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Topologies on a Hyper Sum and Hyper Product of Two Hyper BCK-algebras

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Abstract. Given a hyper BCK-algebra $(H, *, 0)$, each of the families $\mathcal{B}_L(H) = \{L_H(A) : \varnothing \neq A \subseteq H\}$ and $\mathcal{B}_R(H) = \{R_H(A) : \varnothing \neq A \subseteq H\}$ forms a base for some topology on H, where $L_H(A) = \{x \in H : x \ll a, \forall a \in A\}$ and $R_H(A) = \{x \in H : a \ll x, \forall a \in A\}$ for any subset A of H. In this paper, we determine the bases of the topologies induced by the hyper sum $H_1 \oplus H_2$ and hyper product $H_1 \times H_2$, where $(H_1, *_1, 0_1)$ and $(H_2, *_2, 0_2)$ are two hyper BCK-algebras.

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1. Introduction

Although algebra and topology seem to differ generally in their nature, they appear together in some areas of mathematics such as functional analysis, dynamical systems, and representation theory. Previous studies (see [2]) would show the blend of algebraic and of topological structures. Indeed, there are various ways of introducing a a topological structure in a given algebraic structure. For example, in the definition of a topological group, the requirement imposed is that the topology on a given group is the one that makes the multiplication and inversion maps continuous. However, given an algebraic structure (or hyperstructure), it may be possible to find some family of subsets of the underlying set that will serve as base for some topology on the set. This approach can then give rise to a structure that is both algebraic and topological.

The present study considers an algebraic structure which is a decendant of BCK-algebra, an algebraic structure that was introduced and investigated by Y. Imai and K. Iséki [5] in 1966. This variant of BCK-algebra utilizes the hyperstructure theory introduced

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by F. Marty [7] at the 8th Congress of Scandinavian Mathematicians in 1934. Specifically, Y.B. Jun et al. [6] applied the hyperstructure theory to BCK-algebras and introduced the notion of a hyper BCK-algebra. Recently, Patangan and Canoy [8, 9] showed that the families $\mathcal{B}_L(H) = \{L_H(A) : \varnothing \neq A \subseteq H\}$ and $\mathcal{B}_R(H) = \{R_H(A) : \varnothing \neq A \subseteq H\}$, where $L_H(A) = \{x \in H : \ x \ll a, \ \forall a \in A\}$ and $R_H(A) = \{x \in H : \ a \ll x, \ \forall a \in A\}$ for any subset A of H, are bases for some topologies on a hyper BCK-algebra $(H, *, 0)$. Thus, given a hyper BCK-algebra, two different topological structures are generated and investigated.

A *hyper BCK-algebra* is a nonempty set H endowed with a hyperoperation " $*$ " and a constant 0 satisfying the following axioms: for all $x, y, z \in H$,

(H1) $(x * z) * (y * z) \ll x * y$,

(H2) $(x * y) * z = (x * z) * y$,

(H3) $x * H \ll x$,

(H4) $x \ll y$ and $y \ll x$ imply $x = y$,

where for every $A, B \subseteq H$, $A \ll B$ if and only if for each $a \in A$, there exists $b \in B$ such that $0 \in a * b$. In particular, for every $x, y \in H$, $x \ll y$ if and only if $0 \in x * y$. In such case, we call " $\ll$ " the *hyper order* in H.

Throughout this study, $(H_1, *_1, 0_1)$ (or simply H_1) and $(H_2, *_2, 0_2)$ (or simply H_2) are hyper BCK-algebras.

Let H be a hyper BCK-algebra and $A \subseteq H$. The sets $L_H(A)$ and $R_H(A)$ are given as follows:

$$L_H(A) := \{x \in H \mid x \ll a \ \forall a \in A\} = \{x \in H \mid 0 \in x * a \ \forall a \in A\} \quad \text{and}$$
$$R_H(A) := \{x \in H \mid a \ll x \ \forall a \in A\} = \{x \in H \mid 0 \in a * x \ \forall a \in A\}.$$

If $A = \{a\}$, we write $L_H(\{a\}) = L_H(a)$ and $R_H(\{a\}) = R_H(a)$.

Let $(H_1, *_1, 0)$ and $(H_2, *_2, 0)$ be hyper BCK-algebras such that $H_1 \cap H_2 = \{0\}$ and $H = H_1 \cup H_2$. Then $(H, *, 0)$ is a hyper BCK-algebra denoted by $H_1 \oplus H_2$, called the *hyper sum*, where the hyperoperation " $*$ " on H is defined for all $x, y \in H$ by,

$$x * y = \begin{cases} x *_1 y & \text{if } x, y \in H_1 \\ x *_2 y & \text{if } x, y \in H_2 \\ \{x\} & \text{otherwise.} \end{cases}$$

Let $(H_1, *_1, 0_1)$ and $(H_2, *_2, 0_2)$ be hyper BCK-algebras and $H = H_1 \times H_2$. Define a hyperoperation "$*$" on H as follows: for all $(a_1, b_1), (a_2, b_2) \in H$, $(a_1, b_1) * (a_2, b_2) = (a_1 *_1 a_2, b_1 *_2 b_2)$. For $A \subseteq H_1$ and $B \subseteq H_2$, by (A, B) we mean $(A, B) = \{(a, b) : a \in A, b \in B\}$, $0 = (0_1, 0_2)$ and $(a_1, b_1) \ll (a_2, b_2) \iff a_1 \ll a_2$ and $b_1 \ll b_2$. Then $(H, *, 0)$ is a hyper BCK-algebra, and it is called the *hyper product* of H_1 and H_2.

2. Known Results

Proposition 2.1. [1] *Let A and B be subsets of a hyper BCK-algebra H. Then the following hold:*

(i) $L_H(\varnothing) = H$.

(ii) $L_H(A) = \bigcap_{a \in A} L_H(a)$.

(iii) *For any* $A \subseteq H$, $0 \in L_H(A)$. *If* $0 \in A$, *then* $L_H(A) = \{0\}$.

Proposition 2.2. [8] *Let* H *be a hyper BCK-algebra and* $A \subseteq H$. *Then the following hold:*

(i) $R_H(A) = \bigcap_{a \in A} R_H(a)$.

(ii) *For any* $\varnothing \neq A \subseteq H$ *such that* $A \neq \{0\}$, $0 \notin R_H(A)$.

(iii) $R_H(x) \neq \varnothing \ \ \forall x \in H$. *In particular,* $x \in R_H(x)$. *Furthermore,* $R_H(x) = H$ *if and only if* $x = 0 \ \ \forall x \in H$.

Theorem 2.3. [9] *The family* $\mathcal{B}_L(H) = \{L_H(A) : \varnothing \neq A \subseteq H\}$ *where* H *is a hyper BCK-algebra, is a basis for some topology on* H.

Theorem 2.4. [8] *The family* $\mathcal{B}_R(H) = \{R_H(A) : \varnothing \neq A \subseteq H\}$ *where* H *is a hyper BCK-algebra, is a basis for some topology on* H.

3. Bases of $\tau_L(H_1 \oplus H_2)$ and $\tau_R(H_1 \oplus H_2)$

Theorem 3.1. *Let* H *be a hyper sum of hyper BCK-algebras* H_1 *and* H_2 *with* $|H_1| \geq 2$ *and* $|H_2| \geq 2$. *Then* $\mathcal{B}_L(H) = \mathcal{B}_L(H_1 \oplus H_2) = \mathcal{B}_L(H_1) \cup \mathcal{B}_L(H_2)$.

Proof: Since $\mathcal{B}_L(H_1) \subseteq \mathcal{B}_L(H)$ and $\mathcal{B}_L(H_2) \subseteq \mathcal{B}_L(H)$, it follows that $\mathcal{B}_L(H_1) \cup \mathcal{B}_L(H_2) \subseteq \mathcal{B}_L(H)$. Next, let $V \in \mathcal{B}_L(H)$. Then there exists a nonempty set $B \subseteq H$ such that $V = L_H(B)$. Let $B_1 = B \cap H_1$ and $B_2 = B \cap H_2$. If $V = \{0\}$, then by Proposition 2.1(iii), $V = L_{H_1}(0) \in \mathcal{B}_L(H_1) \cup \mathcal{B}_L(H_2)$. So, suppose that $V \neq \{0\}$. Suppose further that $B_1 \neq \varnothing$ and $B_2 \neq \varnothing$. Choose $x, y \in B$ such that $x \in B_1$ and $y \in B_2$. Pick $u \in V \setminus \{0\}$. Then $u \ll x$ and $u \ll y$. If $u \in H_1$, then $u * y = \{u\}$. If $u \in H_2$, $u * x = \{u\}$. In both cases, we get a contradiction since $u \neq 0$. Therefore, either $B_1 = \varnothing$ or $B_2 = \varnothing$, say $B_2 = \varnothing$. Then $B = B_1 \subseteq H_1$. Hence, $V = L_H(B) = L_{H_1}(B) \in \mathcal{B}_L(H_1) \cup \mathcal{B}_L(H_2)$. Therefore, $\mathcal{B}_L(H) = \mathcal{B}_L(H_1) \cup \mathcal{B}_L(H_2)$. □

Theorem 3.2. *Let* H *be a hyper sum of hyper BCK-algebras* H_1 *and* H_2 *with* $|H_1| \geq 2$ *and* $|H_2| \geq 2$. *Then* $\mathcal{B}_R(H) \setminus \{\varnothing, H\} = (\mathcal{B}_R(H_1) \cup \mathcal{B}_R(H_2)) \setminus \{\varnothing, H_1, H_2\}$.

Proof: Let $P \in \mathcal{B}_R(H_1) \setminus \{\varnothing, H_1\}$. Since $P \neq H_1$, by Proposition 2.2(iii), there exists a nonempty set $A \subseteq H_1 \setminus \{0\}$ such that $P = R_{H_1}(A)$. But $A \subseteq H_1 \setminus \{0\} \subseteq H$, thus, $P = R_{H_1}(A) = R_H(A)$. Since $A \neq \{0\}$ and $P \neq \varnothing$, by Theorem 2.2(iii) and definition of a hyper sum, $P \neq H$ and $P \neq \varnothing$ in H. Consequently, $P = R_H(A) \in \mathcal{B}_R(H) \setminus \{\varnothing, H\}$. Similarly, if $Q \in \mathcal{B}_R(H_2) \setminus \{\varnothing, H_2\}$ then $Q \neq H$ and $Q \neq \varnothing$ in H. Hence, $Q = R_H(B) \in \mathcal{B}_R(H) \setminus \{\varnothing, H\}$. Accordingly, $(\mathcal{B}_R(H_1) \cup \mathcal{B}_R(H_2)) \setminus \{\varnothing, H_1, H_2\} \subseteq \mathcal{B}_R(H) \setminus \{\varnothing, H\}$.

Next, let $U \in \mathcal{B}_R(H) \setminus \{\varnothing, H\}$. Since $U \neq H$, by Proposition 2.2(iii), there exists a nonempty subset $D \subseteq H \setminus \{0\}$ such that $U = R_H(D)$. Let $D_1 = D \cap (H_1 \setminus \{0\})$ and $D_2 = D \cap (H_2 \setminus \{0\})$. Suppose that $D_1 \neq \varnothing$ and $D_2 \neq \varnothing$. Choose any $x \in D_1$ and any $y \in D_2$. Since $x, y \in D$, it follows that $x \ll u$ and $y \ll u$ for all $u \in U$. Pick $w \in U$. By the definition of a hyper sum, if $w \in H_1$, then $y * w = \{y\}$ and if $w \in H_2$, then $x * w = \{x\}$. Since x and y are nonzero, $y \not\ll w$ and $x \not\ll w$, a contradiction. Thus, either $D_1 = \varnothing$ or $D_2 = \varnothing$, that is, either $D = D_1$ or $D = D_2$. If $D = D_1$ then $U = R_{H_1}(D_1)$. Since $D_1 \neq \{0\}$ and $U \neq \varnothing$ in H, by Theorem 2.2(iii), $U \neq H_1$ and $U \neq \varnothing$ in H_1. Thus, $U = R_{H_1}(D_1) \in \mathcal{B}_R(H_1) \setminus \{\varnothing, H_1\} \subseteq [\mathcal{B}_R(H_1) \cup \mathcal{B}_R(H_2)] \setminus \{\varnothing, H_1, H_2\}$. In the same way, if $D = D_2$ then $U = R_{H_2}(D_2) \in \mathcal{B}_R(H_2) \setminus \{\varnothing, H_2\} \subseteq [\mathcal{B}_R(H_1) \cup \mathcal{B}_R(H_2)] \setminus \{\varnothing, H_1, H_2\}$. Therefore, $\mathcal{B}_R(H) \setminus \{\varnothing, H\} = (\mathcal{B}_R(H_1) \cup \mathcal{B}_R(H_2)) \setminus \{\varnothing, H_1, H_2\}$. □

4. Bases of $\tau_L(H_1 \times H_2)$ and $\tau_R(H_1 \times H_2)$

For any $\varnothing \neq D \subseteq H_1 \times H_2$, the H_1-projection and H_2-projection of D are, respectively, the sets $D_{H_1} = \{x \in H_1 : (x, y) \in D \text{ for some } y \in H_2\}$ and $D_{H_2} = \{y \in H_1 : (z, y) \in D \text{ for some } z \in D_{H_1}\}$. Now, for each $x \in S = D_{H_1}$, let $T_x = \{y \in D_{H_2} : (x, y) \in D\}$. Then $D = \bigcup_{x \in S} [\{x\} \times T_x]$.

Lemma 4.1. *Let $\{A_\alpha : \alpha \in I\}$ be a collection of subsets of a hyper BCK-algebra H. Then*

$$\bigcap_{\alpha \in I} L_H(A_\alpha) = L_H\left(\bigcup_{\alpha \in I} A_\alpha\right).$$

Proof: Let $\{A_\alpha : \alpha \in I\}$ be a collection of subsets of H. Then

$$\begin{aligned} x \in \bigcap_{\alpha \in I} L_H(A_\alpha) &\Leftrightarrow x \in L_H(A_\alpha) \quad \text{for all } \alpha \in I \\ &\Leftrightarrow x \ll a \quad \text{for all } a \in A_\alpha \text{ and for all } \alpha \in I \\ &\Leftrightarrow x \ll a \quad \text{for all } a \in \bigcup_{\alpha \in I} A_\alpha \\ &\Leftrightarrow x \in L_H\left(\bigcup_{\alpha \in I} A_\alpha\right). \end{aligned}$$

Therefore, the equality is true. □

Theorem 4.2. *Let H be a hyper product of hyper BCK-algebras H_1 and H_2. Then the following properties hold:*

(i) $L_H(A \times B) = L_{H_1}(A) \times L_{H_2}(B)$ *for* $A \subseteq H_1$ *and* $B \subseteq H_2$.

(ii) *If $\{A_\alpha : \alpha \in I\}$ and $\{B_\alpha : \alpha \in I\}$ are collections of subsets of H_1 and H_2, respectively, then*

$$\bigcap_{\alpha \in I} [L_{H_1}(A_\alpha) \times L_{H_2}(B_\alpha)] = L_{H_1}\left(\bigcup_{\alpha \in I} A_\alpha\right) \times L_{H_2}\left(\bigcup_{\alpha \in I} B_\alpha\right).$$

(iii) *If* $D = \bigcup\limits_{x \in S} (\{x\} \times T_x)$*, where* $S \subseteq H_1$ *and* $T_x \subseteq H_2$ *for each* $x \in S$*, then* $L_H(D) = \bigcap\limits_{x \in S} (L_{H_1}(x) \times L_{H_2}(T_x)) = L_{H_1}(S) \times L_{H_2}(\bigcup\limits_{x \in S} T_x)$.

Proof:

(i) Let A and B be subsets of H_1 and H_2, respectively. Then

$$\begin{aligned} L_H(A \times B) &= \{(x, y) \in H_1 \times H_2 : \ (x, y) \ll (a, b) \text{ for all } (a, b) \in A \times B\} \\ &= \{(x, y) \in H_1 \times H_2 : \ x \ll a \text{ and } y \ll b \ \forall a \in A \text{ and } b \in B\} \\ &= \{x \in H_1 : \ x \ll a \ \forall a \in A\} \times \{y \in H_2 : \ y \ll b \ \forall b \in B\} \\ &= L_{H_1}(A) \times L_{H_2}(B). \end{aligned}$$

(ii) Let $\{A_\alpha : \ \alpha \in I\}$ and $\{B_\alpha : \ \alpha \in I\}$ be collections of subsets of H_1 and H_2, respectively, and let $K = \bigcap\limits_{\alpha \in I} [L_{H_1}(A_\alpha) \times L_{H_2}(B_\alpha)]$. Then

$$\begin{aligned} (x, y) \in K &\Leftrightarrow (x, y) \in L_{H_1}(A_\alpha) \times L_{H_2}(B_\alpha) \quad \forall \alpha \in I \\ &\Leftrightarrow x \in L_{H_1}(A_\alpha) \text{ and } y \in L_{H_2}(B_\alpha) \quad \forall \alpha \in I \\ &\Leftrightarrow x \ll a \ \forall a \in A_\alpha \text{ and } y \ll b \ \forall b \in B_\alpha \text{ and } \forall \alpha \in I \\ &\Leftrightarrow x \ll a \ \forall a \in \bigcup_{\alpha \in I} A_\alpha \text{ and } y \ll b \ \forall b \in \bigcup_{\alpha \in I} B_\alpha \\ &\Leftrightarrow x \in L_{H_1}\left(\bigcup_{\alpha \in I} A_\alpha\right) \text{ and } y \in L_{H_2}\left(\bigcup_{\alpha \in I} B_\alpha\right) \\ &\Leftrightarrow (x, y) \in L_{H_1}\left(\bigcup_{\alpha \in I} A_\alpha\right) \times L_{H_2}\left(\bigcup_{\alpha \in I} B_\alpha\right). \end{aligned}$$

Therefore, the assertion is true.

(iii) Let $D = \bigcup\limits_{x \in S} (\{x\} \times T_x)$, where $S \subseteq H_1$ and $T_x \subseteq H_2$ for each $x \in S$. Then by Lemma 4.1, (i), and (ii),

$$\begin{aligned} L_H(D) &= L_H\left[\bigcup_{x \in S} (\{x\} \times T_x)\right] \\ &= \bigcap_{x \in S} [L_H(\{x\} \times T_x)] \\ &= \bigcap_{x \in S} [L_{H_1}(x) \times L_{H_2}(T_x)] \\ &= L_{H_1}(S) \times L_{H_2}\left(\bigcup_{x \in S} T_x\right). \end{aligned}$$

□

Theorem 4.3. *Let H be a hyper product of hyper BCK-algebras H_1 and H_2. Then $\mathcal{B}_L(H) = \mathcal{B}_L(H_1) \times \mathcal{B}_L(H_2)$.*

Proof: Let $U \in \mathcal{B}_L(H)$. Then there exists a nonempty set $D \subseteq H = H_1 \times H_2$ such that $U = L_H(D)$. Let $D = \bigcup_{x\in S}(\{x\} \times T_x)$ where $S \subseteq H_1$ and $T_x \subseteq H_2$ for each $x \in S$. Then $L_H(D) = L_{H_1}(S) \times L_{H_2}(\bigcup_{x\in S} T_x)$ by Theorem 4.2(iii). Hence, $U \in \mathcal{B}_L(H_1) \times \mathcal{B}_L(H_2)$, showing that $\mathcal{B}_L(H) \subseteq \mathcal{B}_L(H_1) \times \mathcal{B}_L(H_2)$. Next, let $V \in \mathcal{B}_L(H_1) \times \mathcal{B}_L(H_2)$. Then there exist nonempty sets $A \subseteq H_1$ and $B \subseteq H_2$ such that $V = L_{H_1}(A) \times L_{H_2}(B) = L_H(A \times B) \in \mathcal{B}_L(H)$ by Theorem 4.2(i). Thus, $\mathcal{B}_L(H_1) \times \mathcal{B}_L(H_2) \subseteq \mathcal{B}_L(H)$. Therefore, $\mathcal{B}_L(H) = \mathcal{B}_L(H_1) \times \mathcal{B}_L(H_2)$. □

Lemma 4.4. *Let $\{A_\alpha : \alpha \in I\}$ be a collection of subsets of a hyper BCK-algebra H. Then*

$$\bigcap_{\alpha\in I} R_H(A_\alpha) = R_H\left(\bigcup_{\alpha\in I} A_\alpha\right).$$

Proof: Let $\{A_\alpha : \alpha \in I\}$ be a collection of subsets of H. Then

$$\begin{aligned} x \in \bigcap_{\alpha\in I} R_H(A_\alpha) &\Leftrightarrow x \in R_H(A_\alpha) \quad \text{for all } \alpha \in I \\ &\Leftrightarrow a \ll x \quad \text{for all } a \in A_\alpha \text{ and for all } \alpha \in I \\ &\Leftrightarrow a \ll x \quad \text{for all } a \in \bigcup_{\alpha\in I} A_\alpha \\ &\Leftrightarrow x \in R_H\left(\bigcup_{\alpha\in I} A_\alpha\right). \end{aligned}$$

Therefore, the equality holds. □

Theorem 4.5. *Let H be a hyper product of hyper BCK-algebras H_1 and H_2. Then the following properties hold:*

(i) $R_H(A \times B) = R_{H_1}(A) \times R_{H_2}(B)$ *for $A \subseteq H_1$ and $B \subseteq H_2$.*

(ii) *If $\{A_\alpha : \alpha \in I\}$ and $\{B_\alpha : \alpha \in I\}$ are collections of subsets of H_1 and H_2, respectively, then*

$$\bigcap_{\alpha\in I} [R_{H_1}(A_\alpha) \times R_{H_2}(B_\alpha)] = R_{H_1}\left(\bigcup_{\alpha\in I} A_\alpha\right) \times R_{H_2}\left(\bigcup_{\alpha\in I} B_\alpha\right).$$

(iii) *If $E = \bigcup_{x\in P}(\{x\} \times T_x)$, where $P \subseteq H_1$ and $T_x \subseteq H_2$ for each $x \in P$, then* $R_H(E) = \bigcap_{x\in P}(R_{H_1}(x) \times R_{H_2}(T_x)) = R_{H_1}(P) \times R_{H_2}(\bigcup_{x\in P} T_x)$.

Proof:

(i) Let A and B be subsets of H_1 and H_2, respectively. Then

$$\begin{aligned} R_H(A\times B) &= \{(x,y)\in H_1\times H_2 :\ (a,b)\ll (x,y) \text{ for all } (a,b)\in A\times B\} \\ &= \{(x,y)\in H_1\times H_2 :\ a\ll x \text{ and } b\ll y\ \forall a\in A \text{ and } b\in B\} \\ &= \{x\in H_1 :\ a\ll x\ \forall a\in A\}\times\{y\in H_2 :\ b\ll y\ \forall b\in B\} \\ &= R_{H_1}(A)\times R_{H_2}(B). \end{aligned}$$

(ii) Let $\{A_\alpha :\ \alpha\in I\}$ and $\{B_\alpha :\ \alpha\in I\}$ be collections of subsets of H_1 and H_2, respectively, and let $Q=\bigcap_{\alpha\in I}[R_{H_1}(A_\alpha)\times R_{H_2}(B_\alpha)]$. Then

$$\begin{aligned} (x,y)\in Q &\Leftrightarrow (x,y)\in R_{H_1}(A_\alpha)\times R_{H_2}(B_\alpha)\quad \forall\alpha\in I \\ &\Leftrightarrow x\in R_{H_1}(A_\alpha) \text{ and } y\in R_{H_2}(B_\alpha)\quad \forall\alpha\in I \\ &\Leftrightarrow a\ll x\ \forall a\in A_\alpha \text{ and } b\ll y\ \forall b\in B_\alpha \text{ and } \forall\alpha\in I \\ &\Leftrightarrow a\ll x\ \forall a\in \bigcup_{\alpha\in I}A_\alpha \text{ and } b\ll y\ \forall b\in\bigcup_{\alpha\in I}B_\alpha \\ &\Leftrightarrow x\in R_{H_1}\left(\bigcup_{\alpha\in I}A_\alpha\right) \text{ and } y\in R_{H_2}\left(\bigcup_{\alpha\in I}B_\alpha\right) \\ &\Leftrightarrow (x,y)\in R_{H_1}\left(\bigcup_{\alpha\in I}A_\alpha\right)\times R_{H_2}\left(\bigcup_{\alpha\in I}B_\alpha\right). \end{aligned}$$

Therefore, the equality holds.

(iii) Let $E=\bigcup_{x\in P}(\{x\}\times T_x)$, where $P\subseteq H_1$ and $T_x\subseteq H_2$ for each $x\in P$. Then by Lemma 4.4, (i), and (ii),

$$\begin{aligned} R_H(E) &= R_H\left[\bigcup_{x\in P}(\{x\}\times T_x)\right] \\ &= \bigcap_{x\in P}[R_H(\{x\}\times T_x)] \\ &= \bigcap_{x\in P}[R_{H_1}(x)\times R_{H_2}(T_x)] \\ &= R_{H_1}(P)\times R_{H_2}\left(\bigcup_{x\in P}T_x\right). \end{aligned}$$

□

Theorem 4.6. *Let H be a hyper product of hyper BCK-algebras H_1 and H_2. Then $\mathcal{B}_R(H)=\mathcal{B}_R(H_1)\times\mathcal{B}_R(H_2)$.*

Proof: Let $D\in\mathcal{B}_R(H)$. Then there exists a nonempty set $E\subseteq H=H_1\times H_2$ such that $D=R_H(E)$. Let $E=\bigcup_{x\in P}(\{x\}\times T_x)$ where $P\subseteq H_1$ and $T_x\subseteq H_2$ for each $x\in P$.

Then $L_H(E) = R_{H_1}(P) \times R_{H_2}(\bigcup_{x\in P} T_x) \in \mathcal{B}_R(H_1) \times \mathcal{B}_R(H_2)$ by Theorem 4.5(iii). Hence, $\mathcal{B}_R(H) \subseteq \mathcal{B}_R(H_1) \times \mathcal{B}_R(H_2)$. Next, suppose that $F \in \mathcal{B}_R(H_1) \times \mathcal{B}_R(H_2)$. Then there exist nonempty sets $O \subseteq H_1$ and $U \subseteq H_2$ such that $F = R_{H_1}(O) \times R_{H_2}(U) = R_H(O \times U)$ by Theorem 4.5(i). Thus, $F \in \mathcal{B}_R(H)$, showing that $\mathcal{B}_R(H_1) \times \mathcal{B}_R(H_2) \subseteq \mathcal{B}_R(H)$. Therefore, $\mathcal{B}_R(H) = \mathcal{B}_R(H_1) \times \mathcal{B}_R(H_2)$. □

Conclusion: This study shows that, indeed, a topological structure may be generated from a given (hyper) algebraic structure by considering some family of subsets of the underlying set of the structure that would qualify as a base for some topology on the set. The topology generated in this way need not coincide with the topology for which continuity is imposed on some hyperoperations associated with the algebraic structure. In this study, the authors, using the construction of a topological structure they introduced, are able to determine the bases of the topologies generated by the hyper sum and hyper product of two hyper BCK-algebras.

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