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Existence of Optimal Control for a nonlinear Partial Differential Equation of Hyperbolic-type

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Abstract. In this paper, we prove the existence of an optimal control for a nonlinear hyperbolic problem, examined in [3]. An estimation is used which makes it possible to extract from a minimizable sequence of controls and from the sequence of corresponding solutions weakly convergent sub sequences. To prove the passage to the limit in a true equality for every element of the minimizable sequence, Lebesgue's theorem on the passage to the limit under the integral sign and the theorem of immersion have been used.

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1. Preliminaries notions

Before proceeding to the formulation of the problem, let us recall some fundamental notions of [2].

1.1. Definition of $\mathcal{C}^{k,\lambda,0}(\bar{\Omega})$ space: (see [4])

Let Ω be a domain of $\mathbb{R}^N, k \in N_0$ and $\lambda \in]0,1[$. We call $\mathcal{C}^{k,\lambda,0}(\bar{\Omega})$ any subset of the functions $u \in \mathcal{C}^{k,\lambda}(\bar{\Omega})$ for which the following condition is satisfied

$$\forall \varepsilon > 0, \exists \delta > 0 : (x, y \in \Omega, 0 < |x-y| < \delta, |\alpha| = k) \Longrightarrow |D^\alpha u(x) - D^\alpha u(y)| \cdot |x-y|^{-\lambda} < \varepsilon$$

where $\alpha = (\alpha_1, \cdots, \alpha_2)$ is the multi-index.

The norm of the $\mathcal{C}^{k,\lambda,0}(\bar{\Omega})$ space is deduced from $\mathcal{C}^{k,\lambda}(\bar{\Omega})$, namely

$$\|u\|_{k,\lambda} = \sum_{|\alpha| \leqslant k} \sup_{x \in \Omega} |D^\alpha u(x)| + \sum_{|\alpha| \leqslant k} \sup_{x \neq y} |D^\alpha u(x) - D^\alpha u(y)| \cdot |x-y|^{-\lambda}.$$

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Theorem 1 ([2], P.11). *Let Ω be a bounded domain of $\mathbb{R}^n$, $0 < \lambda < 1$*
$F\colon \mathbb{R} \times \bar{\Omega} \longrightarrow \mathbb{R}, (u, x) \longmapsto F(u, x)$ a continuous function defined on $\mathbb{R} \times \bar{\Omega}$, differentiable with respect to u on $\mathbb{R}$ for all $x \in \bar{\Omega}$ and also $F'_u\colon \bar{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function on $\bar{\Omega} \times \mathbb{R}$ satisfied

$$|F'_u(u, y) - F'_u(v, z)| \leqslant Q_1|u - v| + Q_2|y - z|^\lambda$$

and

$$|F(u, y) - F(v, z)| \leqslant C_1|u - v| + C_2(y, z)|y - z|^\lambda$$

where Q_1, Q_2, C_1 are the constants and C_2 a bounded function which verifies the condition

$$\forall\ \varepsilon > 0, \exists\ \delta > 0 : (|y - z| < \delta) \Longrightarrow C_2(y, z) < \varepsilon.$$

Then the $\varphi(x) \longmapsto F(\varphi(x), x)$ mapping is defined from $C^{0,\lambda,0}(\bar{\Omega})$ to $C^{0,\lambda,0}(\bar{\Omega})$ and is weakly sequentially continuous.

Theorem 2 ([2], P.13). *Let Ω be a bounded domain of $\mathbb{R}^n$, $0 < \lambda < 1$*
$K\colon \mathbb{R} \times \mathbb{R}, (x, y, u) \longmapsto K(x, y, u)$ a continuous function on $\mathbb{R} \times \bar{\Omega}^2$, differentiable with respect to u on $\mathbb{R}$ for all $(x, y) \in \bar{\Omega}^2$ and also $K'_u\colon \bar{\Omega}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function on $\bar{\Omega}^2 \times \mathbb{R}$ verifying

$$|K'_u(t, y, u) - K'_u(s, y, u)| \leqslant Q_r|t - s|^\lambda, \quad |u| \leqslant r$$

and

$$|K(t, y, u) - K(s, y, u)| \leqslant a_r(t, s, y), \quad |u| \leqslant r$$

with a_r a measurable function,

$$\int_\Omega a_r(t, s, y)dy \leqslant b_r(t, s) \cdot |t - s|^\lambda,$$

and $b_r\colon \bar{Q}_T^2 \longrightarrow \mathbb{R}$ satisfied the following conditions:
b_r is bounded and $\forall\varepsilon > 0, \exists\delta > 0 : (|t - s| < \delta) \Longrightarrow b_r(t, s) < \varepsilon$
Then the mapping

$$G\colon [u(x)] \longmapsto \int_\Omega K(x, y, u(y))dy$$

is defined from $C^{0,\lambda,0}(\bar{\Omega})$ to $C^{0,\lambda,0}(\bar{\Omega})$ and weakly sequentially continuous.

2. Main operators

We shall consider the following problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho u = f(x, t), \quad \rho > 0, \tag{1}$$

$$\frac{\partial u}{\partial \vec{n}}(x, t)_{|\partial\Omega} = 0, \quad t \in (0, T) \tag{2}$$

$$u(x,t)_{|t=0} \;=\; \varphi(x), \quad x \in \Omega,\ \frac{\partial u}{\partial t}(x,t)_{|t=0} = \psi(x), x \in \Omega \tag{3}$$

in the cylinder

$$Q_T = \{x, t : x \in \Omega \subset \mathbb{R}^n, 0 < t \leqslant T < \infty\},$$

where Ω is a bounded domain of $\mathbb{R}^n$ with differentiable boundary $\partial\Omega$, $\vec{n}$ designates the outer normal to $\partial\Omega$ and $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$.

Let

$$H^1(\Omega) = \{v/v \in L_2(\Omega), \frac{\partial v}{\partial x_i} \in L_2(\Omega), i = 1, \cdots, n\}$$

with associated norm

$$\|v\|_{H^1(\Omega)} = \left(\int_\Omega \left[|v|^2 + \sum_{i=1}^{n} |\frac{\partial v}{\partial x_i}|^2 \right] dx \right)^{\frac{1}{2}}.$$

Assume that the functions $f(x,t), \varphi(x), \psi(x)$ are the control and then

$$f(x,t) \in Y \subset L_2(Q_T), \varphi(x) \in X \subset H^1(\Omega), \psi(x) \in W \subset L_2(\Omega) \tag{4}$$

where Y, X, W are respectively the convex sets, bounded and closed of $L_2(Q_T), H^1(\Omega)$ and $L_2(\Omega)$.

Let consider the operator:

$$A\colon L_2(Q_T) \times H^1(\Omega) \times L_2(\Omega) \longrightarrow C^{0,\lambda,0}(\bar{Q}_T)$$

$$[A(f,\varphi,\psi)](x,t) = \int_\Omega K_1(x,t,x',t') f(x',t') dx'dt' + \int_\Omega K_2(x,x')\varphi(x')dx' + \int_\Omega K_3(x,x')\psi(x)dx'$$

where K_1, K_2, K_3 verify the condition of Hölder:

$\lambda + \lambda', 0 < \lambda' < \lambda, \lambda + \lambda' < 1$ respectively in $(x,t), x, x'$ and

$$\begin{aligned}
|K_1(x,t,x',t') - K_1(\tilde{x},\tilde{t},x',t')| &\leqslant c_3(x',t')|(x,t) - (\tilde{x},\tilde{t})|^{\lambda+\lambda'},\\
|K_2(x,x') - K_2(\tilde{x},x')| &\leqslant c_4(x')|x - \tilde{x}|^{\lambda+\lambda'},\\
|K_3(x,x') - K_3(\tilde{x},x')| &\leqslant c_5(x')|x - \tilde{x}|^{\lambda+\lambda'},\\
\sup_{(x,t)\in Q_T} \int_{Q_T} K_1^2(x,t,x',t')dx'dt' &= c_6 < \infty,\\
\sup_{x\in\bar{\Omega}} \int_{Q_T} K_2^2(x,x')dx' &= c_7 < \infty,\\
\sup_{x\in\bar{\Omega}} \int_{Q_T} K_3^2(x,x')dx' &= c_8 < \infty.
\end{aligned}$$

with $c_3(x',t') \in L_2(Q_T), c_4(x'), c_5(x') \in L_2(\Omega)$.

Note that this operator is linear, continuous and therefore it is weakly sequentially continuous (by Theorem 1).

Let consider then the operator

$$[B(f,\varphi,\psi)](x,t) = \int_{Q_T} K(x,t,x',t',[A(f,\varphi,\psi)](x',t')dx'dt'$$

where

- the function $K\colon \bar{Q}_T^2 \times \mathbb{R} \longrightarrow \mathbb{R}$, $K\colon (x,t,x',\xi) \longrightarrow K(x,t,x',t',\xi)$ is continuous on $\bar{Q}_T^2 \times \mathbb{R}$, differentiable with respect to ξ on $\mathbb{R}$ for all $(x,t,x',t') \in \bar{Q}_T^2$;
- the derived function $K'_\xi\colon \bar{Q}_T^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ is also continuous on $\bar{Q}_T^2 \times \mathbb{R}$, and

$$|K'_\xi(x,t,x',t',\xi) - K'_\xi(\tilde{x},\tilde{t},x',t',\xi)| \leqslant Q_T|(x,t)-(\tilde{x},\tilde{t})|^{\lambda+\lambda'}, \quad |\xi| \leqslant r$$

$$|K'_\xi(x,t,x',t',\xi) - K(\tilde{x},\tilde{t},x',t',\xi)| \leqslant a_r(x,t,\tilde{x},\tilde{t},x',t'), \quad |\xi| \leqslant r$$

here a_r is a measurable function verifying

$$\int_{Q_T} a_r(x,t,\tilde{x},\tilde{t},x',t')dx'dt' \leqslant b_r(x,t,\tilde{x},\tilde{t}) \cdot |(x,t)-(\tilde{x},\tilde{t})|^{\lambda+\lambda'}$$

and $b_r\colon \bar{Q}_T^2 \longrightarrow \mathbb{R}$ satisfying the following conditions:

b_r is bounded and

$$\forall\varepsilon > 0, \exists\delta > 0 : (|(x,t)-(\tilde{x},\tilde{t})| < \delta) \Longrightarrow b_r(x,t,\tilde{x},\tilde{t}) < \varepsilon.$$

This operation is a mapping defined from $L_2(Q_T) \times H^1(\Omega) \times L_2(\Omega)$ to $C^{0,\lambda,0}(\bar{Q}_T)$ and it is weakly sequentially continuous (by Theorem 2).

Let $E \in (C^{0,\lambda,0}(\bar{Q}_T))'$.

Remember ([2],P.5) that there exists such Borelian measures (definite positive) μ_1 and μ_2 with bounded variation on $\bar{Q}_T$ and $\bar{Q}_T^2$ respectively for which

$$\langle E,u\rangle = \int_{Q_T} u(x,t)d\mu_1(x,t) + \int_{Q_T}^2 (u(x,t)-u(\tilde{x},\tilde{t})) \cdot |(x,t)-(\tilde{x},\tilde{t})|^{-\lambda} d\mu_2(x,t,\tilde{x},\tilde{t})$$

for $u \in C^{0,\lambda,0}(\bar{Q}_T)$.

In this case, the functionals of the form

$$F^i\colon L_2(Q_T) \times H^1(\Omega) \times L_2(\Omega) \longrightarrow \mathbb{R}$$

$$F^i(f,\varphi,\psi) = \langle E^i, B_i(f,\varphi,\psi)\rangle, \quad \overline{i=0,\ s_1+s_2},$$

are also weakly sequentially continuous.

3. Formulation of the problem

Consider the problem (1)-(3) with the propositions (4). Then consider the functional of the form

$$J_i(f,\varphi,\psi) = \int_{\bar{Q}_T} v_i(x,t,u(x,t))dxdt + F^i(f,\varphi,\psi), \tag{5}$$

$\overline{i=0,\ s_1+s_2}$ where the functions $v_i(x,t,\xi)$ verify the following conditions:

a) the functions $v_i(x,t,\xi)$ are measurable on $Q_T \times \mathbb{R}$,

b) almost for each $(x,t) \in Q_T$, the functions $v_i(x,t,\xi)$ are continuous at ξ on $\mathbb{R}$ and

$$|v_i(x,t,\xi)| \leqslant c_9 + c_{10}|\xi|^2. \tag{6}$$

Note that the functions $J_i(f,\varphi,\psi)$ are weakly sequentially continuous by virtue of the immersion theorem $H^1(Q_T) \subset L_2(Q_T)$, of inequality

$$\|u\|_{H^1(Q_T)} \leqslant c(T)(\|f\|_{L_2(Q_T)} + \|\varphi\|_{H^1(\Omega)} + \|\psi\|_{L_2(\Omega)})$$

[1], and the continuity of the functional $u \longmapsto \int_{Q_T} v_i(x,t,u(x,t))dxdt$ from $L_2(Q_T)$ into $\mathbb{R}$.

We thus pose the following problem:

To find out such measurable functions $f^0(x,t) \in Y,\ \varphi^0(x) \in X,\ \psi^0(x) \in W$ in such a way that, for the solution $u^0(x,t)$ of the problem (1)-(3) corresponding to (f^0,φ^0,ψ^0), inequality-type constraints are verified,

$$J_i(f,\varphi,\psi) \leqslant 0, \quad \overline{i=1,\ s_1}, \tag{7}$$

equality-type constraints,

$$J_i(f,\varphi,\psi) = 0, \quad \overline{i=s_1+1,\ s_1+s_2} \tag{8}$$

and with that

$$J_0(f^0,\varphi^0,\psi^0) = \inf_{Y\times X\times W} J_0(f,\varphi,\psi) \tag{9}$$

4. Existence of an optimal control

Theorem 3. *We suppose there is a control of the above indicated class and* $\inf\limits_{Y\times X\times W} J_i(f,\varphi,\psi) > -\infty$.

Then there exists an optimal control $\hat{f}^0(x,t), \hat{\varphi}^0(x), \hat{\psi}^0(x)$.

Proof. white.

Let $\{f_m(x,t)\}_{m\geqslant 1}, \{\varphi_m(x)\}_{m\geqslant 1}, \{\psi_m(x)\}_{m\geqslant 1}$ be minimizable sequences of controls and $\{u_m(x,t)\}_{m\geqslant 1}$ their corresponding sequence of solution of the problem (1)-(3).

From the inequality

$$\|u_m(x,t)\|_{H^1(Q_T)} + \|u_m(x,t)\|_{L_p(Q_T)} \leqslant const$$

[3], where $p = \rho + 2$, it follows that the $\{u_m(x,t)\}_{m\geqslant 1}$ sequence is uniformly bounded into $H^1(Q_T)$; which allows to subtract a sub-sequence of solutions $\{u_{m_k}(x,t)\}_{k=1}^{\infty}$ that converge weakly to $u(x,t)$ into $H^1(Q_T)$ and $f_{m_k}(x,t), \varphi_{m_k}(x), \psi_{m_k}(x)$ converge weakly in the spaces $L_2(Q_T), H^1(\Omega), L_2(\Omega)$ to $f^0(x,t) \in Y, \varphi^0(x) \in X, \psi^0(x) \in W$.

From the weak converge in $H^1(Q_T)$ of the sequence $u_{m_k}(x,t)$ to $u(x,t)$ and by virtue of the complete continuity of the operator $H^1(Q_T)$ into $L_2(Q_T)$, result the weak convergence into $L_2(Q_T)$ of the sequence $u_{m_k}(x,t)$ to $u(x,t)$.

$$H^1(Q_T) \subset L_2(Q_T)\ \forall\{u_m(x,t)\} \subset H^1(Q_T) : \|u_m(x,t)\|_{H^1(Q_T)} \leqslant c_{11}$$

$\exists\ \{u_{m_k}(x,t)\} \subset \{u_m(x,t)\}$ which is fundamental in $L_2(Q_T)$.

As $L_2(Q_T)$ is complete then $\exists\ x^*(x,t) \in L_2(Q_T)$: $u_{m_k}(x,t) \longrightarrow u^*$ converge strongly into $L_2(Q_T)$.

By virtue of the separation of $L_2(Q_T)$, we have $u = u^*$.

We can consider that ([5],p.162)

$$|u_{m_k}(x,t)| \leqslant z(x,t) \in L_2(Q_T).$$

Then from the inequality (6), we obtain

$$|v_i(x,t,u_{m_k})| \leqslant c_9 + c_{10}z^2(x,t) \in L_1(Q_T).$$

By using the formula of the functional $J_i(f,\varphi,\psi)$ for $u_{m_k}(x,t)$, we have

$$\begin{aligned} J_i(f_{m_k},\varphi_{m_k},\psi_{m_k}) &= \int_{Q_T} v_i(x,t,u_{m_k})dxdt + F^i(f_{m_k},\varphi_{m_k},\psi_{m_k}) \\ i &= \overline{0,\ s_1+s_2} \end{aligned}$$

According to the Lebesgue theorem, we obtain

$$\begin{aligned} J_i(\hat{f}^0,\hat{\varphi}^0,\hat{\psi}^0) &= \int_{Q_T} v_i(x,t,u(x,t))dxdt + F^i(\hat{f}^0,\hat{\varphi}^0,\hat{\psi}^0) \\ i &= \overline{0,\ s_1+s_2} \end{aligned} \tag{10}$$

As the functions $f_m(x,t), \varphi_m(x), \psi_m(x)$ are the minimizable sequences, then

$$J_0(f_m,\varphi_m,\psi_m) \longrightarrow \inf_{X\times Y\times W} J_0(f,\varphi,\psi) := J_* \tag{11}$$

Under the weak sequential continuity, we have

$$J_* = \lim_{m\to\infty} J_0(f_m,\varphi_m,\psi_m) = J_0(\hat{f}^0,\hat{\varphi}^0,\hat{\psi}^0) \tag{12}$$

By the same way, we have

$$\lim_{m\to\infty} J_i(f_m,\varphi_m,\psi_m) \quad = \quad J_i(\hat{f}^0,\hat{\varphi}^0,\hat{\psi}^0), \tag{13}$$

$$i = \overline{1,\ s_1 + s_2}$$

In addition, from (7) and (8), it follows that :

$$\begin{aligned} J_i(f_m, \varphi_m, \psi_m) &\leqslant 0,\ i = \overline{1,\ s_1} \\ J_i(f_m, \varphi_m, \psi_m) &= 0,\ i = \overline{s_1 + 1, s_1 + s_2} \end{aligned}$$

and from this, it follows that :

$$J_i(\hat{f}^0, \hat{\varphi}^0, \hat{\psi}^0) \leqslant 0,\ i = \overline{1,\ s_1} \tag{14}$$

$$J_i(\hat{f}^0, \hat{\varphi}^0, \hat{\psi}^0) = 0,\ i = \overline{s_1 + 1, s_1 + s_2}. \tag{15}$$

From (12), (14), (15), it follows that $\hat{f}^0, \hat{\varphi}^0, \hat{\psi}^0$ is an optimal control.

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