Some Structural Properties of Fully UP-semigroups

Dianne P. Gomisong\(^1,2,\ast\), Rowena T. Isla\(^1,2\)

\(^1\) Department of Mathematics and Statistics, College of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines
\(^2\) Center for Graph Theory, Algebra and Analysis, Premier Research Institute of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines

Abstract. This paper investigates a new class of algebra related to UP-algebras and semigroups called fully UP-semigroups (or \(f\)-UP-semigroups). It establishes some structural properties of \(f\)-UP-semigroups. It also introduces and examines \(f\)-UP-fields, \(f\)-UP-domains, \(f\)-UP-ideals, and quotient \(f\)-UP-semigroups. Moreover, it investigates the relationship between an \(f\)-UP-field and an \(f\)-UP-domain.

2010 Mathematics Subject Classifications: 03G25, 08A99

Key Words and Phrases: UP-algebra, \(f\)-UP-semigroup, \(f\)-UP-field, \(f\)-UP-domain, \(f\)-UP-ideal, Quotient \(f\)-UP-semigroup

1. Introduction

In 1993, Jun, Hong, and Roh [7] introduced a class of algebra related to BCI-algebras and semigroups with distributive laws property, called a BCI-semigroup. Jun et al. [8, 9] renamed the BCI-semigroup as the IS-algebra and studied related properties. In 2018, F. Kareem and E. Hasan [10] introduced the concept of KU-semigroup which is a combination of KU-algebra and semigroup. In the same year, A. Iampan [4] introduced a new class of algebra called a fully UP-semigroup (or $f$-UP-semigroup) which is a combination of UP-algebra and semigroup. In this study, the notion of $f$-UP-semigroup is investigated and some of its properties are established.

2. Preliminaries

An algebra of type $(2,0)$ is an algebra with a binary operation and a constant element.

**Definition 1.** [11] A KU-algebra is an algebra $(X; *, 0)$ of type $(2,0)$ satisfying the following axioms: for all $x, y, z \in X$,

(KU1) $(x * y) * [(y * z) * (x * z)] = 0$,

(KU2) $0 * x = x$,

(KU3) $x * 0 = 0$,

(KU4) $x * y = y * x = 0$ implies $x = y$.

**Example 1.** [11] Let $X = \{0, a, b, c\}$ be a set with a binary operation $*$ defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>a</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>0</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Then, $(X; *, 0)$ is a KU-algebra.

**Definition 2.** [3] A UP-algebra is an algebra $(X; *, 0)$ of type $(2,0)$ satisfying the following axioms: for all $x, y, z \in X$,

(UP1) $(y * z) * [(x * y) * (x * z)] = 0$,

(UP2) $0 * x = x$,

(UP3) $x * 0 = 0$,

(UP4) $x * y = y * x = 0$ implies $x = y$.

**Example 2.** [3] Let $X = \{0, a, b, c\}$ be a set with a binary operation $*$ defined by the following Cayley table:
Then, \((X; \ast, 0)\) is a UP-algebra.

**Definition 3.** [3] Let \(X\) be a UP-algebra. A subset \(S\) of \(X\) is called a UP-subalgebra of \(X\) if the constant zero of \(X\) is in \(S\) and \((S; \ast, 0)\) itself forms a UP-algebra.

**Definition 4.** [1] Define \(x \land y = (y \ast x) \ast x\). Then \(X\) is said to be a commutative UP-algebra if for any \(x, y \in X\), \((y \ast x) \ast x = (x \ast y) \ast y\), that is, \(x \land y = y \land x\).

**Definition 5.** [3] Let \(X\) be a UP-algebra. Then, a subset \(I\) of \(X\) is called a UP-ideal of \(X\) if it satisfies:

(i) the constant zero of \(X\) is in \(I\), and

(ii) for any \(x, y, z \in X\), \(x \ast (y \ast z) \in I\) and \(y \in I\) imply \(x \ast z \in I\).

**Proposition 1.** [3] In a UP-algebra \((X; \ast, 0)\), the following properties hold: for any \(x, y, z \in X\),

(i) \(x \ast x = 0\),

(ii) \(x \ast y = 0\) and \(y \ast z = 0\) imply \(x \ast z = 0\),

(iii) \(x \ast y = 0\) implies \((z \ast x) \ast (z \ast y) = 0\),

(iv) \(x \ast y = 0\) implies \((y \ast z) \ast (x \ast z) = 0\),

(v) \(x \ast (y \ast x) = 0\),

(vi) \((y \ast x) \ast x = 0\) implies \(x = y \ast x\), and

(vii) \(x \ast (y \ast y) = 0\).

The next result gives a relationship between UP-algebras and KU-algebras.

**Theorem 1.** [3] Any KU-algebra is a UP-algebra.

The converse of Theorem 1 does not hold. To see this, consider the UP-algebra \((X; \ast, 0)\) in Example 2. Let \(x = 0\), \(y = a\), and \(z = c\). Observe that \((x \ast y) \ast [(y \ast z) \ast (x \ast z)] = (0 \ast a) \ast [(a \ast c) \ast (0 \ast c)] = a \ast (b + c) = a \ast b = b \neq 0\), so (KU1) is not satisfied. Thus, \((X; \ast, 0)\) is not a KU-algebra.

In view of Theorem 1, the notion of UP-algebras is a generalization of KU-algebras.

**Proposition 2.** [3] A nonempty subset \(S\) of a UP-algebra \((X; \ast, 0)\) is a UP-subalgebra of \(X\) if and only if \(S\) is closed under the \(\ast\) operation.
Let $X$ be a UP-algebra and $A$ be a nonempty subset of $X$. Then $X \star A$ is given by $X \star A = \bigcup_{x \in X, a \in A} (x \star a)$.

**Theorem 2.** [3] Let $X$ be a UP-algebra and $B$ a UP-ideal of $X$. Then $X \star B \subseteq B$. In particular, $B$ is a UP-subalgebra of $X$.

Let $(X; \star, 0)$ be a UP-algebra and $B$ a UP-ideal of $X$. Define the binary relation $\sim_B$ on $X$ as follows: for all $x, y \in X$, $x \sim_B y$ if and only if $x \star y \in B$ and $y \star x \in B$. An equivalence relation $\rho$ on $X$ is called a congruence if for any $x, y, z \in X$, $x \rho y$ implies $(x \star z) \rho (y \star z)$ and $(z \star x) \rho (z \star y)$.

If $x \in X$, then the $\rho$-class of $x$ is $[x]_{\rho}$ defined as $[x]_{\rho} = \{y \in X : y \rho x\}$. The set of all $\rho$-classes is called the quotient set of $X$ by $\rho$, and is denoted by $X/\rho$. That is, $X/\rho = \{[x]_{\rho} : x \in X\}$.

**Theorem 3.** [3] Let $(X; \star, 0)$ be a UP-algebra and $B$ a UP-ideal of $X$. Then the following hold:

(i) the $\sim_B$-class $[0]_{\sim_B}$ is a UP-ideal and a UP-subalgebra of $X$,

(ii) $a \sim_B$-class $[x]_{\sim_B}$ is a UP-ideal of $X$ if and only if $x \in B$,

(iii) $a \sim_B$-class $[x]_{\sim_B}$ is a UP-subalgebra of $X$ if and only if $x \in B$, and

(iv) $(X/\sim_B; *, [0]_{\sim_B})$ is a UP-algebra under the operation $*$ defined by $[x]_{\sim_B} *[y]_{\sim_B} = [x \star y]_{\sim_B}$ for all $x, y \in X$, called the quotient UP-algebra of $X$ induced by the congruence $\sim_B$.

**Definition 6.** [10] A KU-semigroup is a nonempty set $X$ together with two binary operations $*$ and $\cdot$ and a constant $0$ satisfying the following:

(KUS1) $(X; *, 0)$ is a KU-algebra;

(KUS2) $(X, \cdot)$ is a semigroup; and

(KUS3) the operation $\cdot$ is left and right distributive over the operation $*$, that is, $x \cdot (y \star z) = (x \cdot y) \star (x \cdot z)$ and $(x \star y) \cdot z = (x \cdot z) \star (y \cdot z)$.

**Example 3.** [10] Let $X = \{0, a, b, c\}$ be a set with the binary operations $*$ and $\cdot$ defined by the following Cayley tables:

\[
\begin{array}{c|cccc}
* & 0 & a & b & c \\
\hline
0 & 0 & a & b & c \\
a & 0 & 0 & b & c \\
b & 0 & a & 0 & c \\
c & 0 & 0 & 0 & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & b \\
c & 0 & 0 & b & c \\
\end{array}
\]
Then, \((X;\ast,\cdot,0)\) is a KU-semigroup.

**Definition 7.** [4] A *fully UP-semigroup* (or \(f\)-**UP-semigroup**) is a nonempty set \(X\) together with two binary operations \(\ast\) and \(\cdot\) and a constant 0 satisfying the following:

\((f\text{UP}1)\) \((X;\ast,0)\) is a UP-algebra;
\((f\text{UP}2)\) \((X,\cdot)\) is a semigroup; and
\((f\text{UP}3)\) the operation \(\cdot\) is left and right distributive over the operation \(\ast\).

A. Iampan [4] analogously introduced a *left [resp., right] UP-semigroup* as a nonempty set \(X\) together with two binary operations \(\ast\) and \(\cdot\) and a constant 0 satisfying \((f\text{UP}1)\), \((f\text{UP}2)\), and the operation \(\cdot\) is left [resp. right] distributive over the operation \(\ast\). Thus, an \(f\)-UP-semigroup is both a left and a right UP-semigroup.

**Example 4.** [4] Let \(X = \{0, a, b, c\}\) be a set with the binary operations \(\ast\) and \(\cdot\) defined by the following Cayley tables:

\[
\begin{array}{|c|ccc|}
\hline
& 0 & a & b & c \\
\hline 0 & 0 & a & b & c \\
a & 0 & 0 & b & c \\
b & 0 & a & 0 & c \\
c & 0 & a & b & 0 \\
\hline
\end{array}
\quad
\begin{array}{|c|ccc|}
\hline
& 0 & a & b & c \\
\hline 0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & a \\
c & 0 & 0 & 0 & a \\
\hline
\end{array}
\]

Then, \((X;\ast,\cdot,0)\) is an \(f\)-UP-semigroup.

**Example 5.** Let \(X = \{0, a, b, c\}\) be a set with the binary operations \(\ast\) and \(\cdot\) defined by the following Cayley tables:

\[
\begin{array}{|c|ccc|}
\hline
& 0 & a & b & c \\
\hline 0 & 0 & a & b & c \\
a & 0 & 0 & b & c \\
b & 0 & a & 0 & c \\
c & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\quad
\begin{array}{|c|ccc|}
\hline
& 0 & a & b & c \\
\hline 0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & a & b & c \\
\hline
\end{array}
\]

Then, routine calculations show that \((X;\ast,\cdot,0)\) is an \(f\)-UP-semigroup.

**Example 6.** Let \(X = \{0, a, b, c, d\}\) be a set with the binary operations \(\ast\) and \(\cdot\) defined by the following Cayley tables:

\[
\begin{array}{|c|ccc|}
\hline
& 0 & a & b & c \\
\hline 0 & 0 & a & b & c \\
a & 0 & 0 & b & c \\
b & 0 & a & 0 & c \\
c & 0 & a & b & 0 \\
\hline
\end{array}
\quad
\begin{array}{|c|ccc|}
\hline
& 0 & a & b & c \\
\hline 0 & 0 & 0 & 0 & 0 \\
a & 0 & a & b & c \\
b & 0 & b & c & a \\
c & 0 & c & a & b \\
\hline
\end{array}
\]
Then, routine calculations show that \((X; *, \cdot, 0)\) is an \(f\)-UP-semigroup.

Hereinafter, let \(X\) denote the \(f\)-UP-semigroup \((X; *, \cdot, 0)\), unless otherwise indicated.

**Definition 8.** A nonempty subset \(S\) of an \(f\)-UP-semigroup \(X\) is called an \(f\)-UP-subsemigroup of \(X\) if the constant 0 of \(X\) is in \(S\) and \((S; *, \cdot, 0)\) itself forms an \(f\)-UP-semigroup.

Obviously, \(\{0\}\) and \(X\) are \(f\)-UP-subsemigroups of \(X\). In Example 4, the set \(S_1 = \{0, b\}\) is an \(f\)-UP-subsemigroup of \(X\), while the set \(S_2 = \{0, b, c\}\) is not an \(f\)-UP-subsemigroup since \(b \cdot c = a \notin S_2\).

The following remark immediately follows from Definitions 8, 7, and 3.

**Remark 1.** Every \(f\)-UP-subsemigroup of \((X; *, \cdot, 0)\) is a UP-subalgebra of \(X\) with respect to \(*\).

The converse of Remark 1 does not hold. To see this, consider Example 4. It can be easily verified that \(S = \{0, b, c\}\) is a UP-subalgebra of \((X; *, 0)\) but \(S\) is not an \(f\)-UP-subsemigroup of \((X; *, \cdot, 0)\) since \(b \cdot c = a \notin S\).

**Definition 9.** An \(f\)-UP-semigroup \(X\) is said to be commutative if \(a \cdot b = b \cdot a\) for all \(a, b \in X\). If \(X\) is not commutative, then it is called a noncommutative \(f\)-UP-semigroup.

Routine calculations show that the \(f\)-UP-semigroups in Examples 4 and 6 are commutative while the \(f\)-UP-semigroup in Example 5 is noncommutative since \(a \cdot c = 0 \neq a = c \cdot a\).

**Definition 10.** Let \(X\) be an \(f\)-UP-semigroup. An element \(e \in X\) is called a unity in \(X\) if \(x \cdot e = e \cdot x = x\) for all \(x \in X\).

**Proposition 3.** Let \(X\) be an \(f\)-UP-semigroup. If the unity of \(X\) exists, then it is unique.

**Proof.** Let \(X\) be an \(f\)-UP-semigroup with unity. Suppose \(1, 1' \in X\) both satisfy the properties of being a unity. Then, for all \(x \in X\), \(x \cdot 1 = 1 \cdot x = x\) and \(x \cdot 1' = 1' \cdot x = x\). If \(x = 1\), we have \(1 \cdot 1' = 1\). If \(x = 1'\), we have \(1 \cdot 1' = 1'\). Therefore, \(1 = 1'\).

If an \(f\)-UP-semigroup \(X\) has unity, it shall be denoted by 1.

**Definition 11.** Let \(X\) be an \(f\)-UP-semigroup with unity 1. An element \(a \in X\) is called \(1\)-invertible if there exists \(b \in X\) such that \(a \cdot b = 1 = b \cdot a\).

We next introduce the concepts of \(f\)-UP-field and \(f\)-UP-domain analogous to the definitions of JB-field and JB-domain given by J. Endam and J. Vilela [2].

**Definition 12.** Let \(X\) be an \(f\)-UP-semigroup with unity 1. Then \(X\) is called an \(f\)-UP-field if the following hold:

(i) the semigroup \((X, \cdot)\) is commutative; and

(ii) every \(0 \neq a \in X\) is \(1\)-invertible.
**Definition 13.** A nonzero element \( a \) of an \( f \)-UP-semigroup \( X \) is called a 0-divisor if there exists \( b \in X \) such that \( b \neq 0 \) and either \( a \cdot b = 0 \) or \( b \cdot a = 0 \).

Note that 0 is not a 0-divisor.

**Remark 2.** An element cannot be 1-invertible and a 0-divisor at the same time. Thus, an \( f \)-UP-field has no 0-divisors.

**Definition 14.** Let \( X \) be an \( f \)-UP-semigroup with unity 1. Then \( X \) is called an \( f \)-UP-domain if the following hold:

(i) the semigroup \((X, \cdot)\) is commutative; and

(ii) \( X \) has no 0-divisors.

The \( f \)-UP-semigroup in Example 6 is an \( f \)-UP-domain.

**Remark 3.** Every \( f \)-UP-field is an \( f \)-UP-domain.

### 3. Elementary Properties of \( f \)-UP-semigroups

This section presents some elementary properties of \( f \)-UP-semigroups. Throughout this section, \( X \) means an \( f \)-UP-semigroup \((X; \cdot, 0)\).

**Theorem 4.** Let \( a, b, c \in X \). Then the following properties hold:

(i) \( a \cdot 0 = 0 = 0 \cdot a \),

(ii) \( a \cdot (0 * b) = (0 * a) \cdot b = a \cdot b \),

(iii) \( a \cdot (b * (0 * c)) = (a \cdot b) * (a \cdot c) \) and \((b * (0 * c)) \cdot a = (b \cdot a) * (c \cdot a)\),

(iv) \( a \cdot (b \land c) = (a \cdot b) \land (a \cdot c) \) and \((a \land b) \cdot c = (a \cdot c) \land (b \cdot c)\),

(v) If \( a \cdot b = 0 \), then \( a \cdot (b \lor c) = a \cdot c \),

(vi) If \( a \cdot c = 0 \), then \((a \lor b) \cdot c = b \cdot c \).

**Proof.** Let \( a, b, c \in X \).

(i) By Proposition 1(i) and (fUP3), \( a \cdot 0 = a \cdot (0 * 0) = (a \cdot 0) * (a \cdot 0) = 0 \). Similarly, \( 0 \cdot a = 0 \).

(ii) By (UP2), \( a \cdot (0 * b) = a \cdot b = (0 * a) \cdot b \).

(iii) By (UP2) and (fUP3), \( a \cdot (b * (0 * c)) = a \cdot (b \lor c) = (a \cdot b) \lor (a \cdot c) \). Similarly, \((b * (0 * c)) \cdot a = (b \lor c) \cdot a = (b \cdot a) \lor (c \cdot a)\).

(iv) By Definition 4 and (fUP3), \( a \cdot (b \land c) = a \cdot [(c \land b) \land b] = [a \cdot (c \land b)] \land (a \cdot b) = [(a \cdot c) \land (a \cdot b)] \land (a \cdot b) = (a \cdot (b \lor c)) \land (a \cdot b) \land (a \lor b) \cdot c = [(b \lor a) \land (a \cdot c)] \land (a \cdot c) = [(b \lor a) \land (a \cdot c)] \land (a \cdot c) = (a \cdot c) \land (b \lor c)\).
(v) Suppose \( a \cdot b = 0 \). Then by (fUP3) and (UP2), \( a \cdot (b \ast c) = (a \cdot b) \ast (a \cdot c) = 0 \ast (a \cdot c) = a \cdot c \).

(vi) If \( a \cdot c = 0 \), then by (fUP3) and (UP2), \( (a \ast b) \cdot c = (a \cdot c) \ast (b \cdot c) = 0 \ast (b \cdot c) = b \cdot c \). 

The following theorem gives a necessary and sufficient condition for a subset of an \( f \)-UP-semigroup to be an \( f \)-UP-subsemigroup.

**Theorem 5.** A nonempty subset \( S \) of an \( f \)-UP-semigroup \( (X; \ast, \cdot, 0) \) is an \( f \)-UP-subsemigroup of \( X \) if and only if \( x \ast y, x \cdot y \in S \) for all \( x, y \in S \).

**Proof.** Let \( \emptyset \neq S \subseteq X \). Suppose \( S \) is an \( f \)-UP-subsemigroup of \( X \). Then by Definition 8, \( (S; \ast, \cdot, 0) \) is an \( f \)-UP-semigroup. Thus, the binary operations \( \ast \) and \( \cdot \) are closed in \( S \), that is, \( x \ast y, x \cdot y \in S \) for all \( x, y \in S \). Conversely, suppose \( x \ast y, x \cdot y \in S \) for all \( x, y \in S \). Then \( 0 = x \ast x \in S \). By Proposition 2, \( (S; \ast, 0) \) is a UP-subalgebra of \( X \), hence (fUP1) holds. Let \( x, y, z \in S \subseteq X \). Then \( x \cdot y \in S \) by our assumption and \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \) by associativity in \( X \). Hence, \( (S, \cdot) \) is a semigroup and (fUP2) is satisfied. Moreover, (fUP3) holds for all \( x, y, z \in S \subseteq X \). Thus, \( S \) is an \( f \)-UP-subsemigroup of \( X \). 

**Theorem 6.** Let \( X \) be an \( f \)-UP-semigroup and \( \{ A_i : i \in I \} \) a family of \( f \)-UP-subsemigroups of \( X \). Then \( \bigcap_{i \in I} A_i \) is an \( f \)-UP-subsemigroup of \( X \).

**Proof.** Since \( A_i \) is an \( f \)-UP-subsemigroup of \( X \), \( 0 \in A_i \) for all \( i \in I \). Thus, \( 0 \in \bigcap_{i \in I} A_i \) and \( \bigcap_{i \in I} A_i \neq \emptyset \). Let \( x, y \in \bigcap_{i \in I} A_i \). Then for all \( i \in I \), \( x, y \in A_i \) and by Theorem 5, \( x \ast y, x \cdot y \in A_i \). Hence, \( x \ast y, x \cdot y \in \bigcap_{i \in I} A_i \). Therefore, \( \bigcap_{i \in I} A_i \) is an \( f \)-UP-subsemigroup of \( X \). 

The next result shows a relationship between KU-semigroups and \( f \)-UP-semigroups.

**Theorem 7.** Any KU-semigroup is an \( f \)-UP-semigroup.

**Proof.** Let \( X = (X; \ast, \cdot, 0) \) be a KU-semigroup. By Theorem 1, \( (X; \ast, 0) \) is a UP-algebra. By Definition 6, \( (X, \cdot) \) is a semigroup and left and right distributivity hold for \( \cdot \) over \( \ast \), thus \( X \) is an \( f \)-UP-semigroup.

**Remark 4.** The converse of Theorem 7 does not hold.

To see this, let \( X = \{0, a, b, c, d\} \) be a set with the binary operations \( \ast \) and \( \cdot \) defined by the following Cayley tables:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>0</td>
<td>b</td>
<td>b</td>
<td>d</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Then by routine calculations, \((X;\ast,\cdot,0)\) is an \(f\)-UP-semigroup. Let \(x = 0, y = c,\) and \(z = a.\) Observe that \((x\ast y)\ast ([y\ast z] \ast (x\ast z)) = (0\ast c)\ast [(c\ast a)\ast (0\ast a)] = c\ast (b\ast a) = c\ast b = b,\) so \((\text{KU1})\) is not satisfied. Thus, \((X;\ast,\cdot,0)\) is not a KU-semigroup.

**Theorem 8.** Let \(X\) be an \(f\)-UP-semigroup with unity 1 and let \(T\) be the set of all 1-invertible elements of \(X.\) Then

(i) \(1 \in T,\)

(ii) \(0 \notin T,\) and

(iii) \(a \cdot b \in T\) for all \(a,b \in T.\)

**Proof.** Let \(T\) be the set of all 1-invertible elements of \(X.\)

(i) Since \(1 \cdot 1 = 1, 1 \in T.\) Thus, \(T \neq \emptyset.\)

(ii) Suppose \(0 \in T.\) Then there exists \(b \in X\) such that \(0 \cdot b = 1 = b \cdot 0.\) But \(0 \cdot b = 0\) and so, \(0 = 1,\) a contradiction. Thus, \(0 \notin T.\)

(iii) Let \(a,b \in T.\) Then there exist \(c,d \in X\) such that \(a \cdot c = 1 = c \cdot a\) and \(b \cdot d = 1 = d \cdot b.\) Moreover, \(d \cdot c \in X.\) By \((f\text{UP2}), (a \cdot b) \cdot (d \cdot c) = ((a \cdot b) \cdot d) \cdot c = (a \cdot (b \cdot d)) \cdot c = (a \cdot 1) \cdot c = a \cdot c = 1\) and \((d \cdot c) \cdot (a \cdot b) = ((d \cdot c) \cdot a) \cdot b = (d \cdot (c \cdot a)) \cdot b = (d \cdot 1) \cdot b = d \cdot b = 1.\) Hence, \(a \cdot b \in T.\)

The next result establishes a relation between 0-divisors and the cancellation property of an \(f\)-UP-semigroup.

**Theorem 9.** If an \(f\)-UP-semigroup \(X\) has no 0-divisors, then left and right cancellation laws hold, that is, for all \(a,b,c \in X,\) \(a \neq 0,\) \(a \cdot b = a \cdot c\) implies \(b = c\) (left cancellation) and \(b \cdot a = c \cdot a\) implies \(b = c\) (right cancellation). If either left or right cancellation law holds, then \(X\) has no 0-divisors.

**Proof.** Let \(a,b,c \in X\) such that \(a \cdot b = a \cdot c\) and \(a \neq 0.\) Then \(a \cdot (b \ast c) = (a \cdot b) \ast (a \cdot c) = 0\) by Proposition 1(i). Since \(X\) has no 0-divisors and \(a \neq 0,\) we have \(b \ast c = 0.\) Since \(a \cdot b = a \cdot c,\) we have \(0 = a \cdot (b \ast c) = (a \cdot b) \ast (a \cdot c) = (a \cdot c) \ast (a \cdot b) = a \cdot (c \ast b)\) and so, \(c \ast b = 0.\) By \((\text{UP4}), b = c.\) Hence, the left cancellation law holds. Similarly, the right cancellation law holds.

Conversely, suppose one of the cancellation laws holds, say, the left cancellation. Let \(a\) be a nonzero element of \(X\) and \(b \in X.\) Suppose \(a \cdot b = 0.\) Then by Theorem 4(i), \(a \cdot b = a \cdot 0\) and so by left cancellation, \(b = 0.\) Suppose \(b \cdot a = 0\) and \(b \neq 0.\) Then by Theorem 4(i), \(b \cdot a = b \cdot 0\) and so by left cancellation, \(a = 0,\) a contradiction. Therefore, \(b = 0\) and \(X\) has no 0-divisors. Similarly, the right cancellation law implies that \(X\) has no 0-divisors.

**Theorem 10.** A finite commutative \(f\)-UP-semigroup \(X\) with more than one element and without 0-divisors is an \(f\)-UP-field.
Proof. Let \( a_1, a_2, \ldots, a_n \) be the distinct elements of \( X \). Let \( a \in X \) with \( a \neq 0 \). Now, \( a \cdot a_i \in X \) for all \( i = 1, 2, \ldots, n \) and so \( \{ a \cdot a_1, a \cdot a_2, \ldots, a \cdot a_n \} \subseteq X \). If \( a \cdot a_i = a \cdot a_j \), then by Theorem 9, \( a_i = a_j \). Thus, the elements \( a \cdot a_1, a \cdot a_2, \ldots, a \cdot a_n \) are distinct and so \( X = \{ a \cdot a_1, a \cdot a_2, \ldots, a \cdot a_n \} \). Hence, one of the elements, say \( a \cdot a_1 \), must be equal to \( a \). Since \( X \) is commutative, \( a_i \cdot a = a \cdot a_i = a \). Let \( b \in X \). Then there exists \( a_j \in X \) such that \( b = a \cdot a_j \). Thus, \( b \cdot a_i = a_i \cdot b = a_i \cdot a_j = a \cdot a_j = a \cdot a_j = b \). This implies that \( a_i \) is the unity of \( X \). We denote the unity of \( X \) by 1. Now, \( 1 \in X = \{ a \cdot a_1, a \cdot a_2, \ldots, a \cdot a_n \} \) and so one of the elements, say \( a \cdot a_k \), must be equal to 1. By commutativity, \( a_k \cdot a = a \cdot a_k = 1 \).

Hence, every nonzero element of \( X \) is 1-invertible. Therefore, \( X \) is an \( f \)-UP-field. \( \square \)

As a consequence of Theorem 10, the following corollary holds.

**Corollary 1.** Every finite \( f \)-UP-domain is an \( f \)-UP-field.

### 4. \( f \)-UP-Ideal and the Quotient \( f \)-UP-semigroup

**Definition 15.** A nonempty subset \( I \) of an \( f \)-UP-semigroup \( X \) is called an \( f \)-UP-ideal of \( X \) if the following hold:

- \((f\text{UPI}1)\) the constant 0 of \( X \) is in \( I \),
- \((f\text{UPI}2)\) for any \( x, y, z \in X \), \( x \ast (y \ast z) \in I \) and \( y \in I \) imply \( x \ast z \in I \), and
- \((f\text{UPI}3)\) for any \( a \in I \), \( x \in X \), \( a \cdot x, x \cdot a \in I \).

Obviously, the subsets \( \{0\} \) and \( X \) are \( f \)-UP-ideals of \( X \). Consider the \( f \)-UP-semigroup in Example 4. Routine calculations show that the set \( I_1 = \{0, a, b\} \) is an \( f \)-UP-ideal of \( X \) while the set \( I_2 = \{0, b, c\} \) is not an \( f \)-UP-ideal of \( X \) since \( b \cdot c = a \notin I_2 \).

**Theorem 11.** Let \( (X; \ast, \cdot, 0) \) be an \( f \)-UP-semigroup and \( I \) an \( f \)-UP-ideal of \( X \). Then \( I \) is an \( f \)-UP-subsemigroup of \( X \).

**Proof.** By \((f\text{UPI}1)\), \((X; \ast, \cdot, 0)\) is a UP-algebra and by definition, \( I \) is a UP-ideal of the UP-algebra \( X \). By Theorem 2, \( I \) is a UP-subalgebra of \( X \). Let \( x, y \in I \subseteq X \). Then by Proposition 2, \( x \ast y \in I \). Since \( I \) is an \( f \)-UP-ideal of the \( f \)-UP-semigroup \( X \), \( x \cdot y \in I \) by \((f\text{UPI}3)\). Thus, \( I \) is an \( f \)-UP-subsemigroup of \( X \) by Theorem 5. \( \square \)

**Theorem 12.** Let \( X \) be an \( f \)-UP-semigroup and \( \{A_i : i \in \mathcal{I}\} \) be a nonempty collection of \( f \)-UP-ideals of \( X \). Then \( \bigcap_{i \in \mathcal{I}} A_i \) is an \( f \)-UP-ideal of \( X \).

**Proof.** Suppose \( \{A_i : i \in \mathcal{I}\} \) is a nonempty collection of \( f \)-UP-ideals of \( X \). Since \( 0 \in A_i \) for all \( i \in \mathcal{I} \), \( 0 \in \bigcap_{i \in \mathcal{I}} A_i \) and so \( \bigcap_{i \in \mathcal{I}} A_i \neq \emptyset \). Suppose \( x, y, z \in X \) such that \( x \ast (y \ast z) \in \bigcap_{i \in \mathcal{I}} A_i \) and \( y \in \bigcap_{i \in \mathcal{I}} A_i \). Then \( x \ast (y \ast z) \in A_i \) and \( y \in A_i \) for all \( i \in \mathcal{I} \). Since
each \( A_i \) is an \( f \)-UP-ideal for all \( i \in \mathcal{I} \), it follows that \( x \ast z \in A_i \) for all \( i \in \mathcal{I} \). Hence, 
\[
x \ast z \in \bigcap_{i \in \mathcal{I}} A_i.
\]
Let \( a \in \bigcap_{i \in \mathcal{I}} A_i \) and \( x \in X \). Then \( a \in A_i \) for all \( i \in \mathcal{I} \). Since each \( A_i \) is
an \( f \)-UP-ideal for all \( i \in \mathcal{I} \), \( a \cdot x, x \cdot a \in A_i \) for all \( i \in \mathcal{I} \). Hence, \( a \cdot x, x \cdot a \in \bigcap_{i \in \mathcal{I}} A_i \).

Therefore, \( \bigcap_{i \in \mathcal{I}} A_i \) is an \( f \)-UP-ideal of \( X \).

Let \( (X; \ast, \cdot, 0) \) be an \( f \)-UP-semigroup and \( I \) be an \( f \)-UP-ideal of \( X \). Define the binary relation \( \sim_I \) on \( X \) as follows: for all \( x,y \in X \), \( x \sim_I y \) if and only if \( x \ast y \in I \) and \( y \ast x \in I \). Denote \( [x]_I \) as the equivalence class containing \( x \in X \) and \( X/I \) as the set of all equivalence classes of \( X \) with respect to \( \sim_I \), that is, \( [x]_I = \{ y \in X : x \sim_I y \} \) and \( X/I = \{ [x]_I : x \in X \} \).

**Remark 5.** Let \( X \) be an \( f \)-UP-semigroup and \( I \) be an \( f \)-UP-ideal of \( X \). Then \( x \in [x]_I \) for all \( x \in X \).

**Lemma 1.** Let \( X \) be an \( f \)-UP-semigroup and \( I \) be an \( f \)-UP-ideal of \( X \). Then \( [x]_I = [y]_I \) if and only if \( x \sim_I y \).

**Proof.** Suppose \( [x]_I = [y]_I \). Since \( y \in [y]_I = [x]_I \), we have \( x \sim_I y \). Conversely, suppose \( x \sim_I y \). Let \( z \in [x]_I \). Then \( x \sim_I z \). By symmetric property, \( z \sim_I x \). By transitivity, \( z \sim_I y \) and by symmetric property, \( y \sim_I z \) and so, \( z \in [y]_I \). Thus, \( [x]_I \subseteq [y]_I \). Let \( z \in [y]_I \). Then \( y \sim_I z \). By transitivity, \( x \sim_I z \), that is, \( z \in [x]_I \). Hence, \( [x]_I \subseteq [y]_I \). Hence, \( [x]_I = [y]_I \).

**Proposition 4.** Let \( X \) be an \( f \)-UP-semigroup and \( I \) be an \( f \)-UP-ideal of \( X \). Then

1. \( [0]_I = I \),
2. \( [x]_I = I \) if and only if \( x \in I \), for all \( x \in I \), and
3. \( I \ast [x]_I = [x]_I \) for all \( x \in X \).

**Proof.** Let \( I \) be an \( f \)-UP-ideal of \( X \).

1. If \( x \in [0]_I \), then by definition, \( 0 \sim_I x \) and by (UP2), \( x = 0 \ast x \in I \). Thus, \( [0]_I \subseteq I \).
   Let \( x \in I \). By (UP2), \( 0 \ast x = x \in I \). By (UP3) and \( (fUP1I) \), \( x \ast 0 = 0 \in I \). Thus, \( 0 \sim_I x \) and so, \( x \in [0]_I \). Hence, \( I \subseteq [0]_I \). Therefore, \( [0]_I = I \).

2. Suppose \( [x]_I = I \). Then by Remark 5, \( x \in I \). Conversely, let \( x \in I \). By (UP2), \( 0 \ast x = x \in I \). By (UP3) and \( (fUP1I) \), \( x \ast 0 = 0 \in I \). Thus, \( 0 \sim_I x \), and by Lemma 1, \( [0]_I = [x]_I \). By (i), \( I = [x]_I \).

3. For all \( x \in X \), \( [x]_I = [0 \ast x]_I = [0]_I \ast [x]_I \) as defined in Theorem 3(iv). By (i), \( [x]_I = I \ast [x]_I \).

\( \square \)
Theorem 13. If $X$ is an $f$-UP-semigroup and $I$ an $f$-UP-ideal of $X$, then $(X/I; *, ·, [0]_I)$ is an $f$-UP-semigroup, where $*$ and $·$ are defined by $[x]_I*[y]_I = [x*y]_I$ and $[x]_I·[y]_I = [x*y]_I$, respectively. If $X$ is commutative, then $X/I$ is commutative and if $X$ has unity, then $X/I$ has unity.

Proof. Let $I$ be an $f$-UP-ideal of $X$. Then $I$ is a UP-ideal of the UP-algebra $(X; *, 0)$. By Theorem 3, $(X/I; *, [0]_I)$ is a UP-algebra, where $*$ is defined by $[x]_I*[y]_I = [x*y]_I$. We show that the binary operation $·$ on $X/I$ is well-defined. Let $[x]_I = [x']_I$ and $[y]_I = [y']_I$. Then $x ∼_I x'$ and $y ∼_I y'$ which imply $x*x', x'*x, y*y', y'*y ∈ I$. By Theorem 4(iii), (UP2), and (fUPI3), $(x*y)*(x*y') = x*(y*(0*y')) = x*(y*y') ∈ I$ and $(x*y')*(x*y) = x*(y'(0*y)) = x*(y*0) ∈ I$. Thus, $x·y ∼_I x'·y'$. Similarly, $(x·y')*(x·y') = (x*(0*x'))·y' = (x'·x)·y' ∈ I$. Thus, $x·y ∼_I x'·y'$.

By transitivity, $x·y ∼_I x'·y'$. By Lemma 1, $[x]_I*[y]_I = [x*y]_I = [x'·y']_I = [x']_I*[y']_I$.

Let $[x]_I, [y]_I, [z]_I ∈ X/I$. Since $(X, ·)$ is a semigroup, then

$$[x]_I·([y]_I·[z]_I) = [x]_I·([y·z]_I)$$

$$= [x·(y·z)]_I$$

$$= ((x·y)·z]_I$$

$$= [x·y]_I·[z]_I$$

Hence, $(X/I, ·)$ is a semigroup. Moreover, by distributive property on $X$,

$$[x]_I·([y]_I*[z]_I) = [x]_I·[y·z]_I$$

$$= [x·(y·z)]_I$$

$$= ((x·y)·z]_I$$

$$= [x·y]_I·[z]_I$$

and

$$([x]_I*[y]_I)·[z]_I = [x·y]_I·[z]_I$$

$$= [x·(y·z)]_I$$

$$= ((x·y)·z]_I$$

$$= [x·z]_I·[y·z]_I$$

$$= ([x·z]_I·[y·z]_I) = ([x]_I·[z]_I)·([y]_I·[z]_I).$$

Thus, the distributive property holds on $X/I$. Therefore, $(X/I; *, ·, [0]_I)$ is an $f$-UP-semigroup. Suppose $X$ is commutative. Then $x·y = y·x$ for all $x, y ∈ X$. Let $[x]_I, [y]_I ∈ X/I$. Then $[x]_I·[y]_I = [x·y]_I = [y·x]_I = [y·x]_I$. Hence, $X/I$ is commutative. If $X$ has unity 1, then $X/I$ has unity $[1]_I$ since $[x]_I·[1]_I = [x·1]_I = [x]_I$ and $[1]_I·[x]_I = [1·x]_I = [x]_I$ for any $x ∈ X$.

The $f$-UP-semigroup $(X/I; *, ·, [0]_I)$ in Theorem 13 is called the quotient $f$-UP-semigroup of $X$ by $I$. 

5. Conclusion

This paper investigated fully UP-semigroups, a new class of algebra related to UP-algebras and semigroups, which was introduced by A. Iampan [4] in 2018. It established some structural properties of \( f \)-UP-semigroups. It also introduced and examined \( f \)-UP-fields, \( f \)-UP-domains, \( f \)-UP-ideals, and quotient \( f \)-UP-semigroups. Moreover, the relationship between an \( f \)-UP-field and an \( f \)-UP-domain is determined. In the subsequent study, we introduce and investigate homomorphisms on \( f \)-UP-semigroups, which lead to the isomorphism theorems on \( f \)-UP-semigroups.

Acknowledgements

This research is funded by the Philippine Department of Science and Technology-Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP) and the Mindanao State University-Iligan Institute of Technology. The authors wish to express their sincere thanks to the referees for their valuable suggestions for the improvement of this paper.

References


