u = (u * v) * (0 * v) = v \odot (u * v) \in I$. Therefore, I is a B-ideal. ■

The following result follows from Theorem 4.3 and Theorem 2.10.

Corollary 4.6. *Let $(X, *, \odot, 0)$ be a companion B-algebra. If S is a B-subalgebra of X, then S is a $\odot$-subalgebra of X.*

Consider again the companion B-algebra $(\mathbb{Z}, -, +, 0)$ and $+$-subalgebra $I = \mathbb{Z}^+$. Notice that $3 - 5 = -2 \notin I$. Hence, I is not a B-subalgebra. Thus, we have the following remark.

Remark 4.7. *A $\odot$-subalgebra of X is not necessarily a B-subalgebra.*

Example 4.8. Consider Example 3.2 and $\odot$-subalgebra $I = \{0, 1, 2\}$. It is easy to see that I is a B-subalgebra and $0 * a \in I$, for any $a \in I$.

Theorem 4.9. *Let $(X, *, \odot, 0)$ be a companion B-algebra. Suppose I is a $\odot$-subalgebra and $0 * a \in I$, for any $a \in I$. Then I is a B-subalgebra.*

Proof: Suppose I is a $\odot$-subalgebra and $0 * a \in I$, for any $a \in I$. Let $x, y \in I$. Then $0 * y \in I$. By Theorem 2.3(b) and Lemma 3.10(b), $x * y = x * (0 * (0 * y)) = (0 * y) \odot x \in I$. Thus, I is a B-subalgebra. ■

Consider again the companion B-algebra $(\mathbb{Z}, -, +, 0)$ and $+$-subalgebra $I = \mathbb{Z}^+$. Take $a = 2$ and $b = 3 \in I$. Then $b^{-1} = 0 - b = -3$ and $a + b^{-1} = -1 \notin I$. Hence, I is not a subgroup of the group $(\mathbb{Z}, +, 0)$. So, we have the following remark.

Remark 4.10. *If I is a $\odot$-subalgebra, then I is not necessarily a subgroup.*

Consider the companion B-algebra $(\mathbb{Z}, -, +, 0)$, $H_1 = \mathbb{Z}^+$ and $H_2 = \mathbb{Z}^-$. Then H_1 and H_2 are $+$-subalgebras. However, $H_1 \cap H_2 = \varnothing$ and hence, not a $+$-subalgebra. Thus, we have the following remark.

Remark 4.11. *The intersection of $\odot$-subalgebras need not be a $\odot$-subalgebra.*

The proof of the following theorem is straightforward.

Theorem 4.12. *Let $\{I_k : k \in K\}$ be a nonempty collection of $\odot$-subalgebras of a companion B-algebra. If $I = \bigcap_{k \in K} I_k \neq \varnothing$, then I is a $\odot$-subalgebra.*

Consider Example 3.2. Take $A = \{0, 3\}$ and $B = \{0, 4\}$. Then A and B are $\odot$-subalgebras. However, $A \cup B = \{0, 3, 4\}$ is not a $\odot$-subalgebra since $3 \odot 4 = 1 \notin A \cup B$. Hence, we have the following remark.

Remark 4.13. *The union of $\odot$-subalgebras need not be a $\odot$-subalgebra.*

5. On $\odot$-ideals

Definition 5.1. Let $(X, *, \odot, 0)$ be a companion B-algebra. A nonempty subset I of X is called a *$\odot$-ideal* if it satisfies: for any $x, y \in X$,

(i) $0 \in I$ and (ii) $x \odot y \in I$ and $y \in I$ imply $x \in I$.

Example 5.2. In Example 3.2, $\{0, 3\}$ is a $\odot$-ideal of X. But, $I = \{0, 1\}$ is not a $\odot$-ideal since $2 \odot 1 = 0 \in I$ and $1 \in I$ but $2 \notin I$.

Lemma 5.3. *Let $(X, *, \odot, 0)$ be a companion B-algebra and let I be a $\odot$-ideal. If $x \in I$, then $x^{-1} = 0 * x \in I$.*

Proof: By Remark 3.15, $x^{-1} = 0 * x$ is the inverse of x. Thus, $(0 * x) \odot x = 0 \in I$. Since $x \in I$ and I is a $\odot$-ideal, then $0 * x \in I$. ■

Theorem 5.4. *Let $(X, *, \odot, 0)$ be a companion B-algebra. If I is a $\odot$-ideal of X, then I is a $\odot$-subalgebra.*

Proof: Let $x, y \in I$. Note that by Lemma 5.3, $0 * y \in I$. Observe that by Lemma 3.10(b), Theorems 2.4, 2.3(c), Definition 2.1(I) and Theorem 2.3(f),

$$\begin{aligned}(x \odot y) \odot (0 * y) &= (y * (0 * x)) \odot (0 * y)\\ &= (0 * y) * (0 * (y * (0 * x)))\\ &= (0 * y) * ((0 * x) * y)\\ &= ((0 * y) * (0 * y)) * (0 * x)\\ &= 0 * (0 * x) = x.\end{aligned}$$

Since $x \in I$, $0 * y \in I$ and I is a $\odot$-ideal, $x \odot y \in I$. Therefore, I is a $\odot$-subalgebra. ■

The converse of Theorem 5.4 need not be true in general. Note that $I = \mathbb{Z}^+$ is a $\odot$-subalgebra of $(\mathbb{Z}, -, +, 0)$ since for all $x, y \in I, x + y \in I$. However, $0 \notin I$. Hence, I is not a $\odot$-ideal. Thus, we have the following remark.

Remark 5.5. *If I is a $\odot$-subalgebra, then I is not necessarily a $\odot$-ideal.*

Example 5.6. Consider Example 3.6 and $\odot$-subalgebra $I = \{0, 2\}$. Observe that $0 * 0 = 0 \in I$ and $0 * 2 = 2 \in I$, so, $0 * a \in I$, for any $a \in I$. It is clear that I is also a $\odot$-ideal.

Theorem 5.7. *Let $(X, *, \odot, 0)$ be a companion B-algebra. Suppose I is a $\odot$-subalgebra of X and $0 * a \in I$ for any $a \in I$. Then I is a $\odot$-ideal.*

Proof: Suppose I is a $\odot$-subalgebra and $0 * a \in I$ for any $a \in I$. Let $x \in I$. Then $0 * x \in I$. Since I is $\odot$-subalgebra, $0 = x \odot (0 * x) \in I$. Now, suppose $u \odot v \in I$ and $v \in I$. Then $0 * v \in I$. By Lemma 3.10(e), $u = (u \odot v) \odot (0 * v)$. Since I is a $\odot$-subalgebra, $u \in I$. Therefore, I is a $\odot$-ideal. ■

Theorem 5.8. *Let $(G, *, \odot, 0)$ be a companion B-algebra. A nonempty subset I of G is a $\odot$-ideal of G if and only if I is a subgroup of the group $(G, \odot, 0)$.*

Proof: Let I be a $\odot$-ideal and $a, b \in I$. By Lemma 5.3, $b^{-1} = 0 * b \in I$. Because I is also a $\odot$-subalgebra by Theorem 5.4, $a \odot b^{-1} \in I$. Hence, I is a subgroup.

Conversely, suppose I is a subgroup of the group $(G, \odot, 0)$ and $a, b \in I$. Then $a \odot b^{-1} \in I$. Note that $a \odot a^{-1} = 0$. So, $0 \in I$. Suppose $x \odot y \in I$ and $y \in I$. Then by Lemma 3.10(e), $x = (x \odot y) \odot (0 * y) = (x \odot y) \odot y^{-1} \in I$. Thus, I is a $\odot$-ideal. ■

The following corollary follows from Theorem 5.8 and 5.4.

Corollary 5.9. *Let $(G, *, \odot, 0)$ be a companion B-algebra. If I is a subgroup of the group $(G, \odot, 0)$, then I is a $\odot$-subalgebra.*

The following corollary follows from Theorem 5.8.

Corollary 5.10. *Let $\{I_k : k \in K\}$ be a nonempty collection of $\odot$-ideals of a companion B-algebra. If $I = \bigcap_{k \in K} I_k \neq \varnothing$, then I is a $\odot$-ideal.*

Observe that in Example 3.2, $I_1 = \{0, 3\}$ and $I_2 = \{0, 4\}$ are $\odot$-ideals. But their union, $I = I_1 \cup I_2 = \{0, 3, 4\}$ is not a $\odot$-ideal because $1 \odot 4 = 3 \in I$ and $4 \in I$ but $1 \notin I$. Thus, we have the following remark.

Remark 5.11. *The union of $\odot$-ideals need not be a $\odot$-ideal.*

6. On Companion-*B*-homomorphisms

Definition 6.1. Let $(X, *_X, \odot_X, 0_X)$ and $(Y, *_Y, \odot_Y, 0_Y)$ be companion B-algebras. A map $f : X \to Y$ is called a *companion-B-homomorphism* if for any $a, b \in X$,

$$f(a *_X b) = f(a) *_Y f(b) \text{ and } f(a \odot_X b) = f(a) \odot_Y f(b).$$

Example 6.2. Let $m \in \mathbb{Z}$ be fixed. The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = mx$, $x \in \mathbb{Z}$, is a companion-B-homomorphism.

Remark 6.3. *A companion* B*-homomorphism is a* B*-homomorphism and a group homomorphism.*

Example 6.4. Consider the companion B-algebra $(X, *_1, \odot_1, 0)$ in Example 3.6 and $(Y, *_2, \odot_2, 0)$ in Example 3.8 where $\odot_2 = *_2$. Let $f : X \to Y$ and $f(x) = \begin{cases} 0, & \text{if } x = 0, 2, \\ 3, & \text{if } x = 1, 3. \end{cases}$
Then f is a companion-B-homomorphism.

Theorem 6.5. *Suppose $f : X \to Y$ is a companion B-homomorphism. Then $Kerf$ is a $\odot$-subalgebra of X.*

Proof: Note that by Remark 6.3, $Kerf$ is a subgroup of X. Thus, by Corollary 5.9, $Kerf$ is also a $\odot$-subalgebra. ■

The proof of the following theorem is straightforward.

Theorem 6.6. *Suppose $f : X \to Y$ is a companion* B*-homomorphism. If I is a $\odot$-subalgebra of X, then $f(I)$ is a $\odot$-subalgebra of Y.*

Theorem 6.7. *Suppose $f : X \to Y$ is a companion B-epimorphism and B is a $\odot$-subalgebra of Y. Then $f^{-1}(B)$ is a $\odot$-subalgebra of X.*

Proof: Let $B \subseteq Y$ be a $\odot$-subalgebra of Y. Since $B \neq \varnothing$ and f is onto, there exist $a \in B$ and $x \in X$ such that $f(x) = a$. Hence, $x \in f^{-1}(B)$. So, $f^{-1}(B) \neq \varnothing$. Note that $f^{-1}(B) = \{a \in X : f(a) \in B\} \subseteq X$. Now, let $x, y \in f^{-1}(B)$. Then $f(x), f(y) \in B$. Because B is a $\odot$-subalgebra, $f(x \odot y) = f(x) \odot f(y) \in B$. Hence, $x \odot y \in f^{-1}(B)$. Therefore, $f^{-1}(B)$ is a $\odot$-subalgebra of X. ■

By Theorem 5.8, a $\odot$-ideal is equivalent to a subgroup of $(X, \odot)$. Thus, the following corollary holds:

Corollary 6.8. *Suppose $f : X \to Y$ is a companion B-homomorphism.*

(i) *If I is a $\odot$-ideal of X, then $f(I)$ is a $\odot$-ideal of Y.*

(i) *If $B \subseteq Y$ is a $\odot$-ideal of Y, then $f^{-1}(B)$ is a $\odot$-ideal of X.*

(iii) *$Kerf$ is a $\odot$-ideal of X.*

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