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Identification of parameters of Richards equation using Modified hybrid Grey Wolf Optimizer-Genetic Algorithm (HmGWOGA)

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Abstract. In this paper, it is a question of identification of the parameters in the equation of Richards modelling the flow in unsaturated porous medium. The mixed formulation pressure head-moisture content has been used. The direct problem was solved using Multiquadratic Radial Basis Function (RBF-MQ) method which is a meshless method. The Newton-Raphson's method was used to linearize the equation. The function cost used is built by using the infiltration. The optimization method used is a meta-heuristic called Modified hybrid Grey Wolf Optimizer -Genetic Algorithm (HmGWOGA). A test on experimental data has been carried. We compared the results with genetic algorithms. The results showed that this new method was better than genetic algorithms.

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1. Introduction

The fluid movement in unsaturated porous medium is governed by the Richards equation [2, 24] which contains parameters that take into account type of the considered soil. The calculation of the water balance on a soil-scale requires knowledge of infiltration that is obtained by solving the unsaturated flow equation. However The hydrodynamic parameters of the soils involved in the equation are, in most cases, badly known. The values

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given in the literature are not precise values but intervals, hence the importance of the inverse modeling.

Estimating parameters in unsaturated porous environments is not trivial. Indeed, it requires numerical resolution of the Richards equation. The numerical methods used must allow a good estimate of hydraulic pressure and water content. Several methods have been proposed for solving the Richards equation: the finite difference method [2], the finite volume method [1], the finite element method, the mixed finite element method [6]) and the discontinuous finite element [22]. Theses methods are based on the mesh of the domain in which the problem must be discretized. The mesh must obey certain rules. For example, the elements should not be overwritten to prevent the associated Jacobian from degenerating. This makes their implementation difficult and expensive ins some cases.

To overcome these shortcomings, meshless methods called Radial basis function (RBF) have been developed since the 1970s. The idea is to reconstruct a function defined on a continuous space from the set of discrete values taken by this function on a not connected point cloud of the physical domain. Stevens and Power [23] used a implicit RBF method to solve the pressure h formulation of Richards equation. More recently, F. Motaman and al. [16] used the RBF-DQ method to solve the moisture content θ formulation. These two formulations have limits. In this work, we use the RBF-MQ method to solve the mixed formulation of Richards equation.

There exist many methods to solve the inverse problems [4, 12, 13, 19]. Most computing software in hydrogeology use deterministic methods. However most of these methods require a good knowledge of the solution. Indeed, these algorithms can not detect a global optimum and can stop with a local optimum. Moreover, these algorithms require a certain regularity of the functions to be optimized. However, this regularity is not always checked. Meta-heuristic optimization techniques are adapted better to the problems of optimization in which the size of the space of research is important, where the parameters interact in a complex way and where very little information on the function to be optimized is available [7, 15]. The function to be optimized can thus be the result of a simulation. These algorithms are often much more robust in their capacity to identify the total optimum with less sensitivity to the initial condition.

Modified hybrid Grey Wolf Optimizer-Genetic Algorithm (HmGWOGA) proposed by Sawadogo et al. [20] is a combination of Grey Wolf Optimizer algorithm (GWO) proposed by S. Mirjalili et al [15] and a version of genetic algorithm proposed in [21]. Tests were successfully performed on test functions [20]. In this work we use it for identifying parameters of the Richards equation. The rest of the paper is organized as follows:

the second part is devoted to the equation of Richards in one dimension and his resolution by RBF-MQ method; in the third part, we present the inverse problem to solve; the fourth part is devoted to the identification of the parameters of the equation of Richards by HmGWOGA; the fifth section present the results and discussions.

2. Direct problem

2.1. Mathematical model

There exist several formulations of the equation of Richards which models the flow in unsaturated porous medium but in this work, we use the mixed formulation pressure head-moisture content because the numerical solutions obtained with his mixed formulation are more precise [2, 14].
In one dimension, the mixed formulation is given by :

$$\begin{cases} \frac{\partial \theta(h)}{\partial t} + \frac{\partial}{\partial z} q(h) = f & \text{in } \Omega \times [0, T] \\ q(h) = -K(h)\frac{\partial h}{\partial z} - K(h) & \text{in } \Omega \times [0, T] \\ h = h_{init} & \text{in } \Omega \\ h = g_D & \text{on } \partial\Omega_D \times [0, T] \\ q(h) = g_N & \text{on } \partial\Omega_N \times [0, T] \end{cases} \tag{1}$$

with :

- $\Omega = [a, b] \subset \mathbb{R}$ represents a column of water infiltration;
- z denotes the vertical dimension:
- $h[L]$ the pressure head;
- g_D and g_N are respectively imposed pressure and flow on the boundaries $\partial\Omega_D$ and $\partial\Omega_N$;
- $q(h)$ the flow velocity;
- h_{init} is the initial pressure head and f is a source function;
- $\theta[L^3/L^3]$ the moisture content given by:

  $$\theta(h) = \frac{\theta_s - \theta_r}{(1 + (\alpha|h|)^n)^m} + \theta_r \tag{2}$$

  where θ_r the moisture content to saturation ($L^3.L^{-3}$), θ_s the residual moisture content ($L^3.L^{-3}$), α a parameter of form related to the mean size of the pores (L^{-1}), n a parameter related to the distribution of the sizes of pores ($-$). According to Mualem [17], we have $m = 1 - 1/n$.

- $K(h)$ is the insaturated hydraulic conductivity $[L/T]$. We use the relation of Van Genuchten [24] given by

  $$K(S_e) = K_s S_e^{1/2}(1 - (1 - S_e^{1/m})^m)^2 \tag{3}$$

  with K_S the effective saturated hydraulic conductivity $[L/T]$.
  S_e the effective saturation given by:

  $$S_e = \begin{cases} \frac{\theta - \theta_r}{\theta_s - \theta_r} & \text{si } h < 0 \\ 1 & \text{si } h \geq 0 \end{cases} \tag{4}$$

h and θ are related by the moisture capacity function $C(h)[1/L]$ defined by

$$C(h) = \frac{\partial \theta}{\partial h} \tag{5}$$

Whats gives

$$C(h) = -\alpha n(\theta_r - \theta_s) sign(h)(\frac{1}{n} - 1)(\alpha|h|)^{n-1}(1 + (\alpha|h|)^n)^{1/n-2} \tag{6}$$

To solve the problem (1), you need to know the parameters α, θ_S, θ_r, n and K_S.

2.2. Numerical resolution of direct problem

2.2.1. Description of RBF-MQ method

Appeared in the 1970s [3], it was Kansa who introduced the PDE resolution using the Radial Basis Funtions (RBF) method [10, 11].
We consider a numerical function $u(x), x \in \mathbb{R}^d$ where d is the space dimension. The MQ-RBF method consist to approximate u by

$$\hat{u}(x) = \sum_{j=1}^{M} \lambda_j \phi(\|x - x_j\|, c), \quad x \in \mathbb{R}^d \tag{7}$$

where
$x_j,\ j = 1, \dots, M$ are the *centers* of the RBF approximation
ϕ the basic radial function of Hardy [8] given by $\phi(r, c) = \sqrt{r^2 + c^2}$
c the precision parameter.
Expansion coefficients $\lambda_j,\ j = 1, \dots, M$ are determined by setting:

$$\sum_{j=1}^{M} \lambda_j \phi(\|x_i - x_j\|, c) = u(x_i), \quad i = 1, \dots, M$$

Which is expressed in the following matrix form:

$$A\lambda = U \tag{8}$$

where

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_M)^\top, \qquad U = (u(x_1), u(x_2), \dots, u(x_M))^\top$$

and

$$A = \begin{pmatrix} \phi(\|x_1 - x_1\|, c) & \phi(\|x_1 - x_2\|, c) & \dots & \dots & \phi(\|x_1 - x_M\|, c) \\ \phi(\|x_2 - x_1\|, c) & \phi(\|x_2 - x_2\|, c) & \dots & \dots & \phi(\|x_2 - x_M\|, c) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \phi(\|x_M - x_1\|, c) & \phi(\|x_M - x_2\|, c) & \dots & \dots & \phi(\|x_M - x_M\|, c) \end{pmatrix}$$

According to [9] λ is given by

$$\lambda = A^{-1}U \tag{9}$$

The k order derivative of $\hat{u}$ at the center x_i is:

$$\frac{\partial^k \hat{u}(x_i)}{\partial x^k} = \sum_{j=1}^{M} \lambda_j \frac{\partial^k}{\partial x^k} \phi(\|x_i - x_j\|, c), \quad i = 1, 2, \ldots, M \tag{10}$$

In matrix form , the derivative (10) is written as :

$$U^{(k)} = A^{(k)}\lambda \tag{11}$$

with

$$U^{(k)} = \left(\frac{\partial^k \hat{u}(x_1)}{\partial x^k}, \frac{\partial^k \hat{u}(x_2)}{\partial x^k}, \ldots, \frac{\partial^k \hat{u}(x_M)}{\partial x^k} \right)$$

and

$$A^{(k)} = \begin{pmatrix} \frac{\partial^k}{\partial x^k}\phi(\|x_1 - x_1\|, c) & \frac{\partial^k}{\partial x^k}\phi(\|x_1 - x_2\|, c) & \ldots & \ldots & \frac{\partial^k}{\partial x^k}\phi(\|x_1 - x_M\|, c) \\ \frac{\partial^k}{\partial x^k}\phi(\|x_2 - x_1\|, c) & \frac{\partial^k}{\partial x^k}\phi(\|x_2 - x_2\|, c) & \ldots & \ldots & \frac{\partial^k}{\partial x^k}\phi(\|x_2 - x_M\|, c) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \frac{\partial^k}{\partial x^k}\phi(\|x_M - x_1\|, c) & \frac{\partial^k}{\partial x^k}\phi(\|x_M - x_2\|, c) & \ldots & \ldots & \frac{\partial^k}{\partial x^k}\phi(\|x_M - x_M\|, c) \end{pmatrix} \tag{12}$$

Using the expression of λ , we get:

$$U^{(k)} = D^{(k)}U \tag{13}$$

where

$$D^{(k)} = A^{(k)}A^{-1} \tag{14}$$

2.2.2. Space approximation of the Richards equation by the RBF-MQ method

In this section, we present the numerical resolution of problem (1). This approach was proposed by Ouédraogo et al. [18].
Let $\{z_i\}_{1 \leqslant i \leqslant M}$ a set of points of Ω considered as centers. At each center z_i the approximation value $\hat{h}$ of the pressure head h by the RBF-MQ is given by:

$$\hat{h}(z_i, t) = \sum_{j=1}^{M} \lambda_j(t)\phi(\|z_i - z_j\|, c), \quad t \in]0, T] \tag{15}$$

where ϕ the basic radial function of Hardy.
that can be rewritten in matrix form

$$\mathbf{h} = A\ \lambda(t), \quad t \in]0, T] \tag{16}$$

where $\mathbf{h} = (h(z_1,t), h(z_2,t), \dots, h(z_M,t))^\top$ and $\lambda(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_M(t))^\top$.

the flow velocity $q(h)$ is given by

$$q \circ \hat{h}(z_i,t) = \sum_{j=1}^{M} \gamma_j(t)\phi(\|z_i - z_j\|, c), \quad t \in]0,T],\ z_i,\ i = 1,2,\dots,M$$

Where $\gamma_j(t),\ j = 1,2,\dots,M$ the expansion coefficients of $q(h)$.
In matrix form, we have

$$\mathbf{q}(\mathbf{h}) = A\ \Gamma(t), \quad t \in]0,T] \tag{17}$$

where $\Gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_M(t))^\top$.

Using expression of $\hat{h}$ we have

$$\mathbf{q}(\mathbf{h},t) = -\mathbf{K}_D(\mathbf{h},t)D^{(1)}\mathbf{h}(t) - \mathbf{K}(\mathbf{h},t), \quad t \in]0,T] \tag{18}$$

where

$$\mathbf{K}(\mathbf{h}) = (K \circ \mathbf{h}_1, K \circ \mathbf{h}_2, ,\dots, K \circ \mathbf{h}_M,)^\top$$

et

$$\mathbf{K}_D(\mathbf{h}) = \begin{pmatrix} K \circ \mathbf{h}_1 & & & \\ & K \circ \mathbf{h}_2 & 0 & \\ & 0 & \ddots & \\ & & & K \circ \mathbf{h}_M \end{pmatrix}$$

with $\mathbf{h}_i = h(z_i,t),\ i = 1,\dots,M,\ t \in]0,T]$ the ith value of the vector $\mathbf{h}$ (16).
using 11, we have

$$\frac{\partial}{\partial z} q(h) \simeq \mathbf{q}^{(1)}(\mathbf{h}) = -D^{(1)}\mathbf{K}_D(\mathbf{h})D^{(1)}\mathbf{h} - D^{(1)}\mathbf{K}(\mathbf{h}) \tag{19}$$

Let $\Theta(\mathbf{h}) = (\theta \circ \mathbf{h}_1, \theta \circ \mathbf{h}_2, ,\dots, \theta \circ \mathbf{h}_M)^\top$ and $\mathbf{f} = (f(z_1,t), f(z_2,t), \dots, f(z_M,t)$ be the values respectively of the moisture content $\theta(h)$ and the source function f at the centers $z_i,\ i = 1,2,\dots,M$.
The numerical resolution of Richards' equation (1) can then resume to the resolution of the following problem in time:

$$\begin{cases} \frac{\mathrm{d}\Theta(\mathbf{h})}{\mathrm{d}t} = \mathcal{F}(\mathbf{h}), \quad t \in]0,T] \\ \mathbf{h}(0) = \mathbf{h}_0 \end{cases} \tag{20}$$

with

$$\mathcal{F}(\mathbf{h}) = \mathbf{q}^{(1)}(\mathbf{h}) + \mathbf{f} \tag{21}$$

and

$$\mathbf{h}_0 = (h_{init}(z_1), h_{init}(z_2), \dots, h_{init}(z_M))^\top$$

2.2.3. Resolution by Newton-Raphon's method

Let $t_n = n\delta t,\ n = 0, 1, \dots, N$ a discretization of $[0, T]$, $\delta t = T/(N-1)$ the time-step size and Θ^n, $\mathcal{F}^n$ the approximations of $\Theta(\mathbf{h}^n, t_n)$ and $\mathcal{F}(\mathbf{h}^n, t_n)$ with $\mathbf{h}^n = \mathbf{h}(t_n)$, $n = 0, 1, \dots, N$.

The approximation of the equation (20) by a implicit Euler scheme gives

$$\frac{\Theta^{n+1} - \Theta^n}{\delta t} = \mathcal{F}^{n+1}, \quad n = 0, 1, \dots, N-1 \tag{22}$$

The terms Θ^{n+1} and $\mathcal{F}^{n+1}$ cause equation (22) to be highly nonlinear, we use Newton-Raphson's method to solve it.

Let's denote $\Theta^{n+1,m+1}, \mathbf{K}_D^{n+1,m+1}$ and $\mathbf{K}^{n+1,m+1}$ the approximated values $\Theta(\mathbf{h}^{n+1,m+1}), \mathbf{K}_D(\mathbf{h}^{n+1,m+1})$ and $\mathbf{K}(\mathbf{h}^{n+1,m+1})$ in which $\mathbf{h}^{n+1,m+1}$ is the searched value of $\mathbf{h}^{n+1}$ in the step $m+1$ of Newton's iterative process. Let's also denote $\mathbf{h}^{n+1,m}$ the value of $\mathbf{h}^{n+1}$ at the previous step m,

$$\mathcal{F}(\mathbf{h}^{n+1,m}, \mathbf{h}^{n+1,m+1}) = -D^{(1)}\mathbf{K}_D^{n+1,m}D^{(1)}\mathbf{h}^{n+1,m+1} - D^{(1)}\mathbf{K}^{n+1,m} + \mathbf{f}^{n+1} \tag{23}$$

and

$$\mathcal{R}(\mathbf{h}^{n+1,m}, \mathbf{h}^{n+1,m+1}) = \frac{\Theta^{n+1,m+1} - \Theta^n}{\delta t} - \mathcal{F}(\mathbf{h}^{n+1,m}, \mathbf{h}^{n+1,m+1}) \tag{24}$$

Using a one-order Taylor's series development of $\Theta^{n+1,m+1}$, we obtain the following approximation

$$\Theta^{n+1,m+1} \simeq \Theta^{n+1,m} + \frac{\mathrm{d}\Theta^{n+1,m}}{\mathrm{d}\mathbf{h}}\delta\mathbf{h}^{n+1} \tag{25}$$

where

$$\delta\mathbf{h}^{n+1} = \mathbf{h}^{n+1,m+1} - \mathbf{h}^{n+1,m} \tag{26}$$

We denote $\mathbf{C}^{n+1,m}$ the value of C in the approximated vector $\mathbf{h}^{n+1,m}$ then the approximation (25) can be rewritten as following:

$$\Theta^{n+1,m+1} \simeq \Theta^{n+1,m} + \mathbf{C}^{n+1,m}\delta\mathbf{h}^{n+1} \tag{27}$$

We then replace $\Theta^{n+1,m+1}$ in equation (25) by its expression given by (27) and therefore we obtain the new expression of $\mathcal{R}(\mathbf{h}^{n+1,m}, \mathbf{h}^{n+1,m+1})$ as following:

$$\mathcal{R}(\mathbf{h}^{n+1,m}, \mathbf{h}^{n+1,m+1}) = \frac{1}{\delta t}\mathbf{C}^{n+1,m}\delta\mathbf{h}^{n+1} + \frac{\Theta^{n+1,m} - \Theta^n}{\delta t} - \mathcal{F}(\mathbf{h}^{n+1,m}, \mathbf{h}^{n+1,m+1}) \tag{28}$$

The resolution of nonlinear problem (22) with the Newton-Raphson's iterative method consists in solving at each time-step $n+1$ and at each stage $m+1$, the equation

$$J\ \delta\mathbf{h} = -\mathcal{R}(\mathbf{h}^{n+1,m}, \mathbf{h}^{n+1,m}) \tag{29}$$

where J is Jacobian matrix of $\mathcal{R}$ in $\mathbf{h}^{n+1,m}$ expressed as

$$J = \frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\mathbf{h}}(\mathbf{h}^{n+1,m}, \mathbf{h}^{n+1,m}) = \frac{1}{\delta t}\mathbf{C}^{n+1,m} - D^{(1)}\mathbf{K}_D(\mathbf{h}^{n+1,m})D^{(1)} \tag{30}$$

until $\|\delta \mathbf{h}^{n+1}\|$ is below a certain tolerance *tol* or that m exceeds a maximum value *maxiter*. Algorithm 1 describes how the Richards equation is solved at each time-step $n+1$ by Newton's iterative method.

Algorithm 1 Newton-Raphson's iterative method

Require: $\mathbf{h}^n, maxiter, tol$
  $\mathbf{h}^{n+1,0} = \mathbf{h}^n$
  while $m \leqslant maxiter$ **and** $\|\delta \mathbf{h}^{n+1}\| > tol$ **do**
    Solve the system (29) to obtain $\delta \mathbf{h}^{n+1}$
    $\mathbf{h}^{n+1,m+1} = \mathbf{h}^{n+1,m} + \delta \mathbf{h}^{n+1}$
    $m = m + 1$
  end while
  $\mathbf{h}^{n+1} = \mathbf{h}^{n+1,m+1}$
Ensure: $\mathbf{h}^{n+1}$

3. Inverse problem

3.1. Calculation of infiltration

One of the objectives of the modeling of the flow in unsaturated porous medium is the estimate of the quantity of water which infiltrates to reach the saturated zone. The infiltration describes the process of water penetrating in the ground starting from its surface. In a general way, for a variable initial condition $\theta(0, z)$, the cumulative infiltration I_{cum} is defined by:

$$I_{cum}(t) = \int_0^Z q(t, z)dz$$

$q(z, t)$ is the rate of infiltration and Z is the depth of the ground considered. If the initial condition θ_{init} is constant, we have:

$$I_{cum}(t) = \int_0^Z (\theta(t, z) - \theta_{ini})dz \tag{31}$$

$\theta(t, z)$ is the moisture content. In discrete form $I_{cum}(t_j)$ is obtained by making an approximation of (31) by the formula of the trapezoids:

$$I_{cum}(t_j) = \Delta z \left[\frac{1}{2}\left(\theta_{sup} - 2\theta_{ini} + \theta_{inf}\right) + \sum_{i=1}^{N_z}\left(\theta_i^j - \theta_{ini}\right)\right] \tag{32}$$

θ_{inf} is the moisture content at the bottom and θ_{sup} is the moisture content at the top.

3.2. Function cost

Let thus M observations of values of infiltration $I_{obs}(t_j)$ at the moments t_j, $j = 1, \ldots, M$. Let thus J the functional defined by

$$J(U) = \frac{\Delta t}{2}\sum_{j=1}^{M}(I_{cum}(t_j) - I_{obs}(t_j))^2$$

$$= \frac{\Delta t}{2}\sum_{j=1}^{2}(\Delta z\left[\frac{1}{2}(\theta_{sup} - 2\theta_{ini} + \theta_{inf}) + \sum_{i=1}^{N_z}(\theta_i^j - \theta_{ini})\right] - I_{obs}(t_j))^2 \tag{33}$$

U is the vector of parameters to determinate $(\alpha, n, \theta_r, \theta_s, K_s)$.
The inverse problem consists in solving

$$\min_{U \subset \mathcal{D}} J(U) \tag{34}$$

where $\mathcal{D}$ a bounded subset of $\mathbb{R}^5$.

4. Problem solving by Modified hybrid Grey Wolf Optimizer-Genetic Algorithm (HmGWOGA)

In this section we present the HmGWOGA algorithm proposed by Sawadogo et al. [20]. This algorithm is a hybridization of two meta heuristics. Grey Wolf Optimizer algorithm proposed by S. Mirjalili et al. [15], and a version of genetic algorithm proposed by Sawadogo et al. [21].
We consider the following problem:

$$\min_{x \in \mathcal{D}} f(x) \tag{35}$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, f a positive numeric function of $\mathbb{R}^n$, $\mathcal{D} = \prod_{i=1}^{n}[a_i, b_i]$, a_i and b_i are reals.

4.1. Presentation of the genetic algorithm used

The genetic algorithm used is an adaptation of Non-Dominated Sorting Genetic Algorithm-II (NSGA-II) proposed by Deb et al. [5].
This algorithm consist to create at each iteration t a population of children (Q_t) of size (N) by using selection, crossing, and mutation operators. This population is add to a population of parents (P_t) of size (N) to form a population $(R_t = P_t \cup Q_t)$. This process ensures elitism. The size population $(2N)$ is then sorted according to a non-dominance criterion. A new parent population (P_{t+1}) is formed keeping the N best individuals. The real-type coding used consists in directly representing the actual values of the variable.
We used the selection by caster of Goldberg [7]. The parents are selected according to their score. In this method the probability p with which an individual i represented by a

variable x_i of fitness f_i (evaluation of the function in x_i) reintroduced in a new population of size N is:

$$p = \frac{f_i}{\sum_{j=1}^{N} f_j}$$

The barycentric crossing is used but we did not use a probability of crossing. Mutation of a Gaussian type is applied to the population. One selects an individual x under a probability p. If p is lower than the probability of mutation p_m, one adds a Gaussian noise to x.

4.2. Grey Wolf Optimizer algorithm description

The GWO algorithm is a meta-heuristic which mimics the leadership hierarchy and hunting mechanism of grey wolves in nature. This algorithm has been proposed by S. Mirjalili et al [15]. Four types of grey wolves are employed for the simulating the leadership hierarchy: alpha (α), beta (β), (δ) and omega (ω).
Alpha is the leader of the group. Beta is the second. They help the alpha make decision. Delta are is the third category. Its members are scouts, sentinels and hunters. Last category omega wolves always have to submit to all the other dominant wolves.
Prey encircling is modeled by:

$$\begin{cases} \vec{D} = |\vec{C}.\vec{X}_p(t) - \vec{X}(t)| \\ \vec{X}(t+1) = \vec{X}_p(t) - \vec{A}.\vec{D} \end{cases} \tag{36}$$

where t indicates the current iteration, $\vec{A} = 2a.\vec{r_1}$, $\vec{C} = 2.\vec{r_1}$; $\vec{a}$ are decreased from 2 to 0 over the course of iterations and $\vec{r_1}$, $\vec{r_2}$ are random vectors in $[0, 1]$. $\vec{X}_p$ is the position vector of the prey, and $\vec{X}$ indicates the position vector of a grey wolf.
For better exploration of candidate solutions which tend to diverge when $|\vec{A}| > 1$ and to converge when $|\vec{A}| < 1$.
Grey wolves have the ability to recognize the location of prey and encircle them. Over the course of iterations, the first three fittest solutions we obtain so far are considered as α, β and δ respectively, which guide the optimization processes (the hunting) and are assumed to take the position of the optimum (the prey). The approximate distance between the current solution and alpha, beta and delta is given by the following formula:

$$\begin{cases} \vec{D_\alpha} = |\vec{C_1}.\vec{X}_\alpha - \vec{X}| \\ \vec{D_\beta} = |\vec{C_2}.\vec{X}_\beta - \vec{X}| \\ \vec{D_\delta} = |\vec{C_3}.\vec{X}_\delta - \vec{X}| \end{cases} \tag{37}$$

where:

- $\vec{C}_1$,$\vec{C}_2$ and $\vec{C}_3$ are random vectors.
- $\vec{X}_\alpha$, $\vec{X}_\alpha$ and $\vec{X}_\delta$, the positions of alpha, beta and delta respectively.
- $\vec{X}$ the position of prey(current solution).

Finally the next position of the solution is given by:

$$\vec{X}(t+1) = 0.7 \times \vec{X}_1 + 0.2 \times \vec{X}_2 + 0.1 \times \vec{X}_3 \tag{38}$$

where

$$\begin{cases} \vec{X}_1 = \vec{X}_\alpha - \vec{A}_1.\vec{D}_\alpha \\ \vec{X}_2 = \vec{X}_\beta - \vec{A}_2.\vec{D}_\beta \\ \vec{X}_3 = \vec{X}_\delta - \vec{A}_3.\vec{D}_\delta \end{cases} \tag{39}$$

$\vec{A}_1$,$\vec{A}_2$ and $\vec{A}_3$ are random vectors.

4.3. Problem solving algorithm

The hybridization consist to apply the operators of the genetic algorithm before applying the GWO steps. In summary the resolution of our problem by HmGWOGA algorithm is given by the algorithm below:

Algorithm 2 Problem solving algorithm

Initialize the input parameters for HmGWOGA $(N, d, lb, ub, Maxiter, p_m, sigma)$
Initialize Alpha, Beta and Delta Position and Score.
Initialize the random position of search agents.
$k \leftarrow 0$
while $k < Maxiter$ **do**
  solving direct problem (1)
  evaluate the score of each search agent (P_k) using objective function (33)
  Apply a selection operator
  Apply a crossover operator to generate a new population of child Q_k (The criterion used at this step is $1/(fitness + \epsilon)$ with $(\epsilon \in \mathbb{R}_+^*)$
  $R_k = P_k \cup Q_k$ (add Q_k to P_k) to obtain 2N search agents
  for each agent in R_k **do**
    choose a random number u in $[0, 1]$
    if $u \leq p_m$ **then**
      Apply a mutation operator
    end if
  end for
  Classify search agents of R_k from increasing order according to the score of each agent
  Keep N best individuals of R_k to form a new search agent.
  for i i=1 to N **do**
    $fitness \longleftarrow$ Score_ agent_i
    if $fitness < AlphaScore$ **then**
      Update alpha
    end if
    if $fitness > AlphaScore$ and $fitness < BetaScore$ **then**
      Update beta
    end if
    if $fitness > AlphaScore$ and $fitness > BetaScore$ and $fitness < DeltaScore$ **then**
      Update delta
    end if
  end for
  for i=1 to N **do**
    Update the Position of search agents including omegas using equation (37-39)
    Update the position of prey using equation(38)
  end for
  $k \leftarrow k + 1$
end while
Return the position of α as the fittest optimum

5. Results and discussions

5.1. Application 1

Either an unsaturated medium represented by a domain $\Omega = [0, 20]$ and a simulation time interval $[0, 600]$. Dirichlet conditions were imposed. According to [22], an analytical solution of the problem (1) is given by:

$$h(z,t) = 20.4 \tanh(0.5(z + t/12 - 15)) - 41.5 \tag{40}$$

The source term f is chosen using the analytical solution.

To verify the efficiency of our algorithm data was generated using the analytical solution.
The simulation conditions and the results are given below:

Parameters	Range	used values	identified values
θ_s	$[0;4]$	0.357	0.364
θ_r	$[0;4]$	0.108	0.106
α	$[0;1]$	0.0335	0.032
n	$[0;10]$	1.8	1.87
K_s	$[0;15] \times 10^{-3}$	8.13×10^{-3}	8.25×10^{-3}

Value of objective function: 7.5×10^{-5}.
Figure 1 shows the infiltration curve. In this figure we see that the identified infiltration is very close to the infiltration obtained with the analytical solution.

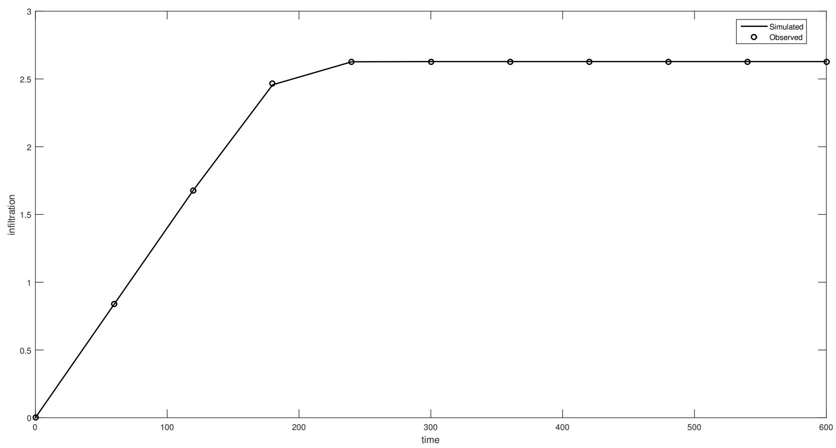

Figure 1: Application 1: Curves of infiltration observed and simulated

5.2. Application 2

In this second application, we use data used in [21]. These were measured on a clay soil on a column of $1m$ long. The values of the infiltration were recorded all the 5 mn during 2 hours. [21] In the direct problem was resolution using the finite difference method and the genetic algorithm was used to identify the parameters. The results of the identification are given in the table below

Parameters	Interval	Genetic algorithm	HmGWOGA algorithm
θ_s	$[0;1]$	0.0238	0.0255
θ_r	$[0;2]$	0.379	0.373
α	$[0;1]$	0.0879	0.0869
n	$[0;3]$	1.1359	1.1395
K_s	$[0;3] \times 10^{-5}$	1.75×10^{-5}	1.84×10^{-5}

Figure 2 is a comparative representation of the simulated and observed infiltration curves. In these figures, we can see the quality of the estimation of the parameters. We also see that the curve obtained by HmGWOGA method is closer to the observed data than that obtained by GA. Which is confirmed by the values of the function cost:
Genetic algorithm: 0.0027 [21].
HmGWOGA algorithm:0.00012.

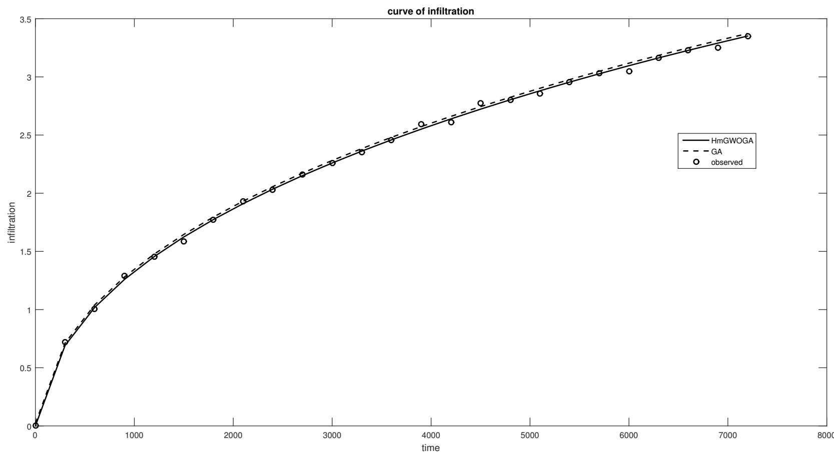

Figure 2: Application2: Curves of infiltration observed and simulated

As in [21], the values of the infiltrations observed at the moments $t = 30mn$, $t = 1h$ and $t = 1h30mn$ were not used in the process of identifications. They were used like values test. The table 1 presents the results. This table shows also that the HmGWOGA algorithm is better than the genetic algorithm which is confirmed by the convergence curve in figure 3.

Times	Observed	Genetic algorithm	Error	HmGWOGA algorithm	Error
$30mn$	1.5943	1.5732	0.0211	1.5927	0.0016
$1h$	2.322	2.2901	0.0319	2.3262	0.0042
$1h30mn$	2.8143	2.7939	0.0204	2.8216	0.0073

Table 1: Comparison to test points

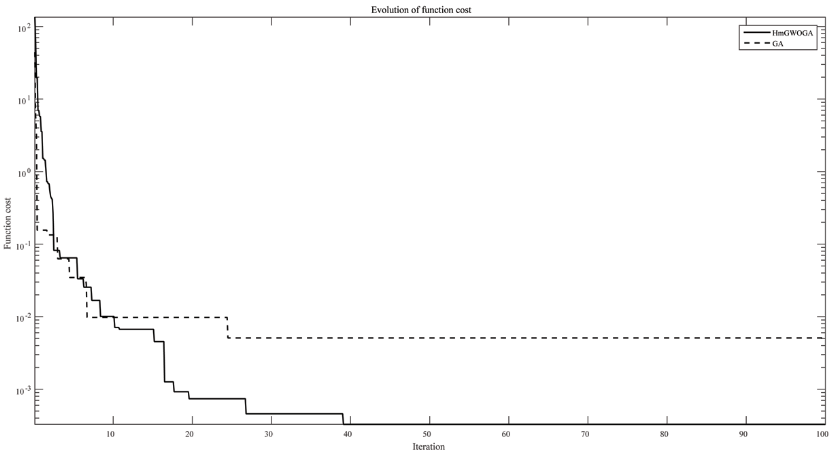

Figure 3: Application 2:Curves of convergence

To measure the performance of both methods in terms of computation time, we ran the simulation 30 times for each example. The characteristics of the computer used are: Processor: Intel(R)Xeon (R) CPU E5-2603 v4@1.7 GHz 1.7GHz, RAM: 24 GB, Operating System: windows 10, 64-bit.
The statistical results in second are given in the table 2.

	HmGWOGA				**GA**			
	Min	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**
Application 2	608.125	609.468	608.757	1.481	421.23	422.39	426.787	1.67

Table 2: Computation time

These results show that although HmGWOGA is more accurate, it is slower than GA.

6. Conclusion

In this work it was about identification of the parameters in Richard's equation. The direct problem was solved using Multiquadratic Radial Basis Function (RBF-MQ) method

which is a meshless method. We used Modified Grey Wolf Optimizer-Genetic Algorithm (HmGWOGA) to determine parameters of Richards equation using synthetic data and real data. Comparison with the genetic algorithm showed that the HmGWOGA algorithm was more effective in identifying parameters involved in the Richards equation.
A comparison of the execution times of the two algorithms shows the HmGWOGA algorithm despite its effectiveness remains slower than GA. In order to reduce execution time, we intend in the future to propose a parallel version of this algorithm.

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