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Topology on a BE-algebra Induced by Right Application of BE-ordering

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Abstract. This study deals with the topology generated by the family of subsets determined by the right application of BE-ordering of a BE-algebra and investigates some of its properties. Characterizations of some elementary topological concepts as well as the concepts of continuous, open, and closed maps associated with this topological space are obtained.

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1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [3] in 1966 as a generalization of the concept of set theoretic difference and propositional calculi. In [5], K. H. Kim and Y. H. Yon introduced the dual BCK-algebra and study its relation to MV-algebra. As a generalization of dual BCK-algebra, H. S. Kim and Y. H. Kim [4] introduced the BE-algebra. Today, BE-algebras have been studied by many authors and many branches of mathematics have been applied to BE-algebras, such as probability theory, topology, fuzzy set theory and so on. Various authors studied the topological aspects of BE-algebras. In [7], S. Mehrshad and J. Golzarpoor studied some properties of uniform topology and topological BE-algebras and compare these topologies. In [8], the author produced a basis for a topology using left and right stabilizers of a BE-algebra. It is proved that the generated topological space is a Bair, locally connected and separable space. Some other topological properties are studied using left and right stabilizers. Motivated by these works, this paper introduces the topology induced by a BE-algebra using the right application of BE-ordering and investigates some of its properties.

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An algebra $(X; *, 1_X)$ is called a *BE-algebra* if the following hold: for all $x, y, z \in X$, (BE1) $x * x = 1_X$; (BE2) $x * 1_X = 1_X$; (BE3) $1_X * x = x$; and (BE4) $x * (y * z) = y * (x * z)$. A relation "$\leq$" on X, called *BE-ordering*, is defined by $x \leq y$ if and only if $x * y = 1_X$. Throughout this paper, we denote a BE-algebra $(X, *, 1_X)$ simply by X if no confusion arises. A non-empty subset S of X is said to be a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A BE-algebra X is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. It is called *commutative* if satisfies $(x * y) * y = (y * x) * x$ for all $x, y \in X$. It is said to be a *transitive BE-algebra* if it satisfies the condition: $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. If X is a transitive BE-algebra, then the relation "$\leq$" is transitive. Let F be a non-empty subset of X. Then F is said to be a *filter* of X if: (F1) $1_X \in F$; and (F2) $x * y \in F$ and $x \in F$ imply $y \in F$. A non-empty subset I of X is called an *ideal* of X if it satisfies: for all $x \in X$ and for all $a, b \in I$, (I1) $x * a \in I$, that is, $X * I \subseteq I$; and (I2) $(a * (b * x)) * x \in I$. The set $[a, 1_X] = \{x \in X \mid a * x = 1_X\}$ for all $a \in X$ is called the *final segment* of X. An element $a \neq 1_X$ of a BE-algebra X is said to be a *dual atom* of X if $a \leq x$ implies either $a = x$ or $x = 1_X$ for all $x \in X$. We will denote by $\mathcal{A}(X)$ the set of all dual atoms of X unless otherwise mentioned. Hence, $\mathcal{A}(X) = \{x \in X \mid x \text{ is a dual atom}\}$. We will consider $\mathcal{A}_1(X) = \mathcal{A}(X) \cup \{1_X\}$. A BE-algebra X is called *dual atomistic* if every non-unit element of X is a dual atom in X, that is, $X = \mathcal{A}_1(X)$, see [1, 8].

Example 1. [8] *Let $N_0 = \mathbb{N} \cup \{0\}$ and let $*$ be the binary operation on N_0 defined by*

$$x * y = \begin{cases} 0 & \text{if } y \leq x \\ y - x & \text{if } x < y. \end{cases}$$

Then $(N_0; *, 0)$ is a commutative BE-algebra where $1_{N_0} = 0$. It can be seen that $\mathcal{A}(N_0) = \{1\}$.

Lemma 1. [9] *Let $(X; *, 1_X)$ be a BE-algebra and let I be a non-empty subset of X. Then I is an ideal of X if and only if it satisfies* (i) *$1_X \in I$; and* (ii) *for all $x, z \in X$ and for all $y \in I$, $(x * (y * z)) \in I$ implies $x * z \in I$.*

Let Y be a non-empty set. A collection τ of subsets of Y is a *topology* on Y if it satisfies the following axioms: (G1) $\varnothing$ and Y belong to τ; (G2) if G_1 and G_2 are elements of τ, then $G_1 \cap G_2 \in \tau$; and (G3) if $\{G_i : i \in I\} \subseteq \tau$, then $\bigcup_{i \in I} G_i \in \tau$. If τ is a topology on Y, then the ordered pair (Y, τ) is called a *topological space*. An element O of τ is called a τ*-open* set (or simply *open set*) and the complement of O is called a τ*-closed* set (or simply *closed set*). The *discrete topology on Y is* $\mathcal{D}_Y = \mathcal{P}(Y)$. A class $\mathcal{B} \subseteq \tau$ is a *basis* for τ if each open set is the union of members of $\mathcal{B}$. The elements of a basis are called *basic open sets*. The topology τ is said to be generated by a basis $\mathcal{B}$ if the family τ consists $\varnothing$, Y, and all unions of members of $\mathcal{B}$. A class $\mathcal{S}$ of open subsets of Y, that is, $\mathcal{S} \subseteq \tau$, is a *subbase* or *subbasis* for the topology τ on Y if and only if finite intersections of members of $\mathcal{S}$ form a basis for τ. Suppose that $x \in Y$ and $U \subseteq Y$. U is a *neighborhood* of x (briefly nbd $U(x)$) if $x \in U$ and $U \in \tau$. Throughout this paper, we denote a topological space (Y, τ) by Y, unless otherwise specified. Let A be a subset of a topological space Y. A point $x \in Y$ is *adherent* to A or *closure point* of A if each neighborhood of x contains at least one point of A (which maybe x itself). The set of all points in Y adherent to A, denoted by $\overline{A}$, is called the *closure* of A, that is, $\overline{A} = \{x \in Y |\ \forall\ U(x) : U(x) \cap A \neq \varnothing\}$. A point $p \in A$ is called an *interior point* of A if p belongs to an open set G in Y contained in A, that is, $p \in G \subseteq A$. The set of all interior points of A, denoted by $Int(A)$, is called the *interior* of A, that is, the interior of A is the largest open set contained in A, or, $Int(A) = \bigcup\{U |\ U \text{ is open and } U \subseteq A\}$. $D \subseteq Y$ is *dense* in Y if $\overline{D} = Y$. Let $Z \subseteq Y$. The topology τ_Z on Y defined as $\tau_Z = \{Z \cap O :\ O \in \tau\}$ is called the *relative topology* on Y. In this case, (Z, τ_Z) is called a *subspace* of (Y, τ). Let Y and Z be topological spaces. A function $f : Y \to Z$ is said to be *continuous* if the inverse image of each open set in Z is open in Y; *open* if the image of each open set in Y is open in Z; and *closed* if the image of each closed set in Y is closed in Y. A space Y is *connected* if it is not the union of two non-empty disjoint open sets. A subset B of Y is *connected* if it is connected as a subspace of Y. A space Y is *disconnected* if $Y = A \cup B$ where $\varnothing \neq A, B \in \tau$ such that $A \cap B = \varnothing$. Then $A \cup B$ is a *decomposition* of Y. Let $p \in Y$. The topology τ_p given by $\tau_p = \{\varnothing\} \cup \{A \subseteq Y :\ p \in A\}$ is called a *particular point topology* on Y. All topological concepts above are found in [2, 6, 10].

Theorem 1. [6] *Let $\mathcal{B}$ be a class of subsets of a nonempty set Y. Then $\mathcal{B}$ is a base for some topology on Y if and only if it possesses the following two properties:*

(i) $Y = \bigcup\{B :\ B \in \mathcal{B}\}$.

(ii) *For any $B, B^* \in \mathcal{B}$, $B \cap B^*$ is the union of members of $\mathcal{B}$, or, equivalently, if $p \in B \cap B^*$ then there exists B_p such that $p \in B_p \subseteq B \cap B^*$.*

Theorem 2. [2] *Let Y be a topological space, and $\mathcal{B} \subseteq \tau$. Then $\mathcal{B}$ is a basis for τ if and only if for each $G \in \tau$ and for each $x \in G$, there exists $U \in \mathcal{B}$ such that $x \in U \subseteq G$.*

Theorem 3. [2] *Let (Y, τ) be a topological space and (Z, τ_Z) be a subspace. If $\{U_\alpha :\ \alpha \in \mathcal{A}\}$ is a basis (subbasis) for τ, then $\{Z \cap U_\alpha :\ \alpha \in \mathcal{A}\}$ is a basis (subbasis) for τ_Z.*

Theorem 4. [2] *Let Y, Z be topological spaces and $f : Y \to Z$ a map. The following statements are equivalent:*

(i) *f is continuous.*

(ii) *The inverse image of each closed set in Z is closed in Y.*

(iii) *The inverse image of each member of a subbasis (basis) for Z is open in Y (not necessarily a member of subbasis, or basis for Y).*

2. Some Properties of $r_X(A)$

Definition 1. *Let X be a BE-algebra. For any $A \subseteq X$, the set $r_X(A) = \{x \in X \mid a * x = 1_X, \forall a \in A\}$ is called the subset of X determined by right application of BE-ordering on A. Note that $r_X(\{a\}) = [a, 1_X]$ for all $a \in X$.*

Theorem 5. *Let A and B be subsets of X. Then the following hold:*

(i) $r_X(\varnothing) = X$.

(ii) *If $A \subseteq B$, then $r_X(B) \subseteq r_X(A)$.*

(iii) *If X is a transitive BE-algebra, then $r_X(r_X(A)) \subseteq r_X(A)$.*

Proof. To prove (i), suppose $r_X(\varnothing) \neq X$. Then there exists $x \in X$ such that $x \notin r_X(\varnothing)$. Thus, there exists $a \in \varnothing$ such that $a * x \neq 1_X$, a contradiction. Therefore, $r_X(\varnothing) = X$.

To prove (ii), let $x \in r_X(B)$. Then $b * x = 1_X$ for all $b \in B$. Since $A \subseteq B$, $a * x = 1_X$ for all $a \in A$. Thus, $x \in r_X(A)$. Hence, $r_X(B) \subseteq r_X(A)$.

To prove (iii), let $x \in r_X(r_X(A))$. Then $b * x = 1_X$ for all $b \in r_X(A)$. Since $a * b = 1_X$ for all $a \in A$ and X is transitive, it follows that $a * x = 1_X$ for all $a \in A$. Thus, $x \in r_X(A)$. Hence, $r_X(r_X(A)) \subseteq r_X(A)$. whitehsdgkjgaskdj □

Theorem 6. *Let X be a BE-algebra and $A \subseteq X$. Then $r_X(A) = \bigcap_{a \in A} [a, 1_X]$ and $1_X \in r_X(A)$. Furthermore, if $1_X \in A$, then $r_X(A) = \{1_X\}$.*

Proof. Note that $r_X(A) = \{x \in X \mid a * x = 1_X, \forall a \in A\} = \{x \in X \mid x \in r_X(\{a\}), \forall a \in A\} = \bigcap_{a \in A} r_X(\{a\}) = \bigcap_{a \in A} [a, 1_X]$. Let $a \in A$. Then $a * 1_X = 1_X$ for all $a \in A$. Thus, $1_X \in r_X(A)$. Now, $r_X(\{1_X\}) = \{y \in X \mid 1_X * y = 1_X\} = \{1_X\}$. Thus, if $1_X \in A$, then $r_X(A) \subseteq r_X(\{1_X\}) = \{1_X\}$, that is, $r_X(A) = \{1_X\}$ □

Theorem 7. *Let X be a self distributive BE-algebra and A be a nonempty subset of X. Then $r_X(A)$ is an ideal and a filter.*

Proof. By Theorem 6, $1_X \in r_X(A)$.

Let $x, y, z \in X$. Suppose that $y \in r_X(A)$. Then $a * y = 1_X$ for all $a \in A$. Let $x * (y * z) \in r_X(A)$. Then $a * (x * (y * z)) = 1_X$ for all $a \in A$. Since $x * (y * z) = y * (x * z)$, $a * (y * (x * z)) = 1_X$ for all $a \in A$. Since X is self distributive, $(a * y) * (a * (x * z)) = 1_X$ for all $a \in A$. Since $a * y = 1_X$ for all $a \in A$, $1_X * (a * (x * z)) = 1_X$ for all $a \in A$. This implies that $a * (x * z) = 1_X$. Hence, $x * z \in r_X(A)$. By Lemma 1, $r_X(A)$ is an ideal.

Suppose that $x \in r_X(A)$ and $x * y \in r_X(A)$. Then $a * x = 1_X$ and $a * (x * y) = 1_X$ for all $a \in A$. Since X is self distributive, $a * y = 1_X * (a * y) = (a * x) * (a * y) = a * (x * y) = 1_X$ for all $a \in A$. Thus, $y \in r_X(A)$. Therefore, $r_X(A)$ is a filter. □

3. A Basis $\mathcal{B}_r(X)$ for a topology on X

Lemma 2. *Let X be a BE-algebra and let $\{A_\alpha : \alpha \in I\}$ be a collection of subsets of X. Then $\bigcap_{\alpha \in I} r_X(A_\alpha) = r_X\left(\bigcup_{\alpha \in I} A_\alpha\right)$.*

Proof. Let $x \in \bigcap_{\alpha \in I} r_X(A_\alpha)$. Then $x \in r_X(A_\alpha)$ for all $\alpha \in I$. Thus, $a * x = 1_X$ for all $a \in A_\alpha$ and for all $\alpha \in I$. Hence, $a * x = 1_X$ for all $a \in \bigcup_{\alpha \in I} A_\alpha$. So, $x \in r_X\left(\bigcup_{\alpha \in I} A_\alpha\right)$ and $\bigcap_{\alpha \in I} r_X(A_\alpha) \subseteq r_X\left(\bigcup_{\alpha \in I} A_\alpha\right)$. The other inclusion is proved similarly. Therefore, the equality is true. □

Theorem 8. *Let X be a BE-algebra. Then $\mathcal{B}_r(X) = \{r_X(A) : \varnothing \neq A \subseteq X\}$ is a basis for some topology on X.*

Proof. Clearly, $X = \bigcup_{a \in X} r_X(\{a\})$. Suppose that $\varnothing \neq A, B \subseteq X$. By Lemma 2, $r_X(A) \cap r_X(B) = r_X(A \cup B) \in \mathcal{B}_r(X)$. By Theorem 1, $\mathcal{B}_r(X)$ is a basis for some topology on X. □

We denote by $\tau_r(X)$ the topology generated by $\mathcal{B}_r(X)$.

Example 2. *Consider the BE-algebra N_0 in Example 1. Let $z \in N_0$. Then $r_{N_0}(z) = \{0, 1, 2, \ldots, z\}$. It is easy to see that $\mathcal{B} = \{r_{N_0}(z) : z \in N_0\} \subseteq \mathcal{B}_r(N_0)$. Suppose that $\varnothing \neq A \subseteq N_0$ and $w = \min A$. By Theorem 6, $r_{N_0}(A) = \bigcap_{a \in A} r_{N_0}(\{a\})$. It follows that $r_{N_0}(A) = \{0, 1, 2, \ldots, w\} = r_{N_0}(w) \in \mathcal{B}$. Hence, $\mathcal{B}_r(N_0) = \mathcal{B} = \{r_{N_0}(z) : z \in N_0\}$. Let $\varnothing \neq G \in \tau_r(N_0)$. Then $G = \bigcup_{x \in K} r_{N_0}(x)$ for some $\varnothing \neq K \subseteq N_0$. Clearly, $K \subseteq G$. Suppose first that $|G| < \infty$ and let $v = \max K$. Then $G = r_{N_0}(v)$. Next, suppose that G is an infinite set. Suppose further that $G \neq N_0$, say $m \in N_0 \setminus G$. Then $m \notin r_{N_0}(x)$ for all $x \in K$. This implies that $x < m$ for all $x \in K$. Hence, $G \subseteq r_{N_0}(m)$, contrary to the assumption that G is an infinite set. Therefore, $G = N_0$. Accordingly, $\tau_r(N_0) = \{\varnothing, N_0\} \cup \{r_{N_0}(z) : z \in N_0\} = \{\varnothing, N_0\} \cup \mathcal{B}_r(N_0)$.*

Theorem 9. *Let X be a BE-algebra. Then $(X, \tau_r(X))$ is connected.*

Proof. Let $\varnothing \neq G \in \tau_r(X)$. By Theorem 2, there exists $A \subseteq X$ such that $r_X(A) \subseteq G$. By Theorem 6, $1_X \in G$. Thus, if U is a nonempty open set such that $U \neq G$, then $G \cap U \neq \varnothing$ since $1_X \in U$. Hence, X cannot have a decomposition, that is, $(X, \tau_r(X))$ is connected. □

Lemma 3. *Let X be a BE-algebra and $x \in X$. Then $\{x\} \in \tau_r(X)$ if and only if $x = 1_X$.*

Proof. Suppose that $x = 1_X$. Then $\{1_X\} = r_X(1_X) \in \mathcal{B}_r(X)$. Hence, $\{1_X\} \in \tau_r(X)$. Suppose that $\{x\} \in \tau_r(X)$. Then there exists $\varnothing \neq A \subseteq X$ such that $r_X(A) = \{x\}$. Since $1_X \in r_X(A)$, it follows that $x = 1_X$. □

Corollary 1. *Let X be a BE-algebra. Then $\tau_r(X)$ is the discrete topology on X if and only if $X = \{1_X\}$.*

Proof. Suppose that $X = \{1_X\}$. Then $\mathcal{B}_r(X) = \{r_X(1_X)\} = \{\{1_X\}\}$. Hence, $\tau_r(X) = \{\varnothing, X\}$, the discrete topology on X. Conversely, suppose that $\tau_r(X)$ is the discrete topology on X. Then $\{x\} \in \tau_r(X)$ for all $x \in X$. By Lemma 3, $X = \{1_X\}$. □

Theorem 10. *If X is a finite BE-algebra, then $\mathcal{S}_r(X) = \{r_X(\{a\}) : a \in X\}$ is a subbase of $\tau_r(X)$.*

Proof. Clearly, $\mathcal{S}_r(X) \subseteq \tau_r(X)$. By Theorem 6, $r_X(A) = \bigcap_{a\in A} r_X(\{a\})$ for each $\varnothing \neq A \subseteq X$. Since X is finite, it follows that every element of $\mathcal{B}_r(X)$ is a finite intersection of members of $\mathcal{S}_r(X)$. Thus, $\mathcal{S}_r(X)$ is a subbase of $\tau_r(X)$. whitejgvgvkhg □

Lemma 4. *Let X be a BE-algebra and $a \in X \setminus \{1_X\}$. Then $a \in \mathcal{A}(X)$ if and only if $r_X(\{a\}) = \{1_X, a\}$.*

Proof. Suppose that $a \in \mathcal{A}(X)$ and let $x \in r_X(\{a\})$. Then $a \leq x$. Since $a \in \mathcal{A}(X)$, $x = 1_X$ or $x = a$. Thus, $r_X(\{a\}) = \{1_X, a\}$. Conversely, suppose that $r_X(\{a\}) = \{1_X, a\}$. Then $a \leq x$ implies that $x = 1_X$ or $x = a$. Therefore, $a \in \mathcal{A}(X)$. white hgzgjykuiahlu □

Theorem 11. *Let X be a BE-algebra with $|X| \geq 2$. Then*

$$\mathcal{B}_r(X) = \{\{1_X, a\} : a \in \mathcal{A}(X)\} \bigcup \{r_X(A) :\ A \cap \mathcal{A}(X) = \varnothing\}.$$

Proof. By Lemma 4, $r_X(a) = \{1_X, a\} \in \mathcal{B}_r(X)$ for each $a \in \mathcal{A}(X)$. Let $\varnothing \neq A \subseteq X$ such that $A \cap \mathcal{A}(X) \neq \varnothing$, say $z \in A \cap \mathcal{A}(X)$. If $1_X \in A$, then by Theorem 6, $r_X(A) = \{1_X\}$. Suppose that $1_X \notin A$. Since $z \in \mathcal{A}(X)$ and by Theorem 5(ii), $r_X(A) \subseteq r_X(z) = \{1_X, z\}$ and $1_X \in r_X(A)$, it follows that $r_X(A) = \{1_X\}$ or $r_X(A) = \{1_X, z\}$. This proves the assertion. □

Corollary 2. *Let X be a BE-algebra with $|X| \geq 2$. If $\mathcal{A}(X) = \{a\}$, then $\mathcal{B}_r(X) = \{\{1_X, a\}\} \cup \{r_X(A) :\ a \notin A\}$.*

Example 3. *Consider the BE-algebra N_0 in Example 2. For any $x \in N_0$, $r_{N_0}(x) = \{0, 1, \ldots, x\}$. Hence, $\mathcal{A}(N_0) = \{1\}$. By Corollary 2, $\mathcal{B}_r(N_0) = \{\{0, 1\}\} \cup \{r_{N_0}(A) :\ 1 \notin A\} = \{r_{N_0}(y) :\ y \in N_0\}$. Therefore, $\tau_r(N_0) = \{\varnothing, N_0\} \cup \{r_{N_0}(y) :\ y \in N_0\}$.*

Theorem 12. *Let X be a BE-algebra with $|X| \geq 2$. Then*
$\mathcal{B}_r(X) = \{\{1_X\}\} \cup \{\{1_X, a\} : a \in X \setminus \{1_X\}\}$ if and only if X is dual atomistic.

Proof. Suppose that X is dual atomistic. By Lemma 4, $r_X(a) = \{1_X, a\}$ for all $a \in X \setminus \{1_X\}$. The only $\varnothing \neq A \subseteq X$ such that $A \cap \mathcal{A}(X) = \varnothing$ is $A = \{1_X\}$. By Theorem 6, $r_X(A) = \{1_X\}$. Thus, $\mathcal{B}_r(X) = \{r_X(\{a\}) :\ a \in X\} = \{1_X\} \cup \{\{1_X, a\} :\ a \in X \setminus \{1_X\}\}$.

Conversely, suppose that $\mathcal{B}_r(X)$ is the given family of subsets of X. Let $a \in X \setminus \{1_X\}$. Then $r_X(\{a\}) = \{1_X, a\}$. Hence, if $x \in X$ and $a \leq x$, then $x = a$ or $x = 1_X$. Thus, $a \in \mathcal{A}(X)$. Accordingly, X is dual atomistic. □

4. Characterizations Involving the Topology $\tau_r(X)$

This section gives some characterizations of the elementary concepts associated with the topological space $(X, \tau_r(X))$.

Theorem 13. *Let X be a BE-algebra with $|X| \geq 2$. Then $\tau_r(X)$ is the particular point 1_X topology τ_{1_X} on X if and only if X is dual atomistic.*

Proof. Suppose that $\tau_r(X) = \tau_{1_X} = \{\varnothing\} \cup \{A \subseteq X : 1_X \in A\}$ and let $A \in \tau_{1_X} \setminus \{\varnothing\}$ such that $|A| \geq 2$. Then $A = \bigcup_{a \in A} \{1_X, a\}$, $a \neq 1_X$. If $|A| = 1_X$, then $A = \{1_X\}$. This implies that $\mathcal{B}_r(X) = \{\{1_X\}\} \cup \{\{1_X, a\} : a \in X \setminus \{1_X\}\}$ is a basis for $\tau_{1_X} = \tau_r(X)$. Hence, by Theorem 12, X is dual atomistic.

Conversely, suppose that X is a dual atomistic. By Theorem 12, $\mathcal{B}_r(X) = \{1_X\} \cup \{\{1_X, a\} : a \in X \setminus \{1_X\}\}$. Let $A \in \tau_r(X)$. Since $\mathcal{B}_r(X)$ is a basis for $\tau_r(X)$, $A = \{1_X\}$ or $A = \bigcup_{a \in A} \{1_X, a\}$. Thus, $1_X \in A$ implying that $A \in \tau_{1_X}$. Hence, $\tau_r(X) \subseteq \tau_{1_X}$. Now, let $A \in \tau_{1_X}$. Then $1_X \in A$. Since $\mathcal{B}_r(X)$ is a basis for $\tau_r(X)$, $A = \{1_X\}$ or $A = \bigcup_{a \in A} \{1_X, a\}$. Therefore, $A \in \tau_r(X)$ and $\tau_{1_X} \subseteq \tau_r(X)$. Consequently, $\tau_r(X) = \tau_{1_X}$. □

In a dual atomistic BE-algebra X with respect to $\tau_r(X)$, every set that contains 1_X is open and every set that does not contain 1_X is closed. Hence, the following corollary is true.

Corollary 3. *Let X be a dual atomistic BE-algebra with $|X| \geq 2$ and let $O, C \subseteq X$. Then with respect to $\tau_r(X)$, we have*

(i)
$$Int(O) = \begin{cases} \varnothing & \text{if } 1_X \notin O \\ O & \text{if } 1_X \in O, \end{cases} \qquad \text{and}$$

(ii)
$$\overline{C} = \begin{cases} X & \text{if } 1_X \in C \\ C & \text{if } 1_X \notin C. \end{cases}$$

Theorem 14. *Let X be a BE-algebra and let $D \subseteq X$. Then with respect to $\tau_r(X)$, we have*

(i) *$z \in Int(D)$ if and only if there exists $\varnothing \neq B \subseteq X$ such that $b * z = 1_X$ for all $b \in B$ and for all $x \in X$, $x \in D$ whenever $b * x = 1_X$ for all $b \in B$.*

(ii) *$y \in \overline{D}$ if and only if for each $\varnothing \neq A \subseteq X$ with $a * y = 1_X$ for all $a \in A$, there exists $d \in D$ such that $a * d = 1_X$ for all $a \in A$.*

(iii) *D is dense in X if and only if $1_X \in D$. In particular, $\{1_X\}$ is dense in X.*

Proof.

(i) By definition, $z \in Int(D)$ if and only if there exists $\varnothing \neq B \subseteq X$ such that $z \in r_X(B) \subseteq D$, that is, $b * z = 1_X$ for all $b \in B$ and for all x in X, $x \in D$ whenever $b * x = 1_X$ for all $b \in B$.

(ii) By definition, $y \in \overline{D}$ if and only if for each $\varnothing \neq A \subseteq X$ with $y \in r_X(A)$, we have $D \cap r_X(A) \neq \varnothing$, that is, there exists $d \in D \cap r_X(A)$. Thus, (ii) holds.

(iii) Let D be dense in X. Then $1_X \in \overline{D} = X$. Since $\{1_X\} = r_X(1_X)$, $D \cap r_X(1_X) \neq \varnothing$, it follows that $1_X \in D$. Conversely, suppose that $1_X \in D$. Let $x \in X$ and let $\varnothing \neq A \subseteq X$ such that $x \in r_X(A)$. Since $1_X \in r_X(A)$ by Theorem 6, it follows that $r_X(A) \cap D \neq \varnothing$. Thus, $x \in \overline{D}$, showing that $\overline{D} = X$. whiteyughjgkh □

Lemma 5. *Let S be a subalgebra of a BE-algebra X. Then*

(i) $\mathcal{A}(X) \cap S \subseteq \mathcal{A}(S)$*; and*

(ii) $r_S(T) = r_X(T) \cap S$ *for every* $T \subseteq S$.

Proof.

(i) Let $a \in \mathcal{A}(X) \cap S$. Then $a \in S$ and for all $x \in X$, $a \leq x$ implies that $x = a$ or $x = 1_X$. Hence, in particular, for all $y \in S$, $a \leq y$ implies that $y = 1_X$ or $y = a$. Thus, $a \in \mathcal{A}(S)$.

(ii) Let $T \subseteq S$. Then $z \in r_S(T)$ if and only if $z \in S$ and $t \leq z$ for all $t \in T$. Thus, $z \in r_S(T)$ if and only if $z \in S \cap r_X(t)$ for each $t \in T \subseteq S \subseteq X$. Accordingly, $r_S(T) = S \cap r_X(T)$. □

Lemma 6. *Let S be a subalgebra of a transitive BE-algebra X. Then for any $\varnothing \neq A \subseteq X$,* $r_X(A) \cap S = \bigcup_{x \in r_X(A) \cap S} r_S(x)$.

Proof. Suppose that $\varnothing \neq A \subseteq X$ and let $x \in r_X(A) \cap S$. Then $a * x = 1_X$ for all $a \in A$ and $x \in S$. Let $y \in r_X(x) \cap S$. Then $x * y = 1_X$ and $y \in S$. Since X is transitive, $a * y = 1_X$ for all $a \in A$. Hence, $y \in r_X(A) \cap S$ showing that $r_X(x) \cap S \subseteq r_X(A) \cap S$. Consequently, $\bigcup_{x \in r_X(A) \cap S} (r_X(x) \cap S) \subseteq r_X(A) \cap S$.

Next, let $z \in r_X(A) \cap S$. Clearly, $z \in r_X(z)$. It follows that $z \in r_X(z) \cap S$ showing that $r_X(A) \cap S \subseteq r_X(z) \cap S$. Thus, $r_X(A) \cap S \subseteq \bigcup_{x \in r_X(A) \cap S} (r_X(x) \cap S)$. Therefore, by Lemma 5(ii), $r_X(A) \cap S = \bigcup_{x \in r_X(A) \cap S} (r_X(x) \cap S) = \bigcup_{x \in r_X(A) \cap S} r_S(x)$. □

Theorem 15. *Let S be a subalgebra of a transitive BE-algebra X with $|S| \geq 2$. Then $\tau_r(S)$ coincides with the relative topology τ_S on S.*

Proof. By Theorem 11, a basis for $\tau_r(S)$ is the family

$$\mathcal{B}_r(S) = \{\{1_X\}\} \cup \{\{1_X, a\} : a \in \mathcal{A}(S)\} \cup \{r_S(A) : A \subseteq S \text{ and } A \cap \mathcal{A}(S) = \varnothing\}.$$

By Theorem 3, a basis for the relative topology τ_S on S is given by

$\mathcal{B}_S = \{\{1_X\}\} \cup \{\{1_X, a\} : a \in S \cap \mathcal{A}(X)\} \cup \{r_X(A) \cap S : A \subseteq S \text{ and } A \cap \mathcal{A}(X) = \varnothing\}$.

Suppose that $\mathcal{A}(S) \setminus \mathcal{A}(X) = \varnothing$. Then $\mathcal{A}(S) \subseteq \mathcal{A}(X)$. Since $\mathcal{A}(S) \subseteq S$, for every $a \in \mathcal{A}(S)$, we have $a \in S \cap \mathcal{A}(X)$. Thus, $\{1_X, a\} \in \mathcal{B}_S$. Now, suppose that $\mathcal{A}(S) \setminus \mathcal{A}(X) \neq \varnothing$. Let $a \in \mathcal{A}(S) \setminus \mathcal{A}(X)$ such that $\{1_X, a\} \in \mathcal{B}_r(S)$. Then $\{a\} \subseteq S$ and $\{a\} \cap \mathcal{A}(X) = \varnothing$. By Lemma 5(ii), $\{1_X, a\} = r_S(\{a\}) = r_X(\{a\}) \cap S \in \mathcal{B}_S$. Next, let $\varnothing \neq A \subseteq S$ such that $A \cap \mathcal{A}(S) = \varnothing$. Then $r_S(A) \in \mathcal{B}_r(S)$. Since by Lemma 5(i), $A \cap \mathcal{A}(X) = (A \cap S) \cap \mathcal{A}(X) = A \cap (S \cap \mathcal{A}(X)) \subseteq A \cap \mathcal{A}(S) = \varnothing$. This implies that by Lemma 5(ii), $r_S(A) = r_X(A) \cap S \in \mathcal{B}_S$. Thus, $\mathcal{B}_r(S) \subseteq \mathcal{B}_S$. By Lemma 5(i), $\{\{1_X\}\} \cup \{\{1_X, a\} : a \in S \cap \mathcal{A}(X)\} \subseteq \{\{1_X\}\} \cup \{\{1_X, a\} : a \in \mathcal{A}(S)\}$. Let $\varnothing \neq A \subseteq S$ such that $A \cap \mathcal{A}(X) = \varnothing$. If $A \cap \mathcal{A}(S) = \varnothing$, then $r_S(A) \in \mathcal{B}_r(S)$. Suppose that $A \cap \mathcal{A}(S) \neq \varnothing$, say $w \in A \cap \mathcal{A}(S)$. By Theorem 5(ii), $r_S(A) \subseteq r_S(w) = \{1_X, w\}$. Hence, $r_S(A) = \{1_X\}$ or $r_S(A) = \{1_X, w\}$. Thus, $r_S(A) \in \{\{1_X\}\} \cup \{\{1_X, a\} : a \in \mathcal{A}(S)\} \subseteq \mathcal{B}_r(S)$. Therefore, $\mathcal{B}_S \subseteq \mathcal{B}_r(S)$. Consequently, $\mathcal{B}_r(S) = \mathcal{B}_S$, showing that $\tau_r(S) = \tau_S$. □

Theorem 16. *Let $(X_1, *_{X_1}, 1_{X_1})$ and $(X_2, *_{X_2}, 1_{X_2})$ be BE-algebras. Then a function $f : (X_1, \tau_r(X_1)) \to (X_2, \tau_r(X_2))$ is continuous on X_1 if and only if for each $B \subseteq X_2$ and for each $x \in X_1$ such that $b \leq f(x)$ for all $b \in B$, there exists $A \subseteq X_1$ satisfying the following conditions:*

(i) *$a \leq x$ for all $a \in A$*

(ii) *$b \leq f(z)$ for all $b \in B$ whenever $a \leq z$ for all $a \in A$.*

Proof. By Theorem 4, f is continuous on X_1 if and only if $f^{-1}(G) \in \tau_r(X_1)$ for each $G \in \mathcal{B}_r(X_2)$. By Theorem 8, f is continuous if and only if for each $B \subseteq X_2$, $f^{-1}(r_{X_2}(B)) \in \tau_r(X_1)$, that is, $b \leq f(x)$ for all $b \in B$. Now, $f^{-1}(r_{X_2}(B)) \in \tau_r(X_1)$ if and only if for each $x \in f^{-1}(r_{X_2}(B))$ there exists $A \subseteq X_1$ (hence $r_{X_1}(A) \in \mathcal{B}_r(X_1)$) such that $x \in r_{X_1}(A) \subseteq f^{-1}(r_{X_2}(B))$. Since $x \in r_{X_1}(A)$, $a \leq x$ for all $a \in A$. Now, suppose that $a \leq z$ for all $a \in A$. Then $z \in r_{X_1}(A) \subseteq f^{-1}(r_{X_2}(B))$. Thus, $z \in f^{-1}(r_{X_2}(B))$. Hence, $f(z) \in r_{X_2}(B)$. Therefore, $b \leq f(z)$ for all $b \in B$. □

Theorem 17. *Let $(X_1, *_{X_1}, 1_{X_1})$ and $(X_2, *_{X_2}, 1_{X_2})$ be BE-algebras and let $f : (X_1, \tau_r(X_1)) \to (X_2, \tau_r(X_2))$ be a function. Then*

(i) *f is open if and only if for each $A \subseteq X_1$ and for each $x \in X_1$ with $a \leq x$ for all $a \in A$, there exists $B \subseteq X_2$ satisfying the following properties:*

   (a) *$b \leq f(x)$ for all $b \in B$*

   (b) *there exists $z \in X_1$ with $a \leq z$ for all $a \in A$ and $f(z) = y$ whenever $a \leq f^{-1}(y)$ for all $a \in A$ and $b \leq y$ for all $b \in B$.*

(ii) *f is closed if and only if for each $\tau_r(X_1)$-closed set F and for all $y \in X_2$ with $y \neq f(x)$ for all $x \in F$, there exists $A_y \subseteq X_2$ such that $r_{X_2}(A_y) \cap f(F) = \varnothing$ and $a \leq y$ for all $a \in A_y$.*

Proof.

(i) By definition, f is open if and only if $f(r_{X_1}(A)) \in \tau_r(X_2)$ for each $A \subseteq X_1$. Now, $f(r_{X_1}(A)) \in \tau_r(X_2)$ if and only if for each $x \in r_{X_1}(A)$, there exists $B \subseteq X_2$ such that $f(x) \in r_{X_2}(B) \subseteq f(r_{X_1}(A))$. Since $f(x) \in r_{X_2}(B)$, $b \leq f(x)$ for all $b \in B$. Moreover, if $a \leq f^{-1}(y)$ for all $a \in A$ and $b \leq y$ for all $b \in B$, then $f^{-1}(y) \in r_{X_1}(A)$ and $y \in r_{X_2}(B)$. This implies that $y \in r_{X_2}(B) \subseteq f(r_{X_1}(A))$. Consequently, there exists $z \in X_1$ such that $z \in r_{X_1}(A)$ and $y = f(z)$. Hence, $a \leq z$ for all $a \in A$ and $y = f(z)$.

(ii) Suppose that f is closed and let F be a closed subset of X_1. Then by definition of a closed map, $f(F) = \{f(x) : x \in F\}$ is closed in X_2, that is, $[f(F)]^c = \{f(x) : x \in F\}^c = \bigcup_{A \in \mathcal{P}_2} r_{X_2}(A)$, where $\mathcal{P}_2 \subseteq \mathcal{P}(X_2) \setminus \{\varnothing\}$. Hence, for each $y \in X_2$ such that $y \neq f(x)$ for all $x \in F$, there exists $A_y \subseteq X_2$ such that $r_{X_2}(A_y) \subseteq [f(F)]^c$ and $a \leq y$ for all $a \in A_y$.

Conversely, suppose that for each closed subset F of X_1 and for all $y \in X_2$ with $y \neq f(x)$ for all $x \in F$, there exists $A_y \subseteq X_2$ such that $r_{X_2}(A_y) \cap f(F) = \varnothing$ and $a \leq y$ for all $a \in A_y$. Let F^* be a closed set in X_1 and let $y \in [f(F^*)]^c$. Then $y \in X_2$ and $y \neq f(x)$ for all $x \in F^*$. By assumption, there exists $A_y \subseteq X_2$ such that $r_{X_2}(A_y) \cap f(F^*) = \varnothing$ and $a \leq y$ for all $a \in A_y$, that is, $y \in r_{X_2}(A_y)$. Thus, $[f(F^*)]^c = \bigcup_{y \in [f(F^*)]^c} r_{X_2}(A_y)$. Hence, $[f(F^*)]^c$ is $\tau_r(X_2)$-open showing that $f(F^*)$ is a closed subset of X_2. Therefore, f is a closed map. □

5. Conclusion

The topology generated by the family of subsets determined by the right application of BE-ordering of a BE-algebra is always connected. Investigations for some elementary topological concepts as well as the concepts of continuous, open, and closed maps associated with this topological space are obtained. This paper will lead to some studies on separation axioms associated with this kind of topological space.

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References

[1] S.S. Ahn and K.S. So. On ideals and upper sets in BE-algebras. *Scientiae Mathematicae Japonicae*, 68(2):279–285, 2008.

[2] J. Dugundji. *Topology.* Prentice Hall of India Private Ltd., New Delhi, 1975.

[3] Y. Imai and K. Iséki. On Axiom systems of Propositional Calculi XIV. *Proc. Japan Academy*, 42:19–22, 1996.

[4] H.S. Kim and Y.H. Kim. On BE-algebras. *Scientiae Mathematicae Japonicae Online*, pages 1299–1302, 2004.

[5] K.H. Kim and Y.H. Yon. Dual BCK-Algebra and MV-algebra. *Scientiae Mathematicae Japonicae Online*, pages 393–399, 2007.

[6] S. Lipschutz. *Schaum's Outines: General Topology.* Schaum Pub. Co., New York, 1965.

[7] S. Mehrshad and J. Golzarpoor. On topological BE-algebras. *Mathematica Moravica*, 21(2):1–13, 2017.

[8] S.R. Mukkamala. *A Course in BE-algebra.* Springer Nature Singapore Pte Ltd., Singapore, 2018.

[9] S.Z. Song and et.al. Fuzzy Ideals in BE-algebras. *Bull. Malays. Math. Sci. Soc.*, 33:147–153, 2010.

[10] L.A. Steen and J.A. Seebach Jr. *Counterexamples in Topology.* New York: Springer-Verlag, Berlin, 1978.