



On the paper “On hyperideals of ordered semihypergroups” by Ze Gu in Ital. J. Pure Appl. Math.

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Abstract. Giving the proper citations, it is shown that, except of Lemma 2.4 and Theorem 2.6, almost all the results of the paper in the title have been previously published for ordered hypersemigroups in Eur. J. Pure Appl. Math. and they are not new. There are also two results obtained from ordered semigroups just putting a “ \circ ” instead of “ \cdot ” (that isn’t a correct way to work), without reference to ordered semigroups on which the results on ordered hypersemigroups are based. One of them can be obtained as corollary to a theorem in Eur. J. Pure Appl. Math. as well, and it is not new.

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1. Introduction

The paper consists of Lemma 2.3, Lemma 2.4, Theorem 2.5, Theorem 2.6, Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4 and the Example 3.5.

Lemma 2.3 is known, a reference was needed in it.

Concerning Theorem 2.5 (one of the main theorems of the paper): The Proposition in [7] published for ordered semigroups in 1992, has been transferred to ordered hypersemigroups just putting “ \circ ” instead of “ \cdot ”. Except of the fact that this is not enough to pass from ordered semigroup to ordered hypersemigroup (symbols like $S \circ b \circ S \circ S \circ a \circ S$ have no sense) and that there is no reference to ordered semigroup on which it is based, this theorem is an immediate consequence of Theorem 23 in [10] and it is not new (see also the equivalence (1) \Leftrightarrow (2) in Corollary 24 in [10] and the remark after that).

Theorem 3.1 (one of the main theorems) is not new. It has been published in Theorems 9 and 18 in [10]. Regarding the proof of the implication (4) \Rightarrow (1) of this theorem, this is actually the proof of (3) \Rightarrow (1) of the same theorem.

It is well known that an ordered semigroup S is intra-regular if and only if every ideal of S is semiprime. See, for example, Remark 3 in [8] or the proof of Theorem 2 in [7]. If

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we get the proof from [7], delete the “.” and put “ \circ ” instead, then this is the Theorem 3.2 in [4]; without reference to ordered semigroups and with the problem of using symbols like $S \circ a \circ a \circ S$ in its proof.

Theorem 3.3 is the Theorem 19 in [10]. This is because an ordered hypersemigroup S is semisimple if and only if the ideals of S are idempotent (see the Theorem 18 in [10]).

Theorem 3.4 is the Theorem 23 in [10]. This theorem is correct, but its proof is wrong in [4]. See the proof of Theorem 23 in [10].

The Example 3.5 is the Example B in [10], there is reference to this example. According to the author, one can check that this is really an ordered hypersemigroup. We never check the ordered hypersemigroups given by a table and an order by hand.

Everything is explained in detail in the next section.

2. Remarks

This is from the introduction of the paper in the title:

“Motivated by the previous work on hyperideals of (ordered) semihypergroups, we attempt in the present paper to study hyperideals of ordered semihypergroups in detail. In this article, we introduce the notion of weakly semiprime and irreducible hyperideals in ordered semihypergroups, and moreover establish the relationship between the five classes of hyperideals. Finally semisimple ordered semihypergroups and intra-regular ordered semihypergroups are characterized in terms of these hyperideals. Partial results which are consistent with the conclusions in [N. Kehayopulu, On ordered hypersemigroups with idempotent ideals, prime or weakly prime ideals, European Journal of Pure and Applied Mathematics, 11 (2018), 10–22] are reorganized and proved.”

We have to do some comments on this paper. We call “ideal” what the author in [4] calls “hyperideal” and “hypersemigroup” what the author calls “semihypergroup”. We use the symbol “ $*$ ” when the operation is between sets (however, we keep the “ \circ ” for the results given by Gu to avoid any misunderstanding); for further information we refer to [10].

The results of this paper, except of Lemma 2.4 and Theorem 2.6, have been published by the author of the present paper, most of them in [10], in an attempt to show how a right paper on an hypersemigroup or on ordered hypersemigroup should be written. The purpose was to show that we can never indicate the operations between elements and subsets of an hypersemigroup by the same symbol, in which case we get a result on an ordered semigroup, delete the multiplication “.” of the semigroup and put “ \circ ” in its place to pass from an ordered semigroup to an ordered hypersemigroup. The notions of prime and weakly prime subsets of an hypergroupoid or ordered hypergroupoid (natural extension of the concept of weakly prime subset of a groupoid or ordered groupoid [6, 7]) have been given in Definition 3 in [10], and it is obvious now what the notion of a weakly semiprime subset is. An weakly semiprime ideal of an ordered hypersemigroup S is clearly an ideal that is at the same time an weakly semiprime subset of S .

The concept of irreducible ideal of hypersemigroups [1] and the concept of irreducible ideal of ordered hypersemigroups [4] are the same.

In addition, all these concepts have been introduced many years ago for more general structures. For the concepts of weakly prime, weakly semiprime and prime ideal elements in *poe*-semigroups see [5]; for the concepts of meet-irreducible and join-irreducible elements of ordered sets see [3]. The irreducible ideal in the paper by Gu [4] is the meet-irreducible ideal element in the sense of Grätzer [3].

It might be mentioned here that the \vee -irreducible element in a \vee -semilattice has been defined much earlier, in 1953, by Dubreil et al. [2, p. 117] (and can be defined in any ordered set, as well).

So the concepts of weakly semiprime and irreducible ideals cannot due to the author and a proper reference list for these concepts was needed.

To be easier for the readers to follow this note, we will write below ZG, NK, for the papers due to Gu and Kehayopulu, respectively.

Lemma 2.3 in [4](ZG) *Let S be an ordered hypersemigroup. Then*

- (1) $A \subseteq (A]$, $((A]) = (A]$ for all $A \subseteq S$.
- (2) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.
- (3) $(A] \circ (B] \subseteq (A \circ B]$, $((A] \circ (B]) = (A \circ B]$.
- (4) $(T] = T$ for every ideal T of S .
- (5) If A, B are ideals of S , then $(A \circ B]$, $A \cap B$ and $A \cup B$ are ideals of S .
- (6) $(S \circ A \circ S]$ is an ideal of S for all $A \subseteq S$.

According to [4], this lemma can be easily obtained, giving the expression that it is new; since it is not new, a proper reference for this lemma was needed.

For the obvious properties (1), (2) and (4) of this lemma see, for example, the Lemma 1 in [7], as these properties hold in ordered groupoids in general (the hyperoperation does not play any role in them). For a detailed proof of the rest (that can be naturally transferred from ordered semigroups) using the symbols “ \circ ” as the “operation” between elements and “ $*$ ” as the operation between sets, see the Lemma 2.8 in [9], the Proposition 11 in [10], the Corollary 15 in [10], the Proposition 7 in [10].

As an example, let us look at property (3) (that is part of Proposition 11 in [10]) and at the proof of Proposition 11 in [10] in which the role of the two operations \circ and $*$ is indicated in a clear way.

Proposition 11 in [10](NK): *Let (H, \circ, \leq) be an ordered hypergroupoid and A, B nonempty subsets of H . Then we have $(A * B] = ((A] * (B]) = ((A] * B] = (A * (B])$.*

Lemma 2.4 in [4](ZG) *Let S be an ordered hypersemigroup and I an hyperideal of S . Then I is the intersection of all irreducible hyperideals of S containing I .*

We sketch the proof of this lemma, to make it clear. Let us assume $\{I_\alpha \mid \alpha \in \Gamma\}$ is the set of all irreducible ideals of S containing I , and let $a \in \bigcap_{\alpha \in \Gamma} I_\alpha$ such that $a \notin I$. The set $\Omega := \{H \mid H \text{ ideal of } S, H \supseteq I, a \notin H\}$ has a maximal element, say M . The set M is irreducible. Indeed: Let A, B be ideals of M such that $A \cap B = M$. Since $a \notin M$, we have $a \notin A \cap B$. If $a \notin A$ then, since A is an ideal of S and $A \supseteq M \supseteq I$, we have $A \in \Omega$. Since $A \supseteq M$ and M is maximal in Ω , we have $A = M$. Similarly $B = M$. Since M is an irreducible ideal of S , we have $a \in I_\beta$ for some $\beta \in \Gamma$. Since $a \in \bigcap_{\alpha \in \Gamma} I_\alpha$, we have $a \in I_\beta$.

Then $a \in M$ that is impossible.

Regarding the proof that the maximal subset M of Ω is irreducible, the proof is the same with the proof of Proposition 3.5 in [1].

To apply Zorn’s Lemma it should be mentioned that every totally ordered subset of Ω has an upper bound in Ω as in the proof of Proposition 3.5 in [1].

This lemma holds for ordered hypergroupoids and hypergroupoids (without order) and it is not needed for the rest of the paper.

The Proposition in [7](NK): *An ideal of a po-semigroup is prime if and only if it is both semiprime and weakly prime. In commutative po-semigroups the prime and weakly prime ideals coincide.*

Theorem 2.5 in [4](ZG): *Let S be an ordered semihypergroup and I a hyperideal of S . Then I is prime if and only if it is semiprime and weakly prime. In particular, if S is commutative, then the prime and weakly hyperideals coincide.*

If we get the proof of Proposition in [7], delete the “.” and write “o” instead, then this the proof of Theorem 2.5 in [4] (is this a right way to work without any explanation if we have the right to combine elements with sets?). The paper in [7] is not cited in [4].

Later, at the end of this note, we will see that the first part of Theorem 2.5 in [4] is not new, and there is a remark for the second part.

Theorem 3.1 in [4](ZG): *Let S be an ordered semihypergroup. Then the following statements are equivalent:*

- (1) S is semisimple.
- (2) $A \cap B = (A \circ B)$ for all hyperideals A, B of S .
- (3) $(A^2) = A$ for every hyperideal A of S .
- (4) Every hyperideal of S is weakly semiprime.

Theorem 18 in [10](NK): *An ordered hypersemigroup (H, \circ, \leq) is semisimple if and only if the ideals of H are idempotent.*

Theorem 9 in [10](NK): *Let (H, \circ, \leq) be an ordered hypersemigroup. The ideals of H are idempotent if and only if for any two ideals A and B of H , we have $A \cap B = (A * B)$.*

So, the equivalence (1) \Leftrightarrow (3) in [4, Theorem 3.1], is the Theorem 18 in [10].

The equivalence (2) \Leftrightarrow (3) in [4, Theorem 3.1], is the Theorem 9 in [10].

The implication (4) \Rightarrow (3) in [4, Theorem 3.1], is obvious as for any ideal A of S , the set $(A * A)$ is an ideal of S that contains $A * A$. So the proof of (4) \Rightarrow (1) in Theorem 3.1 in [4] is actually the proof of (3) \Rightarrow (1) in Theorem 18 in [10].

The proofs of Theorem 2.5 and Theorem 3.1 in [4] should be corrected as notations of the form $S \circ b \circ S \circ S \circ a \circ S$, a being an element, S being a set and \circ being an “operation” between elements have no sense; or a satisfactory explanation should be given in the paper.

In addition, the proof of Theorem 3.1 in [4] is a modification of the proof of Lemma 2 in [7], which is the following; not cited in [4].

Lemma 2 in [7](NK): *Let S be a po-semigroup. The following are equivalent:*

- (1) $(A^2) = A$ for every ideal A of S .
- (2) $A \cap B = (AB)$ for all ideals A, B of S .
- (3) $I(a) \cap I(b) = (I(a)I(b))$ for all $a, b \in S$.
- (4) $I(a) = ((I(a))^2)$ for every $a \in S$.
- (5) $a \in (SaSaS)$ for every $a \in S$ (that means that S is semisimple).

Theorem 3.2 in [4](ZG): *Let S be an ordered semihypergroup. Then S is intra-regular if and only if every hyperideal of S is semiprime.*

It is well known that an ordered semigroup S is intra-regular if and only if the ideals of S are semiprime. Let us give a proof of it: \Rightarrow . Let I be an ideal of S and $a \in S$ such that $a^2 \in I$. Since S is intra-regular, we have $a \in (Sa^2S) \subseteq (SIS) \subseteq (I) = I$. \Leftarrow . Let $a \in S$. The set (Sa^2S) is an ideal of S and $a^4 \in (Sa^2S)$. Since (Sa^2S) is semiprime, we have $a^2 \in (Sa^2S)$ and $a \in (Sa^2S)$, so S is intra-regular.

If we get this proof, delete the “.” and put “ \circ ” instead, then this is the Theorem 3.2 in [4]. No mention about the theorem on ordered semigroups is given in [4]. But anyway, it’s not enough to put “ \circ ” instead of “.” to pass from an ordered semigroup to an ordered hypersemigroup.

Theorem 19 in [10](NK): *Let (H, \circ, \leq) be an ordered hypersemigroup. The ideals of H are weakly prime if and only if they are idempotent and they form a chain.*

Theorem 3.3 in [4](ZG): *Let S be an ordered hypersemigroup and Θ be the set of all hyperideals of S . Then I is weakly prime for every $I \in \Theta$ if and only if S is semisimple and Θ is a chain.*

As an ordered hypersemigroup S is semisimple if and only if the ideals of S are idempotent ([4, Theorem 3.1] or [10, Theorem 18]), the Theorem 3.3 in [4] is the Theorem 19 in [10]; and it is not new.

According to [4], from Theorems 3.1, 3.2 and 3.3 in [4], the Theorem 3.4 in [4] can be easily obtained.

Theorem 3.4 in [4](ZG): *Let S be an ordered hypersemigroup and Θ be the set of all hyperideals of S . Then I is prime for every $I \in \Theta$ if and only if S is intra-regular and Θ is a chain.*

Theorem 23 in [10](NK): *Let S be an ordered hypersemigroup. If the ideals of S are weakly prime and semiprime, then they form a chain and S is intra-regular. “Conversely”, if the ideals of S form a chain and S is intra-regular, then the ideals of S are prime.*

The “ \Rightarrow ”-part in [4, Theorem 3.4] holds, more generally for ordered hypersemigroups in which the ideals are weakly prime and semiprime (see [10, Theorem 23]). Regarding the “converse” statement, let S be intra-regular and Θ be a chain. Since S is intra-regular, by Theorem 3.2 in [4], every ideal of S is semiprime, and so weakly semiprime as well. Since every ideal of S is weakly semiprime, by Theorem 3.1 in [4], S is semisimple. Since S is semisimple and the ideals of S form a chain, by Theorem 3.3 in [4], every ideal of S is weakly prime.

So the “ \Leftarrow ”-part of Theorem 3.4 in [4] cannot be easily obtained by Theorems 3.1, 3.2 and 3.3 in [4] as the author says. The proof of the “ \Leftarrow ”-part of this theorem in [4] is wrong; the right way to say it is to give an example, but this is out of the aim of the present note.

However, the Theorem 3.4 in [4] is correct, and its proof has been given in Theorem 23 in [10]. The proof of the “ \Leftarrow ”-part, that needs many technical details to transfer the proof from ordered semigroups to ordered hypersemigroups, is the main part of the theorem in [10].

From Theorem 23 in [10], we have the following corollary. We add the word “ordered” that, by mistake, was missing in [10].

Corollary 24 in [10](NK): *Let S be an ordered hypersemigroup. The following are equivalent:*

- (1) *The ideals of S are prime.*
- (2) *The ideals of S are weakly prime and semiprime.*
- (3) *The ideals of S form a chain and S is intra-regular.*

(The above Corollary, for hypersemigroups -without order also holds).

Let us give once more the Theorem 2.5 in [4] to compare it with the Corollary 24 in [10].

Theorem 2.5 in [4](ZG): *Let S be an ordered semihypergroup and I a hyperideal of S . Then I is prime if and only if it is semiprime and weakly prime. In particular, if S is commutative, then the prime and weakly hyperideals coincide.*

As we see now, the first part of the Theorem 2.5 in [4] is the equivalence (1) \Leftrightarrow (2) in Corollary 24 in [10] proved independently of the Proposition in [7].

Regarding the second part of the Theorem 2.5 in [4] (if S is commutative...), the following is from the last four lines of Remark 2.5 in [10]:

“It might be finally mentioned that in commutative ordered hypersemigroups the prime and weakly prime ideals coincide –the proof is the same with the Proposition in [N. Kehayopulu, On prime, weakly prime ideals in ordered semigroups, *Semigroup Forum* 44, no. 3 (1992), 341–346], we just have to replace the operation “ \cdot ” of the semigroup by the hyperoperation “ \circ ” of the hypersemigroup.”

The Example 3.5 is the Example B in [10] (cited [19] in the References in [4]) and, according to [4] one can check that this is an ordered hypersemigroup. We never check examples on ordered hypersemigroups given by a table by hand. We have ordered semigroups using computer programs. The Example 3.5 has been constructed in [10] from one of them using the methodology described in [11].

There are many papers on hyper... in the References of [4] not related with the contain of [4], while the paper is based only on the 3–4 papers on ordered semigroups indicated in the present note.

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