



Simultaneous Approximation of New Sequence of Integral Type Operators with Parameter δ_0

Ali Jassim Mohammad¹, Hadeel Omar Muslim^{2,*}

¹ *Department of Mathematics, Faculty of Education for Pure Sciences, University of Basra, Basra, Iraq.*

² *Department of Thermal Mechanical Engineering, Faculty of Engineering Technology, Southern Technical University, Basra, Iraq.*

Abstract. In this paper, we define a new sequence of linear positive operators of integral type $W_n(f; x)$ to approximate functions in the space $C_\alpha[0, \infty)$, $\alpha > 0$. First, we study the basic convergence theorem in simultaneous approximation and then study Voronovskaja-type asymptotic formula. Then, we estimate an error occurs by this approximation in the terms of the modulus of continuity. Next, we give numerical examples to approximate two test functions in the space $C_\alpha[0, \infty)$ by the sequence $W_n(f; x)$. Finally, we compare the results with the classical sequence of Szász operators $S_n(f; x)$ on the interval $[a, b]$. It turns out that, the sequence $W_n(f; x)$ gives better results than the results of the sequence $S_n(f; x)$ for the two test functions using in the numerical examples.

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1. Introduction

Bernstein in 1912, using a sequence known by his name, Bernstein sequence, which is defined as:[1]

$$B_n(f; x) = \sum_{k=0}^n b_{n,k}(x) f\left(\frac{k}{n}\right) \quad (1.1)$$

where, $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and $f \in C[0, 1]$.

Next, Voronovskaja in 1932 shown that the order of approximation is $O(n^{-1})$. Also, she showed that this order of approximation by Bernstein sequence cannot be improved beyond $O(n^{-1})$. [18]. Many papers interested in the classical sequences of Bernstein and gave some modifications of them [5], [11]. In addition, the numerical application for this

*Corresponding author.

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sequence are very limited [3], [13].

Szász in 1950, generalized the Bernstein sequence to approximate the space of continuous functions on the interval $[0, \infty)$ as [16]

$$S_n(f; x) = \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right) \quad (1.2)$$

where $q_{n,k}(x) = \frac{(nx)^k}{k!e^{nx}}$, $x \in [0, \infty)$.

Several new modifications of Szász sequence were constructed and studied, here we refer to [4, 14, 17, 20]. Also, many authors have discussed the approximation behavior of different summation-integral type operators (see [6, 7, 10, 12])

The sequence of integral type operators obviously appeared in the proof of Weierstrass theorem (the fundamental theorem in approximation theory), these sequences variety via the effort of some of the researchers who re-proved the Weierstrass theorem by using different sequences of integral type [8, 9, 19].

In the equation (1.1) Bernstein used the finite discrete sequence of a linear positive operator to give another proof of Weierstrass theorem. The Bernstein sequence gives a better result than the previous sequences in applications because it is simplest, finite and discrete sequence [3], [13]. We believe that the same case occurs when we replaced Bernstein by Szász sequence so, we will use the classical Szász sequence to compare with the numerical results of our sequence.

For $x \geq 0$ is arbitrary but fixed, we define that $C_\alpha[0, \infty) = \{f \in C[0, \infty) : |f(t)| = O(e^{\alpha t}), \text{ for some } \alpha > 0\}$ and the norm $\|f\|_{C_\alpha}[0, \infty) = \sup_{t \in [0, \infty)} |f(t)|e^{-\alpha t}$.

For $f \in C_\alpha[0, \infty)$, we define and study the following sequence of linear positive operators:

$$W_n(f; x) = \int_0^x K_n(t; x) f(t) dt \quad (1.3)$$

$K_n(t; x)$ is the Kernel of $W_n(f; x)$ which is define as:

$K_n(t; x) = \frac{n \cosh(nt)}{(\delta_0 + \sinh(nx))}$, $x \in [0, \infty)$, arbitrary but fixed, $n \in N := 1, 2, 3, \dots$ and δ_0 positive parameter.

Firstly, we introduce some preliminary results for $W_n(f; x)$. Then, we study point-wise convergence in simultaneous approximation and give a Voronovskaja-type asymptotic formula for the sequence $W_n(f; x)$. After that, we proceed to estimate an error occurring by the approximation by this sequence in terms of the modulus of continuity. Finally, we give numerical examples for our sequence to approximate two test functions $g_1(t) = \sin(10t)e^{-2t}$, $g_2(t) = \sqrt{1 - (t-1)^2}$ and evaluate the maximum errors occurring by

this approximation, also, compare the results with the sequence of classical Szász operators $S_n(g_i(t); x)$, $i = 1, 2$ on the interval $[a, b]$.

2. Preliminary Results

In this section, we give some preliminary results for the operators $W_n(f; x)$ which we need in our study.

Lemma 2.1.

For $x \in [0, \infty)$ the following conditions hold:

$$(i) \quad W_n(1; x) = \frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \rightarrow 1 \text{ as } n \rightarrow \infty;$$

$$(ii) \quad W_n(t; x) = \frac{x \sinh(nx)}{(\delta_0 + \sinh(nx))} - \frac{\cosh(nx)}{n(\delta_0 + \sinh(nx))} + \frac{1}{n(\delta_0 + \sinh(nx))} \rightarrow x \text{ as } n \rightarrow \infty;$$

$$(iii) \quad W_n(t^2; x) = \frac{x^2 \sinh(nx)}{(\delta_0 + \sinh(nx))} - \frac{2x \cosh(nx)}{n(\delta_0 + \sinh(nx))} + \frac{2 \sinh(nx)}{n^2(\delta_0 + \sinh(nx))} \rightarrow x^2 \text{ as } n \rightarrow \infty.$$

Proof. By the direct computation, the proof of this lemma follows immediate. In addition, from the above lemma and the Korovkin theorem [9], we have that:

$$\lim_{n \rightarrow \infty} W_n(f(t); x) = f(x). \quad (2.1)$$

Further, if f exists and is continuous on $(a - \eta, b + \eta) \subset (0, \infty)$, $\eta > 0$, the limit (2.1) holds uniformly on $[a, b]$.

Our next definition is the m -th order moment for $W_n(f; x)$.

Definition 2.1. For $m \in \mathbb{N}^0$ the m -th order moment $T_{n,m}(x)$ for the operator $W_n(f(t); x)$ is define as:

$$T_{n,m}(x) = W_n((t-x)^m; x) = (-1)^m W_n((x-t)^m; x) = \frac{n(-1)^m}{(\delta_0 + \sinh(nx))} \int_0^x \cosh(nt)(x-t)^m dt \quad (2.2)$$

the recurrence relations for $T_{n,m}(x)$ are given in the next lemma.

Lemma 2.2.

For the function $T_{n,m}(x)$, we have:

$$(i) \quad T_{n,0}(x) = \frac{\sinh(nx)}{(\delta_0 + \sinh(nx))};$$

$$(ii) \quad T_{n,1}(x) = \frac{1}{n(\delta_0 + \sinh(nx))} - \frac{\cosh(nx)}{n(\delta_0 + \sinh(nx))};$$

(iii) $T_{n,2}(x) = \frac{2\sinh(nx)}{n^2(\delta_0 + \sinh(nx))} - \frac{2x}{n(\delta_0 + \sinh(nx))}$.

Then, we have the following recurrence relation:

$$T_{n,m}(x) = \frac{m(m-1)}{n^2}T_{n,m-2}(x) - \frac{m(-1)^m}{n(\delta_0 + \sinh(nx))}x^{m-1}, m \geq 2. \tag{2.3}$$

Further, we have:

- (1) $T_{n,m}(x)$ approximate a polynomial in x of degree $< m$, whenever n is sufficiently large.
- (2) for every $x \in [0, \infty)$, $T_{n,m}(x) = O(n^{-m})$.

Proof. By direct computation and using lemma (2.1), we have (i),(ii),(iii). Now, we prove (2.3), for $x \in [0, \infty)$, and all $m \geq 2$, we have:

$$\begin{aligned} T_{n,m}(x) &= \frac{n(-1)^m}{(\delta_0 + \sinh(nx))} \int_0^x \cosh(nt)(x-t)^m dt \\ &= \frac{n(-1)^m}{(\delta_0 + \sinh(nx))} \left[\frac{m}{n} \int_0^x \sinh(nt)(x-t)^{m-1} dt \right] \\ &= \frac{n(-1)^m}{(\delta_0 + \sinh(nx))} \left[-\frac{m}{n^2}x^{m-1} + \frac{m(m-1)}{n^2} \int_0^x \cosh(nt)(x-t)^{m-2} dt \right] \\ &= \frac{m(m-1)}{n^2}T_{n,m-2}(x) - \frac{m(-1)^m}{n(\delta_0 + \sinh(nx))}x^{m-1}. \end{aligned}$$

Therefore,(2.3) satisfied.

The consequence (1) can be proved easily by using (2.3) and the induction on m , so the details are omitted.

Lemma 2.3. For $m \geq 1$, we have:

$$W_n(t^m; x) = \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) x^m - \left(\frac{m\cosh(nx)}{n(\delta_0 + \sinh(nx))} \right) x^{m-1} + O(n^{-2}).$$

Clearly, we have $\lim_{n \rightarrow \infty} W_n(t^m; x) = x^m$.

Proof.

$$\begin{aligned} W_n(t^m; x) &= \frac{n}{(\delta_0 + \sinh(nx))} \int_0^x \cosh(nt)t^m dt \\ &= \frac{n}{(\delta_0 + \sinh(nx))} \left[\left(\frac{1}{n} \sinh(nx) \right) x^m - \frac{m}{n} \int_0^x \sinh(nt)t^{m-1} dt \right] \\ &= \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) x^m - \left(\frac{m\cosh(nx)}{n(\delta_0 + \sinh(nx))} \right) x^{m-1} \end{aligned}$$

$$\begin{aligned}
 &+ \left(\frac{m(m-1)}{n(\delta_0 + \sinh(nx))} \right) \int_0^x \cosh(nt)t^{m-2} dt \\
 &= \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) x^m - \left(\frac{m \cosh(nx)}{n(\delta_0 + \sinh(nx))} \right) x^{m-1} + O(n^{-2}).
 \end{aligned}$$

Lemma 2.4. *Let δ and α be any two positive real numbers and $[a, b] \subset (0, \infty)$. Then for $\lambda > 0$, we have:*

$$\sup_{x \in [a, b]} \left| \int_{x-t \geq \delta} K_n(t; x) e^{\alpha t} dt \right| = O(n^{-\lambda}).$$

Making use of Taylor’s expansion, Schwartz inequality and lemma 2.2(2), the proof of this lemma easily follows.

Lemma 2.5. [15]

(1) *Let r be a nonnegative integer and assume f and g are r -times differentiable functions of x then:*

$$\frac{d^r}{dx^r} (fg) = \sum_{l=0}^r \binom{r}{l} \frac{d^{r-l}}{dx^{r-l}} (f) \frac{d^l}{dx^l} (g);$$

(2) *Let $g(x)$ be a real or complex valued function that is r -times differentiable then:*

$$\frac{d^r}{dx^r} \left(\frac{1}{g(x)} \right) = \sum_{l=0}^r (-1)^l \binom{r+l}{l+1} \frac{1}{(g(x))^{l+1}} (f) \frac{d^r}{dx^r} (g(x))^l.$$

Lemma 2.6. *For $r \in N$, we have:*

(i) $\lim_{n \rightarrow \infty} \frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} = 1;$

(ii) $\lim_{n \rightarrow \infty} \frac{\cosh(nx)}{(\delta_0 + \sinh(nx))} = 1;$

(iii) $\lim_{n \rightarrow \infty} \frac{d^r}{dx^r} \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) = 0;$

(iv) $\lim_{n \rightarrow \infty} \frac{d^r}{dx^r} \left(\frac{\cosh(nx)}{(\delta_0 + \sinh(nx))} \right) = 0;$

(v) $\lim_{n \rightarrow \infty} \frac{d^r}{dx^r} \left(\frac{1}{(\delta_0 + \sinh(nx))} \right) = 0;$

(vi) $\lim_{n \rightarrow \infty} \frac{d^r}{dx^r} \left(\frac{x \sinh(nx)}{(\delta_0 + \sinh(nx))} \right) = 0, r > 1.$

Lemma 2.7. [2]

For the function $F(x)$ given by:

$$F(x) = \int_{a(x)}^{b(x)} f(x, y) dy$$

Then the chain rule of differentiation of the function $F(x)$ gives:

$$F'(x) = b'(x)f(x, b(x)) - a'(x)f(x, a(x)) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, y) dy$$

3. The main results

First, we prove that:

$$W_n^{(r)}(f(t); x) \rightarrow f^{(r)}(x), \text{ as } n \rightarrow \infty, r \in N.$$

Theorem 3.1. Suppose that $r \in N, f \in C_\alpha[0, \infty)$ and $f^{(r+1)}(x)$ exists at a point $x \in (0, \infty)$, then:

$$\lim_{n \rightarrow \infty} W_n^{(r)}(f(t); x) \rightarrow f^{(r)}(x). \tag{3.1}$$

Further, if $f^{(r+1)}(x)$ exists and is continuous on $(a - \eta, b + \eta) \subset (0, \infty), \eta > 0$, the limit (3.1) holds uniformly on $[a, b]$.

Proof. By using Taylor's expansion of f , when ξ lies between t and x , we get:

$$f(t) = \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} (t-x)^i + \frac{f^{(r+1)}(\xi)}{(r+1)!} (t-x)^{r+1},$$

Operating by the sequence $W_n^{(r)}$, we get:

$$\begin{aligned} W_n^{(r)}(f(t); x) &= \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} (-1)^i W_n^{(r)}((x-t)^i; x) + \frac{f^{(r+1)}(\xi)}{(r+1)!} (-1)^{r+1} W_n^{(r)}((x-t)^{r+1}; x) \\ &:= \Sigma_1 + \Sigma_2 \end{aligned}$$

$$\begin{aligned} \Sigma_1 &:= \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} (-1)^i W_n^{(r)}((x-t)^i; x) \\ &= \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} (-1)^i \sum_{j=0}^i \binom{i}{j} x^{(i-j)} (-1)^j W_n^{(r)}(t^j; x) \\ &= \frac{f^{(r)}(x)}{r!} W_n^{(r)}(t^r; x) \end{aligned}$$

When $j < r$ then $W_n^{(r)}(t^j; x) \rightarrow 0$ as $n \rightarrow \infty$.

Using lemma 2.3, lemma 2.5, and lemma 2.6 we get:

$$\begin{aligned} \Sigma_1 &= \frac{f^{(r)}(x)}{r!} \left\{ r! \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) + rr!x \frac{n \cosh(nx)}{(\delta_0 + \sinh(nx))} \left(1 - \frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) \right. \\ &+ \dots + x^r \frac{d^r}{dx^r} \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) - rr! \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} - \frac{\cosh^2(nx)}{(\delta_0 + \sinh(nx))^2} \right) \\ &\left. - \dots - \frac{r}{n} x^{r-1} \frac{d^r}{dx^r} \left(\frac{\cosh(nx)}{(\delta_0 + \sinh(nx))} \right) \right\} \rightarrow f^{(r)}(x) \text{ as } n \rightarrow \infty. \end{aligned}$$

$$\begin{aligned} \Sigma_2 &= \frac{f^{(r+1)}(\xi)}{(r+1)!} (-1)^{r+1} W_n^{(r)} \left((x-t)^{r+1}; x \right) \\ &= \frac{f^{(r+1)}(\xi)}{(r+1)!} (-1)^{r+1} \frac{d^r}{dx^r} \left\{ \frac{n}{(\delta_0 + \sinh(nx))} \int_0^x \cosh(nt) (x-t)^{r+1} dt \right\}. \end{aligned}$$

Using lemma 2.5, lemma 2.6 and lemma 2.7, we obtain:

$$\begin{aligned} \Sigma_2 &= \frac{f^{(r+1)}(\xi)}{(r+1)!} (-1)^{r+1} \sum_{l=0}^r \binom{r}{l} \frac{d^{r-1}}{dx^{r-1}} \left(\frac{n}{(\delta_0 + \sinh(nx))} \right) \frac{d^l}{dx^l} \left(\int_0^x \cosh(nt) (x-t)^{r+1} dt \right) \\ &= \frac{f^{(r+1)}(\xi)}{(r+1)!} (-1)^{r+1} \frac{d^r}{dx^r} \left(\frac{n}{(\delta_0 + \sinh(nx))} \right) \int_0^x \cosh(nt) (x-t)^{r+1} dt \\ &+ \frac{f^{(r+1)}(\xi)}{(r+1)!} (-1)^{r+1} \sum_{l=1}^r \binom{r}{l} \frac{d^{r-1}}{dx^{r-1}} \left(\frac{n}{(\delta_0 + \sinh(nx))} \right) \frac{d^l}{dx^l} \left(\int_0^x \cosh(nt) (x-t)^{r+1} dt \right) \\ &= I_1 + I_2. \end{aligned}$$

$$I_1 \leq \left| \frac{f^{(r+1)}(\xi)}{(r+1)!} \right| \sum_{l=0}^r \binom{r+1}{l+1} \left| \frac{n}{(\delta_0 + \sinh(nx))^{l+1}} \frac{d^r}{dx^r} (\delta_0 + \sinh(nx))^l \right| \int_0^x \cosh(nt) \left| -(x-t)^{r+1} \right| dt.$$

By Schwartz inequality, we have:

$$\begin{aligned} |I_1| &\leq \left| \frac{f^{(r+1)}(\xi)}{(r+1)!} \right| \sum_{l=0}^r \binom{r+1}{l+1} \left| \frac{1}{(\delta_0 + \sinh(nx))^l} \frac{d^r}{dx^r} (\delta_0 + \sinh(nx))^l \right| \\ &\times \left(\frac{n}{(\delta_0 + \sinh(nx))} \int_0^x \cosh(nx) dt \right)^{1/2} \left(\frac{n}{(\delta_0 + \sinh(nx))} \int_0^x \cosh(nt) (-x-t)^{2(r+1)} dt \right)^{1/2} \\ &= M_1 n^r O(1) O(n^{-(r+1)}) \\ &= M_1 o(1). \end{aligned}$$

Now, using lemma 2.5 and lemma 2.7, we obtain:

$$|I_2| \leq \left| \frac{f^{(r+1)}(\xi)}{(r+1)!} \right| \sum_{l=1}^r \binom{r}{l} \sum_{l_1=0}^{r-1} \binom{r-l+1}{l_1+1} \left| \frac{n}{(\delta_0 + \sinh(nx))^{l_1+1}} \frac{d^{r-1}}{dx^{r-1}} (\delta_0 + \sinh(nx))^{l_1} \right|$$

$$\begin{aligned}
 & \times \int_0^x \cosh(nt) \left| \frac{d^l}{dx^l} (-(x-t))^{r+1} \right| dt \\
 & = M_2 \left\{ \left[\frac{r^2(r-1)}{2} \frac{n}{(\delta_0 + \sinh(nx))^2} \frac{d^{r-1}}{dx^{r-1}} (\delta_0 + \sinh(nx)) \right. \right. \\
 & + \dots + r \frac{n}{(\delta_0 + \sinh(nx))^r} \frac{d^{r-1}}{dx^{r-1}} (\delta_0 + \sinh(nx))^{r-1} \Big] \\
 & \times \left(\int_0^x \cosh(nt) |-(r+1)(-(x-t))^r| dt \right) \\
 & + \left[\frac{r(r-1)^2(r-2)}{4} \frac{n}{(\delta_0 + \sinh(nx))^2} \frac{d^{r-2}}{dx^{r-2}} (\delta_0 + \sinh(nx)) \right. \\
 & + \dots + \frac{r(r-1)}{2} \frac{n}{(\delta_0 + \sinh(nx))^r} \frac{d^{r-2}}{dx^{r-2}} (\delta_0 + \sinh(nx))^{r-1} \Big] \\
 & \times \left(\int_0^x \cosh(nt) |r(r+1)(-(x-t))^{r-1}| dt \right) \\
 & \left. + \frac{n}{(\delta_0 + \sinh(nx))} \left(\int_0^x \cosh(nt) |(-1)^r(r+1)!(-(x-t))| dt \right) \right\}.
 \end{aligned}$$

By Schwarz inequality, we have:

$$I_2 = M_2 O(n^{-1})$$

Hence,

$$I_2 = o(1).$$

Now, it follows $I_1 \rightarrow 0$ as $n \rightarrow \infty$. also $I_2 \rightarrow 0$ as $n \rightarrow \infty$. Hence $\Sigma_2 = o(1)$. The uniformity assertion follows easily from the fact that $\delta(\varepsilon)$ in the above proof can be chosen to be independent of $x \in [a, b]$ and all the other estimates hold uniformly on $[a, b]$.

Our next results is a Voronovskaja-type asymptotic formula for the operators $W_n^{(r)}(f(t); x), r \in N$.

Theorem 3.2.

Suppose that $r \in N, f \in C_\alpha[0, \infty)$ and $f^{(r+3)}(x)$ exists at a point $x \in (0, \infty)$, then:

$$\lim_{n \rightarrow \infty} n \left(W_n^{(r)}(f(t); x) - f^{(r)}(x) \right) = -f^{(r+1)}(x) \tag{3.2}$$

Further, if $f^{(r+3)}(x)$ exists and is continuous on $(a - \eta, b + \eta) \subset (0, \infty), \eta > 0$, the limit (3.2) holds uniformly on $[a, b]$.

Proof. By using Taylor's expansion of f , when ξ lies between t and x , we get:

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} (t-x)^i + \frac{f^{(r+)}(\xi)}{(r+3)!} (t-x)^{r+3}$$

Operating by the sequence $W_n^{(r)}$, we get:

$$W_n^{(r)}(f(t); x) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} (-1)^i W_n^{(r)}((x-t)^i; x) + \frac{f^{(r+3)}(\xi)}{(r+3)!} (-1)^{r+3} W_n^{(r)}((x-t)^{r+3}; x)$$

$$:= \Sigma_1 + \Sigma_2$$

Using the same technique of theorem 3.1, we get:

$\Sigma_2 \rightarrow 0$ as $n \rightarrow \infty$.

When $i < r$ then from lemma 2.3, we have

$W_n^{(r)}((t-x)^i; x) \rightarrow 0$ as $n \rightarrow \infty$.

Using lemma 2.3, lemma 2.5, and lemma 2.6, we get:

$$\begin{aligned} \Sigma_1 = & \frac{f^{(r)}(x)}{r!} \left\{ r! \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) + rr!x \frac{n \cosh(nx)}{(\delta_0 + \sinh(nx))} \left(1 - \frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) \right. \\ & + \dots + x^r \frac{d^r}{dx^r} \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) - rr! \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} - \frac{\cosh^2(nx)}{(\delta_0 + \sinh(nx))^2} \right) \\ & - \dots - \frac{r}{n} x^{r-1} \frac{d^r}{dx^r} \left(\frac{\cosh(nx)}{(\delta_0 + \sinh(nx))} \right) \left. \right\} - f^{(r)}(x) \\ & + \frac{f^{(r+1)}(x)}{(r+1)!} \left\{ -\frac{r(r+1)!}{2} x^2 \frac{n \cosh(nx)}{(\delta_0 + \sinh(nx))} \left(1 - \frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) \right. \\ & - \dots - r x^{r+1} \frac{d^r}{dx^r} \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) - \frac{(r+1)!}{n} \frac{\cosh(nx)}{(\delta_0 + \sinh(nx))} \\ & + \dots + \frac{(r+1)(r-1)}{n} x^r \frac{d^r}{dx^r} \left(\frac{\cosh(nx)}{(\delta_0 + \sinh(nx))} \right) \left. \right\} \\ & + \frac{f^{(r+2)}(x)}{(r+2)!} \left\{ \frac{r(r+2)!}{6} x^3 \frac{n \cosh(nx)}{(\delta_0 + \sinh(nx))} \left(1 - \frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) \right. \\ & + \dots + \frac{r(r+1)}{2} x^{r+2} \frac{d^r}{dx^r} \left(\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right) - \dots - \frac{r(r-1)(r+2)}{2n} x^{r+1} \\ & \left. \times \frac{d^r}{dx^r} \left(\frac{\cosh(nx)}{(\delta_0 + \sinh(nx))} \right) \right\}. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} n (W_n^{(r)}(f(t); x) - f^{(r)}(x)) = -f^{(r+1)}(x)$$

The uniformity assertion follows easily from the fact that $\delta(\varepsilon)$ in the above proof can be chosen to be independent of $x \in [a, b]$ and all the other estimates hold uniformly on $[a, b]$.

Finally, we give an estimate of the degree of approximation by $W_n^{(r)}(f; x)$.

Theorem 3.3. Let $f \in C_\alpha[0, \infty)$, $\alpha > 0$ for some $h > 0$ and $r \leq q \leq r + 2$. If $f^{(q+1)}$ exists and is continuous on $(a - \eta, b + \eta) \subset (0, \infty)$, $\eta > 0$, then for sufficiently large n ,

$$\begin{aligned} \left\| W_n^{(r)}(f(t); x) - f^{(r)}(x) \right\|_{C[a,b]} &\leq C_1 n^{-1} \sum_{i=r}^q \left\| f^{(i)} \right\|_{C[a,b]} + C_2 n^{-1/2} \omega_{f^{(q+1)}} \\ &\times \left(n^{-1/2}; (a - \eta, b + \eta) \right) + O(n^{-2}), \end{aligned}$$

where, C_1, C_2 are constants independent of f and n .

Proof. By our hypothesis

$$f(t) = \sum_{i=0}^q \frac{f^{(i)}(x)}{i!} (t-x)^i + \frac{f^{(q+1)}(\xi) - f^{(q+1)}(x)}{(q+1)!} (t-x)^{q+1} \chi(t) + h(t, x)(1 - \chi(t)),$$

where, ξ lies between t, x and $\chi(t)$ is the characteristic function of the interval $(a - \eta, b + \eta)$. For $t \in (a - \eta, b + \eta)$ and $x \in (0, \infty)$ we get:

$$f(t) = \sum_{i=0}^q \frac{f^{(i)}(x)}{i!} (t-x)^i + \frac{f^{(q+1)}(\xi) - f^{(q+1)}(x)}{(q+1)!} (t-x)^{q+1}.$$

For $t \in [0, \infty) \setminus (a - \eta, b + \eta)$ and $x \in [a, b]$, we define

$$\begin{aligned} h(t, x) &= f(t) - \sum_{i=0}^q \frac{f^{(i)}(x)}{i!} (t-x)^i. \\ \frac{\partial^r}{\partial x^r} h(t, x) &= \sum_{i=0}^q \frac{f^{(i)}(x)}{i!} (t-x)^i \end{aligned}$$

Now,

$$\begin{aligned} W_n^{(r)}(f(t); x) - f^{(r)}(x) &= \left[\sum_{i=0}^q \frac{f^{(i)}(x)}{i!} (-1)^i W_n^{(r)}((x-t)^i; x) - f^{(r)}(x) \right] \\ &+ (-1)^{q+1} W_n^{(r)} \left(\frac{f^{(q+1)}(\xi) - f^{(q+1)}(x)}{(q+1)!} (x-t)^{q+1} \chi(t); x \right) \\ &+ W_n^{(r)}(h(t, x)(1 - \chi(t)); x) \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned}$$

Using lemma 2.3, we get:

$$\begin{aligned} \Sigma_1 &= \sum_{i=0}^q \frac{f^{(i)}(x)}{i!} (-1)^i \sum_{j=0}^i \binom{i}{j} x^{i-j} (-1)^j W_n^{(r)}(t^j; x) - f^{(r)}(x) \\ &= \sum_{i=r}^q \frac{f^{(i)}(x)}{i!} (-1)^i \sum_{j=0}^i \binom{i}{j} x^{i-j} (-1)^j \frac{d^r}{dx^r} \left[\frac{\sinh(nx)}{(\delta_0 + \sinh(nx))} \right] x^j \end{aligned}$$

$$-\frac{j}{n} \left(\frac{\cosh(nx)}{(\delta_0 + \sinh(nx))} \right) x^{j-1} + O(n^{-2})] - f^r(x)$$

Consequently,

$$\|\Sigma_1\|_{C[a,b]} \leq C_1 n^{-1} \left[\sum_{i=r}^q \|f^{(i)}\|_{C[a,b]} \right] + O(n^{-2}), \text{ uniformly on } [a, b].$$

To estimate Σ_2 we proceed as follows:

$$\begin{aligned} |\Sigma_2| &\leq W_n^{(r)} \left(\frac{|f^{(q+1)}(\xi) - f^{(q+1)}(x)|}{(q+1)!} |-(x-t)^{q+1}\chi(t); x \right) \\ &\leq \frac{\omega_{f^{(q+1)}}(\delta; (a-\eta, b+\eta))}{(q+1)!} W_n^{(r)} \left(\frac{\delta + |-(x-t)|}{\delta} |-(x-t)^{q+1}\chi(t); x \right) \\ &\leq \frac{\omega_{f^{(q+1)}}(\delta; (a-\eta, b+\eta))}{(q+1)!} \frac{d^r}{dx^r} \left[\frac{n}{(\delta_0 + \sinh(nx))} \int_0^x \cosh(nt) |-(x-t)^{q+1} \right. \\ &\quad \left. + \delta^{-1} |-(x-t)^{q+2} dt \right] \delta > 0. \end{aligned}$$

Now, for $k = 0, 1, 2, \dots$ and using lemma 2.5, lemma 2.7., we have:

$$\begin{aligned} &\left| \frac{d^r}{dx^r} \left[\frac{n}{(\delta_0 + \sinh(nx))} \int_0^x \cosh(nt) (-(x-t))^{k+1} dt \right] \right| \tag{3.3} \\ &= \sum_{l=0}^r \binom{r}{l} \left| \frac{d^{r-l}}{dx^{r-l}} \left(\frac{n}{\delta_0 + \sinh(nx)} \right) \right| \left| \frac{d^l}{dx^l} \left(\int_0^x \cosh(nx) (-(x-t))^{k+1} dt \right) \right| \\ &= \left| \frac{d^r}{dx^r} \left(\frac{n}{\delta_0 + \sinh(nx)} \right) \right| \int_0^x \cosh(nt) |-(x-t)|^{k+1} dt \\ &+ \sum_{l=1}^r \binom{r}{l} \left| \frac{d^{r-l}}{dx^{r-l}} \left(\frac{n}{\delta_0 + \sinh(nx)} \right) \right| \left| \frac{d^l}{dx^l} \left(\int_0^x \cosh(nx) |-(x-t)|^{k+1} dt \right) \right| \\ &:= J_1 + J_2 \end{aligned}$$

Using the same technique in I_1, I_2 (theorem 3.1), we get:

$$J_1 = O(n^{r-(k+1)}), \text{ uniformly on } [a, b]$$

and $J_2 = O(n^{-1})$. Choosing $\delta = n^{-1/2}$ and applying 3.3, we are led to:

$$\begin{aligned} \|\Sigma_2\|_{C[a,b]} &\leq \frac{\omega_{f^{(q+1)}}(n^{-1/2}; (a-\eta, b+\eta))}{(q+1)!} \left[O(n^{r-(q+1)}) + n^{1/2} O(n^{r-(q+2)}) + O(n^{-1}) \right] \\ &\leq C_2 n^{-(r-(q+1))} \omega_{f^{(q+1)}}(n^{-1/2}; (a-\eta, b+\eta)). \end{aligned}$$

Since $t \in [0, \infty) \setminus (a-\eta, b+\eta)$, we can choose $\delta > 0$ in such a way that $x-t \geq \delta$ for all $x \in [a, b]$. Thus,

$$|\Sigma_3| \leq \left| \frac{d^r}{dx^r} \left[\frac{n}{(\delta_0 + \sinh(nx))} \int_0^x \cosh(nt) (h(t, x)(1 - \chi(t))) dt \right] \right|$$

$$\begin{aligned}
 &\leq \left| \frac{d^r}{dx^r} \left[\frac{n}{(\delta_0 + \sinh(nx))} \int_{x-t \geq \delta} \cosh(nt)h(t, x)dt \right] \right| \\
 &= \left| \frac{d^r}{dx^r} \left(\frac{n}{(\delta_0 + \sinh(nx))} \right) \right| \int_{x-t \geq \delta} \cosh(nt)h(t, x)dt \\
 &+ \sum_{l=1}^r \binom{r}{l} \left| \frac{d^{r-l}}{dx^{r-l}} \left(\frac{n}{(\delta_0 + \sinh(nx))} \right) \right| \left| \frac{d^l}{dx^l} \left(\int_{x-t \geq \delta} \cosh(nt)h(t, x)dt \right) \right| \\
 &= J_3 + J_4.
 \end{aligned}$$

For $x - t \geq \delta$, we can find a constant $C > 0$ such that $|h(t, x)| \leq Ce^{\alpha t}$ and $\left| \frac{\partial^r}{\partial x^r} h(t, x) \right| \leq Ce^{\alpha t}$

Using lemma 2.4, we get $|\Sigma_3| = O(n^{-\lambda}), \lambda > 0$ uniformly on $[a, b]$.

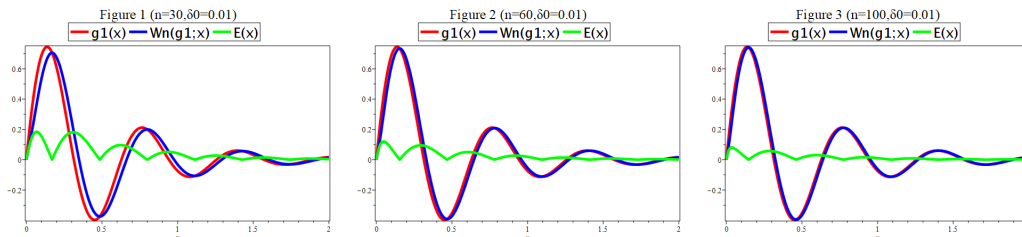
Combining the estimates of $\Sigma_1, \Sigma_2, \Sigma_3$ the required results are immediate.

4. Numerical Examples

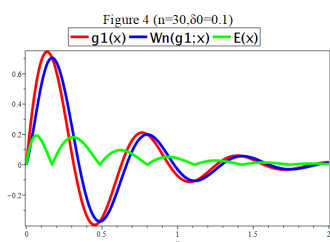
In this section, we give some numerical examples for the sequence of linear positive operators $W_n(.; x)$, by using two test functions $g_1(t) = \sin(10t)e^{-2t}$ and $g_2(t) = \sqrt{1 - (t - 1)^2}, t \in [0, 2]$. we calculate the maximum error and compare the results of the sequence $W_n(.; x)$, with the results of classical Szász sequence $S_n(.; x)$ in the interval $[0, 2]$ and we describe the results by figures (1 – 24) for some $n = 30, 60, 100$ and the positive parameter $\delta_0 = 0.01, 0.1, 1$. respectively.

Definition 4.1. Given a sequence M_n , here ($M_n = W_n$ or S_n), and let f be a function, the error function $E(x)$ occurring by approximate the function f by the sequence M_n is defined as $E(x) = |M_n(f; x) - f(x)|$. Also, the maximum error of the function $E(x)$ is denote and define as $MaxE = \max_{x \in [0, 2]} |E(x)|$.

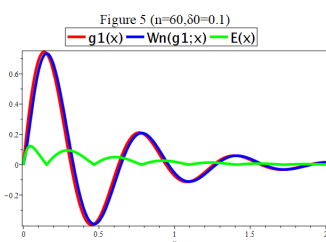
Example 4.1. For $n = 30, 60, 100$ and $\delta_0 = 0.01, 0.1, 1$. respectively the sequences $W_n(g_1; x)$ and $S_n(g_1; x)$ converge to the test function $g_1(x) = \sin(10x)e^{-2x}$, with maximum error ($MaxE$) given in the following figures (1 – 12)



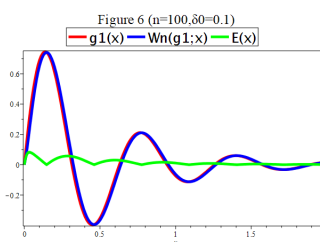
(a) $MaxE := 0.1844077862$ (b) $MaxE := 0.1201880456$ (c) $MaxE := 0.0806800064$



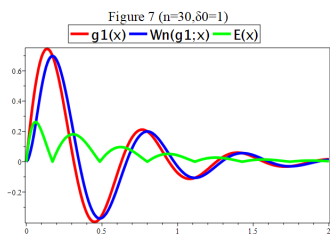
(a) $MaxE := 0.192609996$



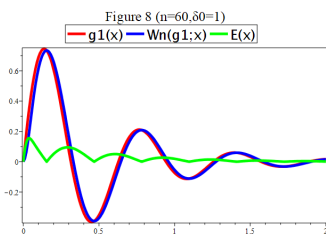
(b) $MaxE := 0.1229776897$



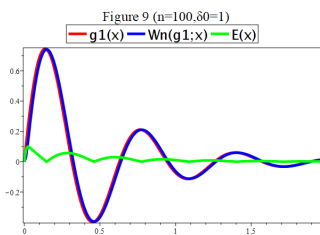
(c) $MaxE := 0.08243795854$



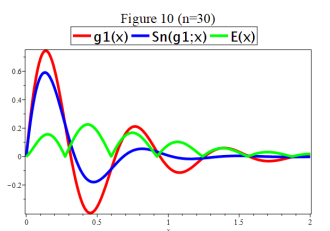
(a) $MaxE := 0.2624406460$



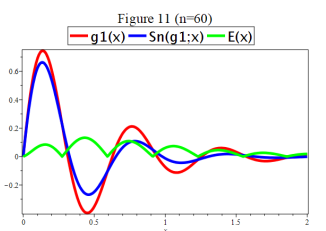
(b) $MaxE := 0.1553141469$



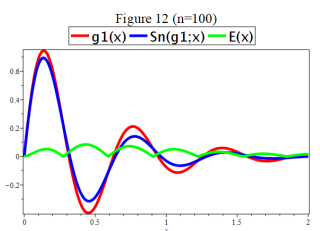
(c) $MaxE := 0.0984378898$



(a) $MaxE := 0.2259276806$

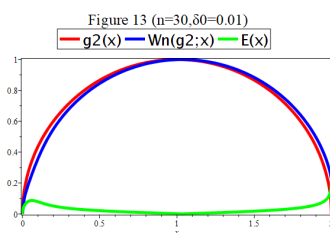


(b) $MaxE := 0.1323992412$

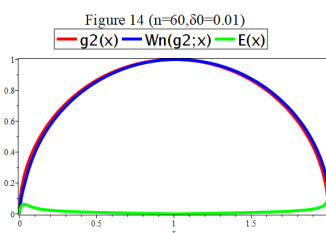


(c) $MaxE := 0.085016286$

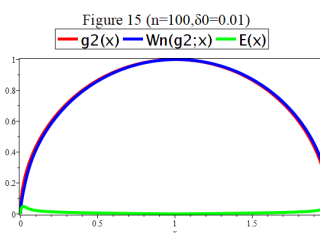
Example 4.2. For $n = 30, 60, 100$ and $\delta_0 = 0.01, 0.1, 1$, respectively the sequences $W_n(g_2; x)$ and $S_n(g_2; x)$ converge to the test function $g_2(t) = \sqrt{1 - (t - 1)^2}$, with maximum error ($MaxE$) given in the following figures (13 – 24)



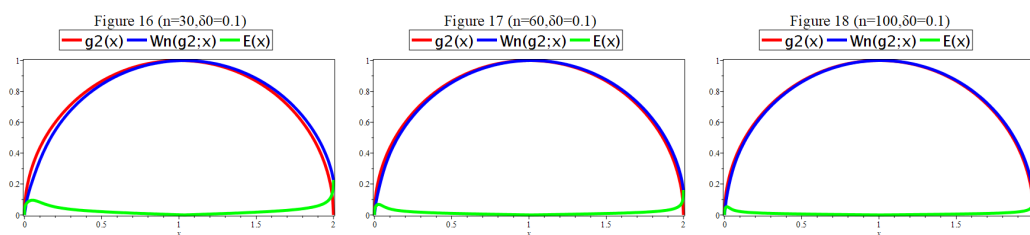
(a) $MaxE := 0.1319652402$



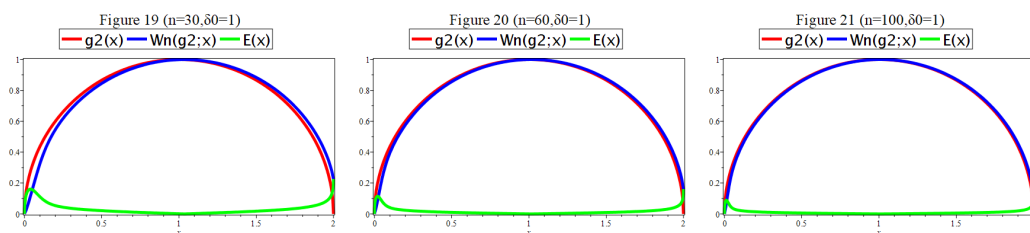
(b) $MaxE := 0.0791847229$



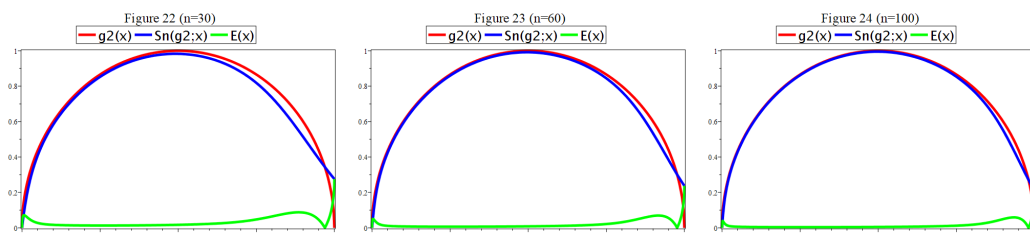
(c) $MaxE := 0.0528549386$



(a) $MaxE := 0.1319652402$ (b) $MaxE := 0.0791847229$ (c) $MaxE := 0.0528549386$



(a) $MaxE := 0.1605861444$ (b) $MaxE := 0.1142832076$ (c) $MaxE := 0.08801539766$



(a) $MaxE := 0.2753815754$ (b) $MaxE := 0.2368816505$ (c) $MaxE := 0.2109249528$

5. Conclusions

In this section, we gave some numerical examples for our sequences $W_n(\cdot; x)$, in cases $n = 30, 60, 100$ and the positive parameter $\delta_0 = 0.01, 0.1, 1$. to approximate two test functions $g_1(t) = \sin(10t)e^{-2t}$ and $g_2(t) = \sqrt{1 - (t-1)^2}$, in the space $C_\alpha[0, \infty)$ and compared the results of the sequence $W_n(\cdot; x)$, with the results of classical Szász sequence $S_n(\cdot; x)$ in the interval $[0, 2]$. it turns out that: If $i = 1$, the sequence $W_n(g_i(t); x)$ gives better results than the result of the Szász sequence $S_n(g_i; x)$ for all value of n and $\delta_0 = 0.01, 0.1$, except $\delta_0 = 1$ the sequence of Szász sequence give little better results than the sequence $W_n(\cdot; x)$. When $i = 2$, the sequence $W_n(g_i(t); x)$ gives better results than the Szász sequence $S_n(g_i; x)$ for all value of n and δ_0 . Hence, we recommend to use the sequence $W_n(\cdot; x)$ instead of the sequence $S_n(\cdot; x)$ in the application.

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