EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 12, No. 4, 2019, 1497-1507
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

The Dual B-Algebra

Katrina E. Belleza[1,*], Jocelyn P. Vilela[2]

[1] *Department of Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines*
[2] *Department of Mathematics and Statistics, College of Science and Mathematics, Center of Graph Theory, Algebra and Analysis, Premier Research Institute of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. This paper introduces and characterizes the notion of a dual B-algebra. Moreover, this study investigates the relationship between a dual B-algebra and a BCK-algebra. Commutativity of a dual B-algebra is also discussed and its relation to some algebras such as CI-algebra and dual BCI-algebra is examined.

2010 Mathematics Subject Classifications: 06F35, 47L45, 08C05
Key Words and Phrases: B-algebra, dual B-algebra, dual algebra

1. Introduction

In 2002, J.Neggers and H.S. Kim [9] introduced and investigated B-algebras which is related to several classes of algebras such as $BCH/BCI/BCK$-algebras and established that B-algebras are related to groups. In the same year, M.Kondo and Y.B. Jun [4] showed that every B-algebra is group-derived. In 2010, N.O. Al-Shehrie [1] introduced the left-right (resp. right-left) derivation on a B-algebra and some related properties were investigated. In 1996, Y.Imai and K.Iseki [2] introduced two classes of algebras: BCK-algebras and BCI-algebras. It is known that a BCI-algebra is a generalization of a BCK-algebra. In 2007, dual BCK-algebra was introduced by K.H. Kim and Y.H. Yon [3] and some properties were also studied. Moreover, K.H. Kim and Y.H. Yon [3] investigated the relationship between a dual BCK-algebra and an MV-algebra. On the other hand, A. Walendziak [12] defined commutative BE-algebras in 2008 and proved that these are equivalent to the commutative dual BCK-algebras. In 2009, the notions of dual BCI-algebra and CI-algebra were introduced by B.L. Meng [5] together with some of their properties. It is shown that CI-algebra is a generalization of dual $BCK/BCI/BCH$-algebras. In 2013, A.B. Saeid [11] established the relationship between CI-algebra and dual Q-algebra.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v12i4.3526
Email addresses: kebelleza@usc.edu.ph (K. Belleza), jocelyn.vilela@g.msuiit.edu.ph (J. Vilela)

This paper aims to characterize a dual B-algebra and to investigate the relationship between a dual B-algebra and BCK-algebra. Moreover, commutativity of a dual B-algebra will also be considered. Relationships between commutative dual B-algebra and other algebras such as CI-algebra and dual BCI-algebra will be investigated in this paper.

2. Preliminaries

An algebra of type (2,0) is an algebra with a binary operation and a constant element.

Definition 1. [9] A *B-algebra* is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms for all x, y, z in X:

$$\text{(B1) } x * x = 0 \quad \text{(B2) } x * 0 = x \quad \text{(B3) } (x * y) * z = x * [z * (0 * y)]$$

Example 1. [8] Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X; *, 0)$ is a B-algebra.

Definition 2. [6] An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCK-algebra* if for all x, y, z in X, the following hold:

(BCK1) $[(x * y) * (x * z)] * (z * y) = 0$
(BCK2) $[x * (x * y)] * y = 0$
(BCK3) $x * x = 0$
(BCK4) $x * y = 0$ and $y * x = 0$ imply $x = y$
(BCK5) $0 * x = 0$

Lemma 1. [2] *In any BCK-algebra $(X, *, 0)$, the following hold for all x, y, z in X:*

$$\text{(i) } x * 0 = x \quad \text{(ii) } (x * y) * z = (x * z) * y$$

Definition 3. [7] A *Q-algebra* is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms: for all x, y, z in X,

$$\text{(Q1) } x * x = 0 \quad \text{(Q2) } x * 0 = x \quad \text{(Q3) } (x * y) * z = (x * z) * y$$

Definition 4. [11] Let $(X, *, 0)$ be a Q-algebra and a binary operation $\circ$ on X is defined as: $x \circ y = y * x$. Then $(X, \circ, 1)$ is called a *dual Q-algebra*. In fact, its axioms are as follows for all x, y, z in X:

$$\text{(DQ1) } x \circ x = 1 \quad \text{(DQ2) } 1 \circ x = x \quad \text{(DQ3) } x \circ (y \circ z) = y \circ (x \circ z)$$

Definition 5. [5] A *CI-algebra* is an algebra $(X, *, 1)$ of type $(2, 0)$ satisfying the following axioms: for all x, y, z in X, (CI1) $x * x = 1$ (CI2) $1 * x = x$ (CI3) $x * (y * z) = y * (x * z)$

Theorem 1. [11] *Any CI-algebra is equivalent to a dual Q-algebra.*

Definition 6. [5] A *dual BCI-algebra* is an algebra $(X, *, 1)$ of type (2,0) satisfying the following axioms: for all x, y, z in X,

(DBCI1) $x * x = 1$
(DBCI2) $x * y = y * x = 1$ implies $x = y$
(DBCI3) $(x * y) * [(y * z) * (x * z)] = 1$
(DBCI4) $x * [(x * y) * y] = 1$

Proposition 1. [5] *Let $(X, *, 1)$ be a dual BCI-algebra. Then for all x, y, z in X, the following hold:*

(i) $x * y = 1$ implies $(y * z) * (x * z) = 1$
(ii) $x * y = 1$ and $y * z = 1$ imply $x * z = 1$
(iii) $y * (z * x) = z * (y * x)$
(iv) $1 * x = x$

3. Dual B-Algebra

Definition 7. A *dual B-algebra* X^D is a triple $(X, \circ, 1)$ where X is a non-empty set with a binary operation " $\circ$ " and a constant 1 satisfying the following axioms for all x, y, z in X^D:

(DB1) $x \circ x = 1$ (DB2) $1 \circ x = x$ (DB3) $x \circ (y \circ z) = ((y \circ 1) \circ x) \circ z$

Remark 1. *If $(X, *, 0)$ is a B-algebra, define " $\circ$ " as follows: $x \circ y = y * x$ for all x, y in X. Then $(X, \circ, 0)$ is a dual B-algebra, called the derived dual B-algebra.*

Example 2. Consider the B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 1. The dual B-algebra of X is $X^D = (X, \circ, 0)$ with the following table:

$\circ$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	2	0	1	4	5	3
2	1	2	0	5	3	4
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

Define " $\cdot$ " as follows: $x \cdot y = y \circ x$. Then $X^{DD} = (X, \cdot, 0)$ is the B-algebra X with Cayley table in Example 1.

Proposition 2. *Let $X^D = (X, \circ, 0)$ be a dual B-algebra. Then $X^{DD} = (X, \cdot, 0)$ is a B-algebra where $x \cdot y = y \circ x$ for all x, y in X^D.*

Proof: Suppose X^D is a dual B-algebra and define " $\cdot$ " as follows: $x \cdot y = y \circ x$ for all x, y in X^D. Then the axioms of $X^{DD} = (X, \cdot, 0)$ coincide with that of a B-algebra. Hence, X^{DD} is a B-algebra. □

Example 3. Let $X = \mathbb{R}$ and $\circ$ be defined as $x \circ y = \dfrac{y}{x}$ for all x, y in X with $x \neq 0$.

Note that X satisfies (DB1): $x \circ x = \dfrac{x}{x} = 1$, (DB2): $1 \circ x = \dfrac{x}{1} = x$, and (DB3): $x \circ (y \circ z) = \dfrac{y \circ z}{x} = \dfrac{z}{xy} = \dfrac{z}{\dfrac{x}{y \circ 1}} = \dfrac{z}{(y \circ 1) \circ x} = ((y \circ 1) \circ x) \circ z$. Hence, $(\mathbb{R}, \circ, 1)$ is a dual B-algebra. Observe that $(\mathbb{R}, \circ, 1)$ is not a B-algebra since $4 \circ 1 = \dfrac{1}{4} \neq 4$. This leads to the next remark.

Remark 2. Not every dual B-algebra is a B-algebra.

Example 4. Let $X = \{e, a, b, c\}$ be the Klein-4 B-algebra with the following table:

∘	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Then the dual X^D of X is itself. Hence, the Klein-4 B-algebra is a dual B-algebra. Observe that the Klein-4 B-algebra has a symmetric Cayley table and is a dual B-algebra itself. Hence, there exists a B-algebra that is also a dual B-algebra. This is generalized in the next theorem.

Let $(X, *, 0)$ be any algebra of type $(2, 0)$ satisfying $x * y = y * x$ for all x, y in X. Then we say that $(X, *, 0)$ satisfies a *symmetric condition.*

Theorem 2. *Let X be a B-algebra satisfying a symmetric condition. Then X itself is a dual B-algebra, that is, $X = X^D$.*

Proof: Suppose X is a B-algebra satisfying a symmetric condition. Then the dual B-algebra axioms hold, namely (DB1): $x * x = 0$ by (B1), (DB2): $0 * x = x * 0 = x$ by (B2), and (DB3): $x * (y * z) = (z * y) * x = z * [x * (0 * y)] = [(y * 0) * x] * z$ by (B3). Hence, X is a dual B-algebra. □

Example 5. Let $X = \{0, 1, 2\}$ be a set with the following table:

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X, *, 0)$ is a B-algebra [9]. Observe that in this example, $1 * (2 * 0) = 1 * 2 = 2 \neq 1 = 1 * 0 = (2 * 1) * 0 = [(2 * 0) * 1] * 0$. This implies that X is not a dual B-algebra.

Remark 3. Not every B-algebra is a dual B-algebra.

Lemma 2. *Let X^D be a dual B-algebra. Then for any x, y, z in X^D, we have*

(i) $x \circ y = [(x \circ 1) \circ 1] \circ y$	(vi) $x \circ 1 = y \circ 1$ implies $x = y$
(ii) $(x \circ 1) \circ (x \circ y) = y$	(vii) $x = (x \circ 1) \circ 1$
(iii) $(y \circ z) \circ x = z \circ [(y \circ 1) \circ x]$	(viii) $(y \circ x) \circ (y \circ 1) = x \circ 1$
(iv) $z \circ x = z \circ y$ implies $x = y$	(ix) $x \circ [(x \circ 1) \circ x] = x$
(v) $x \circ y = 1$ implies $x = y$	(x) $x \circ y = 1$ implies $(x \circ z) \circ (y \circ z) = 1$.

Proof: Let X^D be a dual B-algebra and $x, y, z \in X^D$.

(i) By (DB2) and (DB3), $x \circ y = 1 \circ (x \circ y) = [(x \circ 1) \circ 1] \circ y$.

(ii) By (DB3), (DB1), and (DB2), $(x \circ 1) \circ (x \circ y) = [(x \circ 1) \circ (x \circ 1)] \circ y = 1 \circ y = y$.

(iii) By (i) and (DB3), $(y \circ z) \circ x = \big[((y \circ 1) \circ 1) \circ z\big] \circ x = z \circ [(y \circ 1) \circ x]$.

(iv) Suppose $z \circ x = z \circ y$. Then $(z \circ 1) \circ (z \circ x) = (z \circ 1) \circ (z \circ y)$ implies $x = y$ by (ii).

(v) Suppose $x \circ y = 1$. By (DB1) and (iv), we get $x \circ y = x \circ x$ implying $x = y$.

(vi) Suppose $x \circ 1 = y \circ 1$.By (DB1), (DB2),(DB3), and (i) we have $1 = x \circ x = 1 \circ (x \circ x) = [(x \circ 1) \circ 1] \circ x = [(y \circ 1) \circ 1] \circ x = y \circ x$. Hence, $y = x$ by (v).

(vii) By (DB2), (DB3), and (vi), $x \circ 1 = 1 \circ (x \circ 1) = [(x \circ 1) \circ 1] \circ 1$ implies that $x = (x \circ 1) \circ 1$.

(viii) By (iii) and (DB1), $(y \circ x) \circ (y \circ 1) = x \circ [(y \circ 1) \circ (y \circ 1)] = x \circ 1$.

(ix) Take $y = z = x$ in (iii). Then apply (DB1) and (DB2).

(x) By (v), $x \circ y = 1$ implies $x = y$. Hence by (DB1), $(x \circ z) \circ (y \circ z) = (x \circ z) \circ (x \circ z) = 1$.

□

The following theorem is a characterization of a dual B-algebra given any algebra with a binary operation and a constant element.

Theorem 3. *Let $X = (X, \circ, 1)$ be any algebra of type $(2, 0)$. Then X is a dual B-algebra if and only if for any x, y, z in X,*

$$\text{(i) } x \circ x = 1; \quad \text{(ii) } x = (x \circ 1) \circ 1; \quad \text{(iii) } (x \circ y) \circ (x \circ z) = y \circ z.$$

Proof: Suppose $X = (X, \circ, 1)$ is a dual B-algebra. Then X satisfies (DB1) and Lemma 2(vii). By (DB3), (DB1), and (DB2), $(x \circ y) \circ (x \circ z) = [(x \circ 1) \circ (x \circ y)] \circ z = \big[\big((x \circ 1) \circ (x \circ 1)\big) \circ y\big] \circ z = (1 \circ y) \circ z = y \circ z$. It follows that X satisfies (i), (ii), and (iii). Conversely by (iii), (i), and (ii), $1 \circ x = (x \circ 1) \circ (x \circ x) = (x \circ 1) \circ 1 = x$. Hence, X satisfies (DB2). For X to satisfy (DB3), we have $x \circ (y \circ z) = [(y \circ 1) \circ x] \circ [(y \circ 1) \circ (y \circ z)] = [(y \circ 1) \circ x] \circ (1 \circ z) = [(y \circ 1) \circ x] \circ z$ by (iii) and (DB2). Therefore, X is a dual B-algebra. □

Comparing the axioms of a dual B-algebra and a BCK-algebra, we have the following remark.

Remark 4. (DB1) is equivalent to (BCK3) and Lemma 2(v) is equivalent to (BCK4) where the constant 1 corresponds to the constant 0 in a dual B-algebra and BCK-algebra, respectively.

Example 6. Consider the dual B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2. Note that $(X, \circ, 0)$ is not a BCK-algebra since (BCK2) is not satisfied, that is, $[1 \circ (1 \circ 5)] \circ 5 = (1 \circ 3) \circ 5 = 4 \circ 5 = 1 \neq 0$. Also, $2 \circ 1 = 2 \neq 1 = 1 \circ 2$.

Example 7. Consider the Klein-4 dual B-algebra X^D in Example 4. Observe that this example satisfies the symmetric condition but is not a BCK-algebra since $e \circ x \neq e$ for all $x \in X$.

Lemma 3. *Let $X^D = (X, \circ, 1)$ be a dual B-algebra satisfying a symmetric condition. Then for all x, y, z in X, $(x \circ y) \circ (z \circ y) = x \circ z$.*

Proof: By (DB3), hypothesis, Lemma 2(iii) and (i), (DB1), and (DB2), we have $(x \circ y) \circ (z \circ y) = [(z \circ 1) \circ (x \circ y)] \circ y = [z \circ (x \circ y)] \circ y = [(x \circ y) \circ z] \circ y = \big([(x \circ y) \circ 1] \circ z\big) \circ y = z \circ [(x \circ y) \circ y] = z \circ \big(y \circ [(x \circ 1) \circ y]\big) = z \circ [y \circ (x \circ y)] = z \circ [y \circ (y \circ x)] = z \circ \big(y \circ [(y \circ 1) \circ x]\big) = z \circ \big[\big([(y \circ 1) \circ 1] \circ y\big) \circ x\big] = z \circ [(y \circ y) \circ x] = z \circ (1 \circ x) = z \circ x = x \circ z$. □

Proposition 3. *Let $X^D = (X, \circ, 1)$ be a dual B-algebra satisfying a symmetric condition. Then X^D satisfies* (BCK1), (BCK2), (BCK3), *and* (BCK4) *of a BCK-algebra.*

Proof: Suppose X^D is a dual B-algebra satisfying a symmetric condition. Then by (DB2) and the hypothesis, $x = 1 \circ x = x \circ 1$ for all x in X^D. By Remark 4, it remains to show that X^D satisfies (BCK1) and (BCK2). Let $x, y, z \in X^D$. By (DB3), hypothesis, (DB1) and (DB2), $[x \circ (x \circ y)] \circ y = \big([(x \circ 1) \circ x] \circ y\big) \circ y = [(x \circ x) \circ y] \circ y = (1 \circ y) \circ y = y \circ y = 1$. Thus, X^D satisfies (BCK2). By Lemma 2 (iii) and hypothesis, $[(x \circ y) \circ (x \circ z)] \circ (z \circ y) = (x \circ z) \circ \big([(x \circ y) \circ 1] \circ (z \circ y)\big) = (x \circ z) \circ [(x \circ y) \circ (z \circ y)]$. By the hypothesis, Lemma 3 and (DB1), $[(x \circ y) \circ (x \circ z)] \circ (z \circ y) = [(y \circ x) \circ (z \circ x)] \circ (y \circ z) = (y \circ z) \circ (y \circ z) = 1$. So, X^D satisfies (BCK1). □

Example 8. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra **[10]** with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

Note that $b * a = b \neq a = a * b$. In fact, $0 * b = 0 \neq b$. So, X does not satisfy (DB2) and hence, is not a dual B-algebra.

The following theorem shows that if the symmetric condition holds in a BCK-algebra X, then X is a dual B-algebra.

Theorem 4. *If $(X, \circ, 1)$ is a BCK-algebra satisfying a symmetric condition, then X is a dual B-algebra.*

Proof: Suppose X is a BCK-algebra satisfying $x \circ y = y \circ x$ for all x, y in X. By Remark 4, it remains to show that X satisfies (DB3) and (DB2). By Lemma 1(i) and (ii) of a BCK-algebra, $[(y \circ 1) \circ x] \circ z = (y \circ x) \circ z = (y \circ z) \circ x$. Since $x \circ y = y \circ x$ for all x, y in X, $(y \circ z) \circ x = x \circ (y \circ z)$. Hence, X satisfies (DB3). By Lemma 1(i) and the hypothesis, $x = x \circ 1 = 1 \circ x$. This implies that X satisfies (DB2). □

4. Commutativity in a Dual B-algebra

Definition 8. Let X^D be a dual B-algebra. Define a binary operation " $+$ " on X as follows: $x + y = (x \circ 1) \circ y$ for all x, y in X^D. A dual B-algebra is said to be *commutative* if $x + y = y + x$, that is, $(x \circ 1) \circ y = (y \circ 1) \circ x$ for all x, y in X^D.

Example 9. The dual B-algebra $X = \mathbb{R}$ in Example 3 is commutative since for all x, y in $\mathbb{R}$, $(x \circ 1) \circ y = \dfrac{y}{x \circ 1} = \dfrac{y}{\frac{1}{x}} = xy = \dfrac{x}{\frac{1}{y}} = \dfrac{x}{y \circ 1} = (y \circ 1) \circ x$. However, the dual B-algebra in Example 2 is not commutative since $(1 \circ 0) \circ 4 = 2 \circ 4 = 3 \neq 5 = 4 \circ 1 = (4 \circ 0) \circ 1$. Observe that $(1 \circ 0) \circ (3 \circ 0) = 2 \circ 3 = 5 \neq 4 = 3 \circ 1$ and $(2 \circ 5) \circ 5 = 4 \circ 5 = 1 \neq 2$.

However, for a commutative dual B-algebra, the following proposition holds.

Proposition 4. *Suppose X^D is a commutative B-algebra. Then the following hold for all x, y in X^D:* (i) $(x \circ 1) \circ (y \circ 1) = y \circ x$ (ii) $(y \circ x) \circ x = y$.

Proof: Let X^D be a commutative B-algebra. (i)By Definition 8 and Lemma 2(i), $(x \circ 1) \circ (y \circ 1) = [(y \circ 1) \circ 1] \circ x = y \circ x$. (ii)Applying Lemma 2(iii), Definition 8, (DB3), Lemma 2(i), (DB1), and (DB2), $(y \circ x) \circ x = x \circ [(y \circ 1) \circ x] = x \circ [(x \circ 1) \circ y] = \big([(x \circ 1) \circ 1] \circ x\big) \circ y = (x \circ x) \circ y = 1 \circ y = y$. □

Lemma 4. *If X^D is a commutative dual B-algebra, then the right cancellation law holds, that is, $x \circ z = y \circ z$ implies $x = y$ for all x, y, z in X^D.*

Proof: Suppose X^D is commutative and $x \circ z = y \circ z$ for any x, y, z in X^D. Then by Proposition 4(ii), we can write $x = (x \circ z) \circ z = (y \circ z) \circ z = y$. □

Proposition 5. *If X^D is a commutative dual B-algebra, then the following hold for all x, y, z in X^D:*

(i) $x \circ (y \circ z) = y \circ (x \circ z)$ (iii) $x \circ (y \circ x) = (x \circ y) \circ (x \circ 1)$
(ii) $(x \circ y) \circ z = (z \circ y) \circ x$ (iv) $y \circ [(y \circ x) \circ x] = 1$.

Proof: Suppose X^D is commutative and $x, y, z \in X^D$. (i) By (DB3) and Definition 8, $x \circ (y \circ z) = [(y \circ 1) \circ x] \circ z = [(x \circ 1) \circ y] \circ z = y \circ (x \circ z)$. (ii) Applying Lemma 2(iii) and since X^D is commutative, $(x \circ y) \circ z = y \circ [(x \circ 1) \circ z] = y \circ [(z \circ 1) \circ x] = (z \circ y) \circ x$. (iii) Write $x \circ (y \circ x) = y \circ (x \circ x)$ by (i). Then $y \circ (x \circ x) = y \circ 1 = (x \circ y) \circ (x \circ 1)$ by (DB1) and Lemma 2(viii). (iv) Follows directly from Proposition 4(ii) and (DB1). □

Corollary 1. *If X^D is a dual B-algebra satisfying a symmetric condition, then X^D is commutative.*

Proof: Let X^D be a dual B-algebra satisfying a symmetric condition. Then $(x \circ 1) \circ y = (1 \circ x) \circ y = x \circ y = y \circ x = (1 \circ y) \circ x = (y \circ 1) \circ x$. This implies that X^D is commutative. □

The following corollary follows from Theorem 4 and Corollary 1.

Corollary 2. *Suppose X is a BCK-algebra satisfying a symmetric condition. Then X is a commutative dual B-algebra.*

The following results present the relationship between a commutative dual B-algebra and some algebras, namely, CI-algebra and dual BCI-algebra. Comparing the axioms and properties of commutative dual B-algebra, CI-algebra and dual BCI-algebra, we have the following remarks.

Remark 5.

(i) The class of commutative dual B-algebras is a subclass of CI-algebras since (DB1) is equivalent to (CI1), (DB2) is equivalent to (CI2), and Proposition 5(i) is equivalent to (CI3).

(ii) (DB1) is equivalent to (DBCI1), Lemma 2(v) is equivalent to (DBCI2), Proposition 5(iv) is equivalent to (DBCI4), (DB2) is equivalent to Proposition 1(iv)

Example 10. Consider the non-commutative dual B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2. Now $2 \circ (4 \circ 5) = 2 \circ 1 = 2 \neq 0 = 4 \circ 4 = 4 \circ (2 \circ 5)$. Hence, X does not satisfy (CI3).

The following corollaries follow from Remark 5 and Theorem 1.

Corollary 3. *If X^D is a commutative dual B-algebra, then X^D is a CI-algebra.*

Corollary 4. *Every commutative dual B-algebra is a dual Q-algebra.*

The converse of Corollary 3 is not always true as shown in the following example.

Example 11. Let $X = \{1, a, b, c, d\}$ be a set with the following Cayley table:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	d	d	d	d	1

Then $(X, *, 1)$ is a CI-algebra [5] but is not a dual B-algebra since it does not satisfy (DB3). Indeed, $a \circ (b \circ c) = a \circ a = 1 \neq b = a \circ c = (1 \circ a) \circ c = [(b \circ 1) \circ a] \circ c$.

Theorem 5. *If X is a CI-algebra satisfying a symmetric condition, then X is a commutative dual B-algebra.*

Proof: Suppose X is a CI-algebra satisfying a symmetric condition. By Remark 5, it remains to show that X satisfies (DB3) and that X is commutative. Applying (CI3) and the hypothesis, $x \circ (y \circ z) = y \circ (x \circ z) = (y \circ 1) \circ (z \circ x) = z \circ [(y \circ 1) \circ x] = [(y \circ 1) \circ x] \circ z$. Hence, X satisfies (DB3). By Corollary 1, it follows that X is commutative. □

Example 12. Consider the non-commutative dual B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2. Observe that $(1 \circ 2) \circ [(2 \circ 4) \circ (1 \circ 4)] = 1 \circ (3 \circ 5) = 1 \circ 2 = 1 \neq 0$. Hence, X^D does not satisfy (DBCI3) and so X^D is not a dual BCI-algebra.

However, if commutativity holds for a dual B-algebra, then it is also a dual BCI-algebra as shown in the next theorem.

Theorem 6. *Every commutative dual B-algebra is a dual BCI-algebra.*

Proof: Let X^D be a commutative dual B-algebra. By Remark 5, it remains to show that X^D satisfies (DBCI3). By Proposition 5(ii), Proposition 4(ii), and (DB1), $(x \circ y) \circ [(y \circ z) \circ (x \circ z)] = (x \circ y) \circ \big([(x \circ z) \circ z] \circ y\big) = (x \circ y) \circ (x \circ y) = 1$. Hence, X satisfies (DBCI3). Therefore, X is a dual BCI-algebra. □

Note that the converse of Theorem 6 is not always true as shown in the following example.

Example 13. Let $X = \{0, 1, a, b, c\}$ with binary operation " $*$ " on X defined by the following table on the left:

$*$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	1
c	c	a	1	1	0

$\circ$	0	1	a	b	c
0	0	1	a	b	c
1	0	0	a	a	a
a	a	a	0	1	1
b	a	a	0	0	1
c	a	a	0	1	0

Then $X = (X, *, 0)$ is a BCI-algebra [13]. Note that $(X, \circ, 0)$ is a dual BCI-algebra. Now, $1 \circ (b \circ c) = 1 \circ 1 = 0 \neq 1 = a \circ c = (a \circ 1) \circ c = [(b \circ 0) \circ 1] \circ c$. Thus, X does not satisfy (DB3). Hence, X is not a dual B-algebra.

However if a dual BCI-algebra X satisfies the symmetric condition, then X is also a dual B-algebra as shown in the next theorem.

Theorem 7. *If X is a dual BCI-algebra satisfying a symmetric condition, then X is a commutative dual B-algebra.*

Proof: Suppose X is a dual BCI-algebra satisfying a symmetric condition. Then Proposition 1(iv) becomes $x = 1 \circ x = x \circ 1$. By Remark 5, it remains to show that X satisfies (DB3) and is commutative. Applying the hypothesis, Propositon 1(iii) and (iv), $x \circ (y \circ z) = x \circ (z \circ y) = z \circ (x \circ y) = z \circ [x \circ (1 \circ y)] = [x \circ (1 \circ y)] \circ z = [(1 \circ y) \circ x] \circ z = [(y \circ 1) \circ x] \circ z$. Hence, X satisfies (DB3). Also by the hypothesis and Proposition 1(iii), $(x \circ 1) \circ y = y \circ (x \circ 1) = x \circ (y \circ 1) = (y \circ 1) \circ x$. Therefore, X is commutative. □

5. Conclusion

In this paper, the notion of a dual B-algebra is presented together with some of its properties and characterizations. Not every B-algebra is a dual B-algebra and not every dual B-algebra is a B-algebra. However, there exists an algebra that is both a B-algebra and a dual B-algebra. The different relationships of the dual B-algebra to BCK-algebra, CI-algebra, and dual BCI-algebra is given. The concept of commutativity in a dual B-algebra was introduced and some properties were provided.

Acknowledgements

This research is funded by the Commission on Higher Education (CHED) and Mindanao State University-Iligan Institute of Technology, Philippines.

References

[1] N. Al-Shehrie. Derivations of B-algebras. *JKAU: Sci.*, 22(1):71–83, 2010.

[2] Y. Imai and K. Iseki. On Axiom Systems of Propositional Calculi. *Proceedings of Japan Academy*, 42(1):19–22, 1966.

[3] K. Kim and Y. Yon. Dual BCK-algebra and MV-algebra. *Scientiae Mathematicae Japonicae*, 42(1):393–399, 2007.

[4] M. Kondo and Y.B. Jun. The Class of B-algebras Coincides with the Class of Groups. *Scientiae Mathematicae Japonicae*, 7:175–177, 2002.

[5] B.L. Meng. CI-algebra. *Scientiae Mathematicae Japonicae*, 2009:695–701, 2009.

[6] J. Meng and Y. Jun. *BCK-algebra.* Kyung Moonsa, Seoul, 1994.

[7] J. Neggers and S.S. Ahn. On Q-algebras. *International Journal of Mathematics and Mathematical Sciences*, 27:749–757, 2001.

[8] J. Neggers and H. Kim. A Fundamental Theorem of B-homomorphism for B-algebras. *Inter.Math.J.*, 2:207–214, 2002.

[9] J. Neggers and H. Kim. On B-algebras. *Mat. Vesnik*, 54:21–29, 2002.

[10] E. Roh and Y. Jun. Positive Implicative Ideals of BCK-algebras Based on Intersectional Soft Sets. *Journal of Applied Mathematics*, 2013.

[11] A. Saeid. CI-algebra is Equivalent to Dual Q-Algebra. *Journal of the Egyptian Mathematical Society*, 21:1–2, 2013.

[12] A. Walendziak. On Commutative BE-algebras. *Scientiae Mathematicae Japonicae*, 2008:585–588, 2008.

[13] O. Zahiri and R. Burzooei. Graph of BCI-algebras. *International Journal of Mathematics and Mathematical Sciences*, 2012.