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Generalized Quasilinearization using coupled lower and upper solutions for periodic boundary value problem of an integro differential equation

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Abstract. In this paper we first develop the method of generalized quasilinearization for initial value problem of an integro differential equation and then use it to develop quasilinearization for the periodic boundary value problem of the integro differential equation by using the coupled lower and upper solutions of Type-I.

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1. Introduction

Integro differential equations [1] arise quite frequently as mathematical models in various disciplines of physical, social and biological sciences and engineering. Models involving integro differential equations can be found in unsteady aerodynamics and aero-elastic phenomena etc. The qualitative theory of integro differential equations deals with existence and uniqueness of solutions, stability of solutions etc. The existence and uniqueness results are studied using various approaches like fixed point theory and iterative techniques.

There are various iterative techniques for solving integro differential equations. Some of the iterative methods are monotone iterative technique, quasilinearization and their generalizations. The monotone iterative technique and quasilinearization are two iterative techniques that are widely used to obtain existence and uniqueness results of various types of differential equations. Both monotone iterative technique and quasilinearization [2, 3, 4, 5] along with the method of upper and lower solutions yield monotone iterates which are solutions of certain linear differential equations obtained from the hypothesis of the given problem. These iterates converge to a solution of the original problem.

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The monotone iterative technique had undergone various extensions and generalizations. The right hand side of the problem was considered as a sum of a nondecreasing and nonincreasing function. This gave rise to various notions of coupled solutions and much work has been done in this setup for various types of differential equations.

In [6, 7] the authors obtained the existence of solutions for an integro differential equation with periodic boundary condition using monotone iterative technique. This was a very interesting result because of two reasons: 1) no additional lemmas were needed to prove the result 2) no extra conditions were needed for uniqueness, as the uniqueness of solution was generated by the method itself.

This led to a spurt of publications in monotone iterative technique for various types of differential equations [8, 9].

This idea has been extended to quasilinearization and generalized quasilinearisation had been developed for periodic boundary value problem of a graph differential equation and a matrix differential equation through natural upper and lower solutions [10].

In [11] it was observed that quasilinearization for periodic boundary value problem through coupled lower and upper solutions of the initial value problem can be obtained with certain restrictions.

In this paper, using the approach given in [11] we develop quasilinearization technique, using coupled lower and upper solutions, for initial value problem of an integro differential equation and using this result to obtain existence and uniqueness of solutions for periodic boundary value problem of an integro differential equation.

2. Preliminaries

Consider the periodic boundary value problem of an integro differential equation given by

$$x' = f_1(t, x, Sx) + f_2(t, x, Sx), \tag{1}$$

$$x(0) = x(T). \tag{2}$$

To develop the method of the quasilinearization technique (1) and (2), we first develop quasilinearization technique for the corresponding initial value problem of an integro differential equation given by

$$x' = f_1(t, x, Sx) + f_2(t, x, Sx), \tag{3}$$

$$x(0) = x_0, \tag{4}$$

where $f_1, f_2 \in C[I \times R^n \times R^n, R^n]$, $Sx(t) = \int\limits_0^t K(t,s)x(s)ds$, with $K \in C[I \times I, R_+]$ and I=[0,T].

To do so we first define the various types of lower and upper solution for (3) and (4),

Definition 1. *Let $\alpha_0, \beta_0 \in C^1[I, R^n]$. Then α_0, β_0 are said to be*
(a) natural lower and upper solutions of (3) *and* (4) *if*

$$\left.\begin{array}{r} \alpha_0' \leq f_1(t, \alpha_0, S\alpha_0) + f_2(t, \alpha_0, S\alpha_0), \quad \alpha_0(0) \leq x_0, \\ \beta_0' \geq f_1(t, \beta_0, S\beta_0) + f_2(t, \beta_0, S\beta_0), \ \ \beta_0(0) \geq x_0, \quad t \in I; \end{array}\right\} \tag{5}$$

(b) coupled lower and upper solutions of Type I of (3) *and* (4) *if*

$$\left.\begin{array}{ll}\alpha_0' \leq f_1(t,\alpha_0,S\alpha_0)+f_2(t,\beta_0,S\beta_0), & \alpha_0(0)\leq x_0,\\ \beta_0' \geq f_1(t,\beta_0,S\beta_0)+f_2(t,\alpha_0,S\alpha_0), & \beta_0(0)\geq x_0, \quad t\in I;\end{array}\right\} \tag{6}$$

(c) coupled lower and upper solutions of Type II of (3) *and* (4) *if*

$$\left.\begin{array}{ll}\alpha_0' \leq f_1(t,\beta_0,S\beta_0)+f_2(t,\alpha_0,S\alpha_0), & \alpha_0(0)\leq x_0,\\ \beta_0' \geq f_1(t,\alpha_0,S\alpha_0)+f_2(t,\beta_0,S\beta_0), & \beta_0(0)\geq x_0, \quad t\in I;\end{array}\right\} \tag{7}$$

(d) coupled lower and upper solutions of Type III of (3) *and* (4) *if*

$$\left.\begin{array}{ll}\alpha_0' \leq f_1(t,\beta_0,S\beta_0)+f_2(t,\beta_0,S\beta_0), & \alpha_0(0)\leq x_0,\\ \beta_0' \geq f_1(t,\alpha_0,S\alpha_0)+f_2(t,\alpha_0,S\alpha_0), & \beta_0(0)\geq x_0, \quad t\in I.\end{array}\right\} \tag{8}$$

We observe that whenever we have $\alpha(t)\leq\beta(t),\ t\in I,\ f_1(t,x,Sx)$ is nondecreasing in x and y and $f_2(t,x,Sx)$ is nonincreasing in x and y for each $t\in I$, the lower and upper solutions defined by (5) and (8) reduce to (6) and (7) consequently, hence it is sufficient to investigate the cases (6) and (7).

3. Generalized quasilinearization for initial value problem of an integro differential equation.

In this section we develop the method of generalized quasilinearization for the initial value problem of an integro differential equation and use it in the next section.

we first state a known result from [1] corresponding to an integro differential equation which is useful in developing a sequence of iterates to be constructed while developing the quasilinearization.

Lemma 1. Let $p\in C^1[I,R]$, where $I=[0,T]$ is such that and

$$p\,'(t)\leq -Mp(t)-NSp(t)\ \ on\ I,\quad p(0)\leq 0, \tag{9}$$

where $M>0,\ N\geq 0$ are constants such that

$$Nk_1T(e^{MT}-1)\leq M, \tag{10}$$

where $k_1 = max_{t\in I}\ K(t,s)$. Then $p(t)\leq 0$ on I.

To prove the main theorem we need the following assumptions which are listed below for convenience.

H_1 : (i) Second order Frechet derivatives of $f_1(t,x,\xi)$, $f_2(t,x,\xi)$ with respect to all variables exist and are bounded;

(ii) $f_1(t,x,\xi)$ is convex in x, ξ ;

(iii) $f_{1x}(t,x,\xi)$ is nondecreasing in ξ for each (t,x);

(iv) f_1 is nondecreasing function in x, ξ for each $t \in I$ and f_2 is nonincreasing function in x, ξ for each $t \in I$.

H_2 : (i) $-M_1 \le f_{1x}(t,x,\xi) \le -M, \quad 0 < M < M_1$;

(ii) $-M_2 \le f_{1\xi}(t,x,\xi) \le -N, \quad 0 < N < M_2$;

(iii) $Nk_1T < M$;

where $M > 0, \quad N \ge 0$.

H_3 : α_0, β_0 are coupled lower and upper solutions of (3) and (4).

H_4 : (i) $f_1(t,x,Sx) \ge f_1(t,y,Sy) + f_{1x}(t,y,Sy)(x-y) + f_{1\xi}(t,y,Sy)(Sx-Sy)$;

(ii) $|f_{1x}(t,x,Sx) - f_{1x}(t,y,Sy)| \le L_1(x-y) + M_1(Sx-Sy), \quad L_1, M_1 \ge 0$.

Theorem 1. *Suppose that the assumptions H_1 to H_4 are satisfied. Then there exists monotone sequence $\{\alpha_n\}$, such that $\alpha_n \to \rho$, as $n \to \infty$ uniformly and monotonically to the unique solution $\rho = u$ of an integro differential equation* (3) *and* (4)*on I and the convergence is quadratic.*

Proof: In order to construct a sequence of lower iterates that converge to the solution of the IVP we fix the upper solution β_0. Now consider the following linear problem for $n = 0, 1, 2, 3, ...$

$$\left.\begin{aligned} \alpha'_{n+1} = f_1(t,\alpha_n,S\alpha_n) + f_{1x}(t,\alpha_n,S\alpha_n)[\alpha_{n+1}-\alpha_n] + f_{1\xi}(t,\alpha_n,S\alpha_n)[S\alpha_{n+1}-S\alpha_n] \\ +f_2(t,\beta_0,S\beta_0), \end{aligned}\right\} \tag{11}$$

$$\alpha_{n+1}(0) = x_0. \tag{12}$$

Since the above equation is a linear integro differential equation, it has unique solution $\alpha_{n+1}(t)$ on I for each n. Now we claim that

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le ... \le \alpha_{n-1} \le \alpha_n \le ... \le \beta_0 \tag{13}$$

on I.

We begin by setting $p = \alpha_0 - \alpha_1$. Then

$$\begin{aligned} p' &= \alpha_0' - \alpha_1' \\ &\leq \{f_1(t,\alpha_0,S\alpha_0) + f_2(t,\beta_0,S\beta_0)\} \\ &\quad -\{f_1(t,\alpha_0,S\alpha_0) + f_{1x}(t,\alpha_0,S\alpha_0)[\alpha_1-\alpha_0] \\ &\quad + f_{1\xi}(t,\alpha_0,S\alpha_0)[S\alpha_1 - S\alpha_0] + f_2(t,\beta_0,S\beta_0)\} \\ &\leq f_{1x}(t,\alpha_0,S\alpha_0)[\alpha_0-\alpha_1] + f_{1\xi}(t,\alpha_0,S\alpha_0)[S\alpha_0 - S\alpha_1] \end{aligned}$$

$$p'(t) \leq -Mp(t) - NSp(t).$$

Also $p(0) = \alpha_0(0) - \alpha_1(0) \leq 0$. Hence by Lemma 1 we have $p(t) \leq 0$. So $\alpha_0 \leq \alpha_1$ on I. Next, we show that $\alpha_1 \leq \alpha_2$ on I. For this set $p = \alpha_1 - \alpha_2$. Then

$$\begin{aligned} p' &= \alpha_1' - \alpha_2' \\ &\leq f_1(t,\alpha_1,S\alpha_1) - \{f_1(t,\alpha_1,S\alpha_1) + f_{1x}(t,\alpha_1,S\alpha_1)[\alpha_2-\alpha_1] \\ &\quad + f_{1\xi}(t,\alpha_1,S\alpha_1)[S\alpha_2 - S\alpha_1]\} \\ &= f_{1x}(t,\alpha_1,S\alpha_1)[\alpha_1-\alpha_2] + f_{1\xi}(t,\alpha_1,S\alpha_1)[S\alpha_1 - S\alpha_2] \\ &\leq -Mp(t) - NSp(t). \end{aligned}$$

Also $p(0) = \alpha_1(0) - \alpha_2(0) = 0$. Hence by Lemma 1 we have $p(t) \leq 0$. Thus $\alpha_1 \leq \alpha_2$ on I. Now we show $\alpha_1 \leq \beta_0$ on I by setting $p = \alpha_1 - \beta_0$. Then,

$$\begin{aligned} p' &= \alpha_1' - \beta_0' \\ &\leq \{f_1(t,\alpha_0,S\alpha_0) - f_1(t,\beta_0,S\beta_0)\} + \{f_{1x}(t,\alpha_0,S\alpha_0)[\alpha_1-\alpha_0] \\ &\quad + f_{1\xi}(t,\alpha_0,S\alpha_0)[S\alpha_1 - S\alpha_0]\} + \{f_2(t,\beta_0,S\beta_0) - f_2(t,\alpha_0,S\alpha_0)\} \\ &\leq f_{1x}(t,\alpha_0,S\alpha_0)[\alpha_1-\beta_0] + f_{1\xi}(t,\alpha_0,S\alpha_0)[S\alpha_1 - S\beta_0] \\ &\leq -Mp(t) - NSp(t). \end{aligned}$$

Also $p(0) = \alpha_1(0) - \beta_0(0) \leq 0$. Hence by Lemma 1 we have $p(t) \leq 0$. Which means that $\alpha_1 \leq \beta_0$ on I. Similarly we can show $\alpha_2 \leq \beta_0$ on I.

Thus

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \beta_0,$$

on I.

Now we assume that the result holds for $n = k$ and prove it for $n = k+1$. We now consider the following linear integro differential equation,

$$\alpha_{k+1}' = f_1(t,\alpha_k,S\alpha_k) + f_{1x}(t,\alpha_k,S\alpha_k)[\alpha_{k+1}-\alpha_k] + f_{1\xi}(t,\alpha_k,S\alpha_k)[S\alpha_{k+1}-S\alpha_k] + f_2(t,\beta_0,S\beta_0),$$

$$\alpha_{k+1}(0) = x_0.$$

The above linear integro differential equation has the unique solution α_{k+1} where α_k *and* β_0 are known lower and upper solutions of (3) and (4). Further α_k is the solution of the linear integro differential equation

$$\alpha_k' = f_1(t, \alpha_{k-1}, S\alpha_{k-1}) + f_{1x}(t, \alpha_{k-1}, S\alpha_{k-1})[\alpha_k - \alpha_{k-1}]$$
$$+ f_{1\xi}(t, \alpha_{k-1}, S\alpha_{k-1})[S\alpha_k - S\alpha_{k-1}] + f_2(t, \beta_0, S\beta_0),$$
$$\alpha_k(0) = x_0.$$

We now consider $p = \alpha_k - \alpha_{k+1}$

$$p' = \alpha_k' - \alpha_{k+1}'$$
$$\leq \{f_{1x}(t, \alpha_k, S\alpha_k)[\alpha_k - \alpha_{k+1}] + f_{1\xi}(t, \alpha_k, S\alpha_k)[S\alpha_k - S\alpha_{k+1}]$$
$$\leq -Mp(t) - NSp(t).$$

Also $p(0) = \alpha_k(0) - \alpha_{k+1}(0) = 0$. Hence by Lemma 1 we have $p(t) \leq 0$. Thus $\alpha_k \leq \alpha_{k+1}$ on I. To show $\alpha_{k+1} \leq \beta_0$ on I.

Set $p = \alpha_{k+1} - \beta_0$

$$p' = \alpha_{k+1}' - \beta_0'$$
$$\leq \{f_{1x}(t, \alpha_k, S\alpha_k)[\alpha_k - \beta_0] + f_{1\xi}(t, \alpha_k, S\alpha_k)[S\alpha_k - S\beta_0]\}]$$
$$+ \{f_{1x}(t, \alpha_k, S\alpha_k)[\alpha_{k+1} - \alpha_k] + f_{1\xi}(t, \alpha_k, S\alpha_k)[S\alpha_{k+1} - S\alpha_k]\}$$
$$+ \{f_2(t, \beta_0, S\beta_0) - f_2(t, \alpha_0, S\alpha_0)\}$$
$$\leq \{f_{1x}(t, \alpha_k, S\alpha_k)[\alpha_{k+1} - \alpha_k] + f_{1\xi}(t, \alpha_k, S\alpha_k)[S\alpha_{k+1} - S\alpha_k]\}$$
$$\leq -Mp(t) - NSp(t).$$

Also $p(0) = \alpha_{k+1}(0) - \beta_0(0) \leq 0$. Hence by Lemma 1 we have $p(t) \leq 0$ which implies that $\alpha_{k+1} \leq \beta_0$ on I. Thus

$$\alpha_k \leq \alpha_{k+1} \leq \beta_0,$$

on I. Now using the principle of mathematical induction, we deduce the relation (13) and our claim holds. Also from relation (13), we can seen that the sequences are uniformly bounded. Since f_1, f_2 are uniformly bounded, the sequence $\{\alpha_n\}$ is equicontinuous on $[0, T]$ and therefore by using Ascoli-Arzela Theorem, there exists a subsequence $\{\alpha_{n_k}\}$ that converges uniformly on $[0, T]$. In view of (13) it also follows that the entire sequence $\{\alpha_n\}$ converges uniformly to ρ. Since f_{1_x} exists and is bounded on $[0, T]$, we obtain that f_1 is Lipschitz and hence the solution u is unique.

To show that the convergence is quadratic, we begin by writing $p_{n+1} = u - \alpha_{n+1}$ and consider

$$p_{n+1}' = u' - \alpha_{n+1}'$$

$$
\begin{aligned}
&= [f_1(t,u,Su) + f_2(t,u,Su)] \\
&\quad - [\{f_1(t,\alpha_n,S\alpha_n) + f_{1x}(t,\alpha_n,S\alpha_n)[\alpha_{n+1}-\alpha_n] \\
&\quad + f_{1\xi}(t,\alpha_n,S\alpha_n)[S\alpha_{n+1}-S\alpha_n] + f_2(t,\beta_0,S\beta_0)\}]
\end{aligned}
$$

$$
p'_{n+1} \leq A + B + f_{1x}(t,\alpha_n,S\alpha_n)p_{n+1} + f_{1\xi}(t,\alpha_n,S\alpha_n)Sp_{n+1} \tag{14}
$$

where

$$
\begin{aligned}
A &= f_1(t,u,Su) - f_1(t,\alpha_n,Su) - f_{1x}(t,\alpha_n,S\alpha_n)[u-\alpha_n]; \\
B &= f_1(t,\alpha_n,Su) - f_1(t,\alpha_n,S\alpha_n) - f_{1\xi}(t,\alpha_n,S\alpha_n)[Su-S\alpha_n].
\end{aligned}
$$

Our aim is to simplify each of the term A,B and substitute in (14). In this direction, consider

$$
\begin{aligned}
A &= f_1(t,u,Su) - f_1(t,\alpha_n,Su) - f_{1x}(t,\alpha_n,S\alpha_n)[u-\alpha_n]; \\
&= [f_{1x}(t,\eta_1,Su)(u-\alpha_n) \; - \; f_{1x}(t,\alpha_n,S\alpha_n)](u-\alpha_n) \\
&= [f_{1x}(t,\eta_1,Su) \; - \; f_{1x}(t,\alpha_n,S\alpha_n)](u-\alpha_n) \\
&= [f_{1x}(t,\eta_1,Su) - f_{1x}(t,\alpha_n,Su) \\
&\quad + f_{1x}(t,\alpha_n,Su) - [f_{1x}(t,\alpha_n,S\alpha_n)]p_n(t) \\
&= f_{1xx}(t,\tau_1,Su)p_n[\eta_1-\alpha_n] \\
&\quad + \int_0^1 f_{1x\xi}(t,\alpha_n,sSu+(1-s)S\alpha_n)[Su-S\alpha_n]p_n ds \\
&\leq f_{1xx}(t,\tau_1,Su){p_n}^2 + \int_0^1 f_{1x\xi}(t,\alpha_n,sSu+(1-s)S\alpha_n)[Sp_n]p_n ds \\
&\leq l_1|p_n|^2 + l_2k_1T|p_n||p_n|\int_0^1 ds \\
&\leq l_1|p_n|^2 + l_2k_1T|p_n|^2
\end{aligned}
$$

Next consider

$$
\begin{aligned}
B &= f_1(t,\alpha_n,Su) - f_1(t,\alpha_n,S\alpha_n) - f_{1\xi}(t,\alpha_n,S\alpha_n)[Su-S\alpha_n]; \\
&= \int_0^1 [f_{1\xi}(t,\alpha_n,s(Su)+(1-s)S\alpha_n) - [f_{1\xi}(t,\alpha_n,S\alpha_n)](Su-S\alpha_n)ds.
\end{aligned}
$$

Let $\eta_2(s) = s(Su) + (1-s)S\alpha_n$. Then

$$
\begin{aligned}
B &= \int_0^1 [f_{1\xi}(t,\alpha_n,\eta_2(s)) - f_{1\xi}(t,\alpha_n,S\alpha_n)](Su-S\alpha_n)ds \\
&= \int_0^1\int_0^1 f_{1\xi\xi}(t,\alpha_n,\sigma\;\eta_2(s)+(1-\sigma)S\alpha_n)s(Sp_n)(Su-S\alpha_n)dsd\sigma \\
&= \int_0^1\int_0^1 f_{1\xi\xi}(t,\alpha_n,\sigma\;\eta_2(s)+(1-\sigma)S\alpha_n)s(Sp_n)(Sp_n)dsd\sigma
\end{aligned}
$$

$$= \int_0^1 \int_0^1 f_{1\xi\xi}(t, \alpha_n, \sigma\ \eta_2(s) + (1-\sigma)S\alpha_n)s(Sp_n)^2 ds d\sigma$$
$$\leq l_3 k_1^2 T^2 |p_n^2| \int_0^1 \int_0^1 s\ ds d\sigma$$
$$\leq l_3 k_1^2 T^2 |p_n^2|$$

$$p'_{n+1} \leq \{l_1|p_n|^2 + l_2 k_1 T|p_n|^2\} + \{l_3 k_1^2 T^2 |p_n^2|\} - Mp_{n+1}(t) - NSp_{n+1}(t)$$
$$\leq l - Mp_{n+1}(t) - NSp_{n+1}(t)$$

where $l = \{l_1|p_n|^2 + l_2 k_1 T|p_n|^2\} + \{l_3 k_1^2 T^2 |p_n^2|\}$. Now multiplying throughout by e^{Mt} and setting $\tilde{p}_{n+1} = e^{Mt} p_{n+1}$, we get

$$(p_{n+1}(t)e^{Mt})' \leq Nk_1 \int_0^{h_2} p_{n+1}(s) e^{Mt} ds + le^{Mt}$$
$$\leq Nk_1 \int_0^t \tilde{p}_{n+1}(s) e^{M(t-s)} ds + le^{Mt}$$
$$= w'(t)\ (say)$$

Then choosing w(0)=0, we get $\tilde{p}_{n+1} \leq w(t)$ on I. Clearly $w'(t) \geq 0$, which means that $w(t)$ is nondecreasing on I. Now for $t \in I$,

$$w(t) \leq Nk_1 \int_0^t \int_0^z \tilde{p}_{n+1}(s) e^{M(z-s)} ds dz + \int_0^t le^{Mu} du$$
$$\leq Nk_1 \int_0^t \int_0^z w(z) e^{M(z-s)} ds dz + l[max_{[0,T]}\{\tfrac{e^{Mt}}{M} - \tfrac{1}{M}\}]$$
$$\leq Nk_1 \int_0^t \int_0^z w(z) e^{M(z-s)} ds dz + l[max_{[0,T]}\{\tfrac{e^{Mt}}{M}\}]$$
$$\leq Nk_1 \int_0^t \int_0^z w(z) e^{M(z-s)} ds dz + l\{\tfrac{e^{MT}}{M}\}$$

by setting $c_2 = l\{\frac{e^{MT}}{M}\}$ and $c_1 = \frac{N}{M} k_1 e^{Mt}$ we get

$$w(t) \leq c_1 \int_0^t w(z) dz + c_2.$$

Now by using Gronwall's inequality we get

$$w(t) \leq c_2 e^{\int_0^t c_1 ds}$$

$$\leq c_2 e^{c_1 t}$$

$$\leq c_2 e^{c_1 T}.$$

Hence

$$\tilde{p}_{n+1}(t) \leq w(t) \leq max_{[0,T]} l(t)[e^{Mt}]e^{c_1 T}$$

$$\tilde{p}_{n+1}(t) \leq w(t) \leq max_{[0,T]} l(t)[e^{(M+c_1)t}]$$

$$|p_{n+1}| \leq (e^{(N_1+c_1)T})[M|p_n|^2],$$

Therefore the sequence $\{\alpha_n\}$ converges quadratically on I. Hence the theorem.

4. Generalized quasilinearization for periodic boundary value problem

In this section an existence and uniqueness result is obtained for an PBVP of an integro differential equation using the method of generalized quasilinearization. For this first we define the various types of lower and upper solutions for the periodic boundary value problem of an integro differential equation given by

$$x' = f_1(t, x, Sx) + f_2(t, x, Sx), \tag{15}$$

$$x(0) = x(T), \tag{16}$$

where $f_1, f_2 \in C[I \times R^n \times R^n, R^n]$, $Sx(t) = \int\limits_0^t K(t,s)x(s)ds$, and $K \in C[I \times I, R_+]$, I=[0,T].

Definition 2. *Let $\alpha_0, \beta_0 \in C^1[I, R^n]$. Then α_0, β_0 are said to be*
(a) natural lower and upper solutions of (15) *and* (16) *if*

$$\left.\begin{array}{r} \alpha_0' \leq f_1(t, \alpha_0, S\alpha_0) + f_2(t, \alpha_0, S\alpha_0), \quad \alpha_0(0) \leq \alpha_0(T), \\ \beta_0' \geq f_1(t, \beta_0, S\beta_0) + f_2(t, \beta_0, S\beta_0), \ \ \beta_0(0) \geq \beta_0(T), \quad t \in I; \end{array}\right\} \tag{17}$$

(b) coupled lower and upper solutions of Type I of (15) *and* (16) *if*

$$\left.\begin{array}{r} \alpha_0' \leq f_1(t, \alpha_0, S\alpha_0) + f_2(t, \beta_0, S\beta_0), \quad \alpha_0(0) \leq \alpha_0(T), \\ \beta_0' \geq f_1(t, \beta_0, S\beta_0) + f_2(t, \alpha_0, S\alpha_0), \quad \beta_0(0) \geq \beta_0(T), \quad t \in I; \end{array}\right\} \tag{18}$$

(c) coupled lower and upper solutions of Type II of (15) *and* (16) *if*

$$\left.\begin{array}{r} \alpha_0' \leq f_1(t, \beta_0, S\beta_0) + f_2(t, \alpha_0, S\alpha_0), \quad \alpha_0(0) \leq \alpha_0(T), \\ \beta_0' \geq f_1(t, \alpha_0, S\alpha_0) + f_2(t, \beta_0, S\beta_0), \quad \beta_0(0) \geq \beta_0(T), \quad t \in I; \end{array}\right\} \tag{19}$$

(d) coupled lower and upper solutions of Type III of (15) *and* (16) *if*

$$\left.\begin{array}{r} \alpha_0' \leq f_1(t, \beta_0, S\beta_0) + f_2(t, \beta_0, S\beta_0), \quad \alpha_0(0) \leq \alpha_0(T), \\ \beta_0' \geq f_1(t, \alpha_0, S\alpha_0) + f_2(t, \alpha_0, S\alpha_0), \quad \beta_0(0) \geq \beta_0(T), \quad t \in I. \end{array}\right\} \tag{20}$$

Now we will prove the following theorem related to coupled lower and upper solutions of Type I and we develop the generalized quasilinearization method for the periodic boundary value problem of an integro diffential equation via the initial value problem approach.

Theorem 2. *Suppose that the assumptions of Theorem 1 are satisfied. Then there exists monotone sequence* $\{\alpha_n\}$*, such that* $\alpha_n \to \rho$*, as* $n \to \infty$ *uniformly and monotonically to the unique solution* $\rho = u$ *for PBVP of an integro differential equation* (15) *and* (16)*on* I *and the convergence is quadratic.*

Proof: In order to construct a sequence of lower and upper iterates that converge to the solution of the PBVP we fix the upper solution β_0. Now consider the following linear problem for n= 0,1,2,3...

$$\alpha'_{n+1} = f_1(t, \alpha_n, S\alpha_n) + f_{1x}(t, \alpha_n, S\alpha_n)[\alpha_{n+1} - \alpha_n] + f_{1\xi}(t, \alpha_n, S\alpha_n)[S\alpha_{n+1} - S\alpha_n] + f_2(t, \beta_0, S\beta_0),$$

$$\alpha_{n+1}(0) = \alpha_{n(}T).$$

Since the above equation is a linear integro differential equation, so it has unique solution $\alpha_{n+1}(t)$ on I. Now we claim that

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_{n-1} \leq \alpha_n \leq ... \leq \beta_0 \tag{21}$$

on I.

We begin by setting $p = \alpha_0 - \alpha_1$. Then

$$\begin{aligned} p' &= \alpha'_0 - \alpha'_1 \\ &= f_{1x}(t, \alpha_0, S\alpha_0)p(t) + f_{1\xi}(t, \alpha_0, S\alpha_0)Sp(t) \\ p'(t) &\leq -Mp(t) - NSp(t) \end{aligned}$$

Also $p(0) = \alpha_0(0) - \alpha_1(0) \leq 0$. Hence by Lemma 1 we have $p(t) \leq 0$. Thus $\alpha_0 \leq \alpha_1$ on I.

Now we show $\alpha_1 \leq \beta_0$ on I by setting

$p = \alpha_1 - \beta_0$. Then,

$$\begin{aligned} p' &= \alpha'_1 - \beta'_0 \\ &\leq \{f_1(t, \alpha_0, S\alpha_0) - f_1(t, \beta_0, S\beta_0)\} + \{f_{1x}(t, \alpha_0, S\alpha_0)[\alpha_1 - \alpha_0] \\ &\quad + f_{1\xi}(t, \alpha_0, S\alpha_0)[S\alpha_1 - S\alpha_0]\} + \{f_2(t, \beta_0, S\beta_0) - f_2(t, \alpha_0, S\alpha_0)\} \\ &\leq f_{1x}(t, \alpha_0, S\alpha_0)[\alpha_1 - \beta_0] + f_{1\xi}(t, \alpha_0, S\alpha_0)[S\alpha_1 - S\beta_0] \\ &\leq -Mp(t) - NSp(t). \end{aligned}$$

Also $p(0) = \alpha_1(0) - \beta_0(0) \leq 0$. Hence by Lemma 1 we have $p(t) \leq 0$. Hence $\alpha_1 \leq \beta_0$ on I. Thus

$$\alpha_0 \leq \alpha_1 \leq \beta_0$$

on I.

Now assuming that the result is true for $n = k$ and prove it for $n = k + 1$. In order to prove our claim we consider the following linear integro differential equation.

$$\alpha'_{k+1} = f_1(t, \alpha_k, S\alpha_k) + f_{1x}(t, \alpha_k, S\alpha_k)[\alpha_{k+1} - \alpha_k] + f_{1\xi}(t, \alpha_k, S\alpha_k)[S\alpha_{k+1} - S\alpha_k] + f_2(t, \beta_0, S\beta_0),$$

$$\alpha_{k+1}(0) = \alpha_k(T).$$

The above linear integro differential equation has unique solution α_{k+1}, where α_k and β_0 are known lower and upper solutions of (15) and (16). Further α_k is the solution of the linear integro differential equation

$$\alpha'_k = f_1(t, \alpha_{k-1}, S\alpha_{k-1}) + f_{1x}(t, \alpha_{k-1}, S\alpha_{k-1})[\alpha_k - \alpha_{k-1}] + f_{1\xi}(t, \alpha_{k-1}, S\alpha_{k-1})[S\alpha_k - S\alpha_{k-1}]$$

$$+ f_2(t, \beta_0, S\beta_0),$$
$$\alpha_k(0) = \alpha_{k-1}(T).$$

We now claim that $\alpha_k \le \alpha_{k+1}$ on I. For this set $p = \alpha_k - \alpha_{k+1}$

$$\begin{aligned} p' &= \alpha'_k - \alpha'_{k+1} \\ &\le \{ f_{1x}(t, \alpha_k, S\alpha_k)[\alpha_k - \alpha_{k+1}] + f_{1\xi}(t, \alpha_k, S\alpha_k)[S\alpha_k - S\alpha_{k+1}] \\ &\le -Mp(t) - NSp(t) \end{aligned}$$

Also $p(0) = \alpha_k(0) - \alpha_{k+1}(0) \le 0$. Then by Lemma 1 Thus $p(t) \le 0$. So $\alpha_k \le \alpha_{k+1}$ on I. Next to show $\alpha_{k+1} \le \beta_0$ on I,
Set $p = \alpha_{k+1} - \beta_0$

$$\begin{aligned} p' &= \alpha'_{k+1} - \beta'_0 \\ &\le \{ f_{1x}(t, \alpha_k, S\alpha_k)[\alpha_{k+1} - \alpha_k] + f_{1\xi}(t, \alpha_k, S\alpha_k)[S\alpha_{k+1} - S\alpha_k] \} \\ &\le -Mp(t) - NSp(t) \end{aligned}$$

Also $p(0) = \alpha_{k+1}(0) - \beta_0(0) \le 0$. Using Lemma 1 we get $p(t) \le 0$. Which means that $\alpha_{k+1} \le \beta_0$ on I.

Now using the principle of mathematical induction, we deduce the relation (21) and our claim holds. Also from relation (21), we can seen that the sequences are uniformly bounded. Since f_1, f_2 are uniformly bounded so the sequence $\{\alpha_n\}$ equicontinuous on $[0, T]$ and therefore by using Ascoli-Arzela Theorem, there exists subsequence $\{\alpha_{n_k}\}$ that converges uniformly on $[0, T]$. In view of (21) it also follows that the entire sequence $\{\alpha_n\}$ converges uniformly to ρ. Since f_{1_x} exists and is bounded on $[0, T]$, we obtain that f_1 is Lipschitz and hence the solution u is unique.

To show that the convergence is quadratic, we begin by writing $p_{n+1} = u - \alpha_{n+1}$. Then

$$\begin{aligned} p'_{n+1} &= u' - \alpha'_{n+1} \\ &\leq [f_1(t,u,Su) + f_2(t,u,Su)] \\ &\quad - [\{f_1(t,\alpha_n,S\alpha_n) + f_{1x}(t,\alpha_n,S\alpha_n)[\alpha_{n+1}-\alpha_n] \\ &\quad + f_{1\xi}(t,\alpha_n,S\alpha_n)[S\alpha_{n+1} - S\alpha_n] + f_2(t,\beta_0,S\beta_0)\}] \end{aligned}$$

$$\begin{aligned} &= A + B + f_x(t,\alpha_n,S\alpha_n)p_{n+1} \\ &\quad + f_\xi(t,\alpha_n,S\alpha_n,\alpha_{nt},\alpha_n^t)Sp_{n+1} \end{aligned}$$

$$= A + B + f_{1x}(t,\alpha_n,S\alpha_n)p_{n+1} + f_{1\xi}(t,\alpha_n,S\alpha_n)Sp_{n+1} \tag{22}$$

where

$$\begin{aligned} A &= f_1(t,u,Su) - f_1(t,\alpha_n,Su) - f_{1x}(t,\alpha_n,S\alpha_n)[u-\alpha_n]; \\ B &= f_1(t,\alpha_n,Su) - f_1(t,\alpha_n,S\alpha_n) - f_{1\xi}(t,\alpha_n)[Su - S\alpha_n]. \end{aligned}$$

Our aim is to simplify each of the term A,B and substitute in (22). In this direction, consider

$$A = f_1(t,u,Su) - f_1(t,\alpha_n,Su) - f_{1x}(t,\alpha_n,S\alpha_n)[u-\alpha_n];$$

$$\leq f_{1xx}(t,\tau_1,Su){p_n}^2 + \int_0^1 f_{1x\xi}(t,\alpha_n,sSu + (1-s)S\alpha_n)[Sp_n]p_n ds$$

$$\leq l_1|p_n|^2 + l_2k_1T|p_n||p_n|\int_0^1 ds$$

$$\leq l_1|p_n|^2 + l_2k_1T|p_n|^2$$

Next consider

$$\begin{aligned} B &= f_1(t,\alpha_n,Su) - f_1(t,\alpha_n,S\alpha_n) - f_{1\xi}(t,\alpha_n)[Su - S\alpha_n]; \\ &= \int_0^1 [f_{1\xi}(t,\alpha_n,s(Su) + (1-s)S\alpha_n) - [f_{1\xi}(t,\alpha_n,S\alpha_n)](Su - S\alpha_n)ds. \end{aligned}$$

Let $\eta_2(s) = s(Su) + (1-s)S\alpha_n$. Then

$$B = \int_0^1 [f_{1\xi}(t,\alpha_n,\eta_2(s)) - f_{1\xi}(t,\alpha_n,S\alpha_n)](Su - S\alpha_n)ds$$

$$\leq l_3k_1^2T^2|p_n^2|\int_0^1\int_0^1 s\ dsd\sigma$$

$$\leq l_3k_1^2T^2|p_n^2|$$

$$p'_{n+1} \leq \{l_1|p_n|^2 + l_2k_1T|p_n|^2\} + \{l_3k_1^2T^2|p_n^2|\} - Mp_{n+1}(t) - NSp_{n+1}(t)$$

$$\leq l - Mp_{n+1}(t) - NSp_{n+1}(t)$$

where $l = \{l_1|p_n|^2 + l_2k_1T|p_n|^2\} + \{l_3k_1^2T^2|p_n^2|\}$. Now multiplying throughout by e^{Mt} and setting $\tilde{p}_{n+1} = e^{Mt}p_{n+1}$, we get

$$(p_{n+1}(t)e^{Mt})' \leq Nk_1 \int_0^{h_2} p_{n+1}(s)e^{Mt}ds + le^{Mt}$$

$$\leq Nk_1 \int_0^t \tilde{p}_{n+1}(s)e^{M(t-s)}ds + le^{Mt}$$

$$= w'(t)\ (say)$$

Then choosing w(0)=0, we get $\tilde{p}_{n+1} \leq w(t)$ on I. Clearly $w'(t) \geq 0$, which means that $w(t)$ is nondecreasing on I. Now for $t \in I$,

$$w(t) \leq Nk_1 \int_0^t \int_0^z \tilde{p}_{n+1}(s)e^{M(z-s)}dsdz + \int_0^t le^{Mu}du$$

by setting $c_2 = l\{\frac{e^{MT}}{M}\}$ and $c_1 = \frac{N}{M}k_1e^{Mt}$ we get

$$w(t) \leq c_1 \int_0^t w(z)dz + c_2.$$

Now by using Gronwall's inequality we get

$$w(t) \leq c_2e^{\int_0^t c_1ds}$$

$$\leq c_2e^{c_1t}$$

$$\leq c_2e^{c_1T}$$

Hence

$$\tilde{p}_{n+1}(t) \leq w(t) \leq max_{[0,T]}l(t)[e^{Mt}]e^{c_1T}$$

$$\tilde{p}_{n+1}(t) \leq w(t) \leq max_{[0,T]}l(t)[e^{(M+c_1)t}]$$

$$|p_{n+1}| \leq (e^{(N_1+c_1)T})[M|p_n|^2],$$

Therefore the sequence $\{\alpha_n\}$ converges quadratically on I. Hence the theorem.

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