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On Strong Resolving Domination in the Join and Corona of Graphs

Gerald B. Monsanto[1,*], Penelyn L. Acal[2], Helen M. Rara[3]

[1] *Department of Teacher Education, Visayas State University-Villaba, 6537 Villaba, Leyte, Philippines*
[2] *Department of Mathematical Sciences, University of Science and Technology of Southern Philippines, 9023, Cagayan de Oro City, Philippines*
[3] *Department of Mathematics and Statistics, College of Science and Mathematics, Center of Graph Theory, Algebra, and Analysis-Premier Research Institute of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. Let G be a connected graph. A subset $S \subseteq V(G)$ is a *strong resolving dominating set* of G if S is a dominating set and for every pair of vertices $u, v \in V(G)$, there exists a vertex $w \in S$ such that $u \in I_G[v, w]$ or $v \in I_G[u, w]$. The smallest cardinality of a strong resolving dominating set of G is called the *strong resolving domination number* of G. In this paper, we characterize the strong resolving dominating sets in the join and corona of graphs and determine the bounds or exact values of the strong resolving domination number of these graphs.

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Key Words and Phrases: Strong resolving dominating set, strong resolving domination number, join, corona

1. Introduction

All graphs considered in this study are finite, simple, and undirected connected graphs, that is, without loops and multiple edges. For some basic concepts in Graph Theory, we refer readers to [4].

Let $G = \big(V(G), E(G)\big)$ be a connected graph. The *open neighborhood* $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Any element u of $N_G(v)$ is called a *neighbor* of v. The *closed neighborhood* $N_G[v] = N_G(v) \cup \{v\}$. Thus, the degree of v is given by $deg_G(v) = |N_G(v)|$. Customarily, for $S \subseteq V(G)$, $N_G(S) = \bigcup_{v\in S} N_G(v)$ and $N_G[S] = \bigcup_{v\in S} N_G[v]$.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v13i1.3625
Email addresses: monger2006@yahoo.com (G. Monsanto),
penelyn.acal@g.msuiit.edu.ph (P. Acal), helen.rara@g.msuiit.edu.ph (H. Rara)

A nonempty set $S \subseteq V(G)$ is a *dominating set* in graph G if $N_G[S] = V(G)$. Otherwise, we say S is a *non-dominating set* of G. The *domination number* of a graph G, denoted by $\gamma(G)$, is given by $\gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\}$. If $|S| = \gamma(G)$, then S is said to be a *minimum dominating set* or γ-set of G.

A vertex $w \in S$ *strongly resolves* two different vertices $u, v \in V(G)$ if $v \in I_G[u, w]$ or if $u \in I_G[v, w]$. A set W of vertices in G is a *strong resolving set* of G if every two vertices of G are strongly resolved by some vertex of W. The smallest cardinality of a strong resolving set of G is called the *strong metric dimension* of G and is denoted by $sdim(G)$.

A subset $S \subseteq V(G)$ is a *strong resolving dominating set* of G if it is both strong resolving and dominating. The smallest cardinality of a strong resolving dominating set of G is called the *strong resolving domination number* of G and is denoted by $\gamma_{sr}(G)$. A strong resolving dominating set of cardinality $\gamma_{sr}(G)$ is called a γ_{sr}-set of G.

A *clique* in a graph G is a complete induced subgraph of G. A clique C in G is called a *superclique* if for every pair of distinct vertices $u, v \in C$, there exists $w \in V(G) \setminus C$ such that $w \in N_G(u) \setminus N_G(v)$ or $w \in N_G(v) \setminus N_G(u)$. A superclique C in G is called a *dominated superclique* if for every $u \in C$, there exists $v \in V(G) \setminus C$ such that $uv \in E(G)$ [3]. A superclique (resp. dominated superclique) C is *maximum* in G if $|C| \geq |C^*|$ for all supercliques (resp. dominated supercliques) C^* in G. The *superclique (resp. dominated superclique) number*, $\omega_S(G)$ $\big($resp. $\omega_{DS}(G)\big)$ of G is the cardinality of a maximum superclique (resp. maximum dominated superclique) in G.

In recent years, the concept of domination in graphs has been studied extensively and several research papers have been published on this topic. The said concept was not formally defined mathematically until the publications of the books by Claude Berge [1] in 1958 and Oystein Ore in 1962. In 1977, a survey paper by Cockayne and Hedetniemi [2] began to study the concept of domination.

On the other hand, the problem of uniquely recognizing the possible position of an intruder such as fault in a computer network and spoiled device was the principal motivation in introducing the concept of metric dimension in graphs.

Slater [6] brought in the notion of locating sets and its minimal cardinality as locating number. The same concept was also introduced by Harary and Melter [4] but using the terms resolving sets and metric dimension to refer to locating sets and locating number, respectively.

In 2007, Oellerman and Peter-Fransen [5] introduced the strong resolving graph G_{SR} of a connected graph G as a tool to study the strong metric dimension of G.

This study aims to define and characterize the strong resolving dominating sets and determine the exact values or bounds in the join and corona of two graphs.

2. Preliminary Results

Remark 1. Every strong resolving dominating set of a connected graph G is a dominating set. Hence, $\gamma(G) \leq \gamma_{sr}(G)$.

Remark 2. Every strong resolving dominating set of a connected graph G is a strong resolving set. Thus, $sdim(G) \leq \gamma_{sr}(G)$.

Remark 3. For any connected graph G of order n, $1 \leq \gamma_{sr}(G) \leq n-1$.

Remark 4. Any superset of a strong resolving dominating set is a strong resolving dominating set.

Proposition 1. Let G be a connected graph of order $n \geq 2$. Then,

(i) $\gamma_{sr}(P_n) = \lceil \frac{n+1}{3} \rceil$

(ii) $\gamma_{sr}(K_n) = n-1$

(iii)

$$\gamma_{sr}(C_n) = \begin{cases} 2 & , \quad \text{if } n = 3 \\ n-2 & , \quad \text{if } n > 3 \text{ and } n \text{ is odd} \\ \lceil \frac{n}{2} \rceil & , \quad \text{if } n > 3 \text{ and } n \text{ is even} \end{cases}$$

Proposition 2. Let G be a connected graph of order n and let

$$A = \{x \in V(G) : deg_G(x) = n-1\}.$$

If $A \neq \varnothing$ and C is a superclique in G, then $|C \cap A| \leq 1$. Moreover, if C is a maximum superclique of G, then $|C \cap A| = 1$.

Remark 5. Let G be a nontrivial connected graph with $diam(G) \leq 2$. For distinct vertices $u, v, w \in G, u \in I_G[v, w]$ if and only if $d_G(v, w) = 2$ and $u \in N_G(v) \cap N_G(w)$.

Proposition 3. Let G be a nontrivial connected graph with $diam(G) \leq 2$. Then $S = V(G) \setminus C$ is a strong resolving set of G if and only if $C = \varnothing$ or C is a superclique in G. In particular, $sdim(G) = |V(G)| - \omega_S(G)$.

Proof: Assume that S is a strong resolving set of G. If $S \cap V(G) = V(G)$, then $C = \varnothing$. Suppose $S \subsetneq V(G)$. Let $C = V(G) \setminus S$. Then $S = V(G) \setminus C$. Let $x, y \in C$, where $x \neq y$. Since S is a strong resolving set of G, x and y are strongly resolved by some $z \in S$. We may assume that $x \in I_G[y, z]$. Then $d_G(y, z) = 2$ and $x \in N_G(y) \cap N_G(z)$ by Remark 5. Thus, $z \in N_G(x) \setminus N_G(y)$, showing that C is a superclique in G.

Conversely, assume that $S = V(G) \setminus C$, where C is a superclique in G. Let $x, y \notin S$, where $x \neq y$. Then $x, y \in C$. Since C is a superclique in G, there exists $z \in S$ such that $z \in N_G(x) \setminus N_G(y)$ or $z \in N_G(y) \setminus N_G(x)$. Since $diam(G) = 2, d_G(y, z) = 2$ or $d_G(x, z) = 2$. By Remark 5, $x \in I_G[y, z]$ or $y \in I_G[x, z]$. Hence, S is a strong resolving set of G.

Suppose S is a strong resolving set of G. Then $S = V(G) \setminus C$, where C is a superclique in G and $|C| = \omega_S(G)$. Thus, $sdim(G) = |S| = |V(G)| - |C| = |V(G)| - \omega_S(G)$. □

3. On Strong Resolving Domination in the Join of Graphs

The *join* of two graphs G and H is the graph $G + H$ with vertex set $V(G + H) = V(G) \overset{\bullet}{\cup} V(H)$ and edge set $E(G + H) = E(G) \overset{\bullet}{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Remark 6. For the joins $\langle v\rangle+P_n$ and $\langle w\rangle+C_n$, it can be verified that $\gamma_{sr}\big(\langle v\rangle+P_n\big)=n-1$ for $n\geq 3$ and $\gamma_{sr}\big(\langle w\rangle+C_n\big)=n-2$ for $n\geq 4$.

Proposition 4. Let G be a connected graph with $\gamma(G)\neq 1$ and let $K_1=\langle v\rangle$. Then $C\subseteq V(K_1+G)$ is a superclique of K_1+G if and only if $|C|=1$ or $|C|\geq 2$ and $C\setminus\{v\}$ is a superclique of G.

Proof: The conditions follow immediately if $C\subseteq V(K_1+G)$ is a superclique of K_1+G. For the converse, the case when $|C|=1$ is obvious. Suppose $|C|\geq 2$. Since $C\setminus\{v\}$ is a superclique of G, we only need to consider the pair of distinct vertices $z,v\in C$. Since $\gamma(G)\neq 1$, there exists $w\in V(G)$ such that $zw\notin E(G)$. Since $diam(K_1+G)=2$, $d_G(z,w)=2$. Hence, $z\in N_G(v)\setminus N_G(w)$, showing that C is a superclique of K_1+G. □

Theorem 1. Let G be a nontrivial connected graph of order n with $\gamma(G)\neq 1$ and $K_1=\langle v\rangle$. Then $S\subseteq V(K_1+G)$ is a strong resolving dominating set of K_1+G if and only if $S=V(G)$, or $S=V(K_1+G)\setminus C$ or $S=V(G)\setminus C^*$ where C and C^* are superclique and dominated superclique, respectively, in G.

Proof: Let S be a strong resolving dominating set of K_1+G. Suppose $\gamma(G)\neq 1$. If $v\notin S$, then $S\subsetneq V(G)$. By Proposition 3, $S=V(K_1+G)\setminus(C\cup\{v\})=V(G)\setminus C$, where S is a dominating set in K_1+G and $C\cup\{v\}$ is a superclique in K_1+G. By Proposition 4, C is a superclique in G. Since $\{v\}$ is a superclique in $K_1+G, S=V(K_1+G)\setminus\{v\}=V(G)$. On the other hand, if $v\in S$ and $C=V(K_1+G)\setminus S$, then $S=V(K_1+G)\setminus C$ where C is a superclique in K_1+G by Proposition 3. By Proposition 4, $C\setminus\{v\}=C$ is a superclique in G. Conversely, the case when $S=V(G)$ and $S=V(K_1+G)\setminus C$ follows immediately from Proposition 3. Suppose $S=V(G)\setminus C^*$, where C^* is a dominated superclique in G. By Proposition 4, $C\cup\{v\}$ is a superclique of K_1+G. Since $v\notin S$, $S=V(G)\setminus C^*=V(K_1+G)\setminus\big(C\cup\{v\}\big)$. By Proposition 3, S is a strong resolving dominating set of K_1+G. □

Theorem 2. Let G be a nontrivial connected graph of order n with $\gamma(G)=1$ and $K_1=\langle v\rangle$. Then $S\subseteq V(K_1+G)$ is a strong resolving dominating set of K_1+G if and only if $S=V(G)$ or $S=V(K_1+G)\setminus C$ or $S=(V(G)\setminus C^*)\cup\{x\in C^*:\deg(x)=n-1\}$ where C and C^* are superclique and dominated superclique, respectively, in G.

Proof: Let S be a strong resolving dominating set of K_1+G. Suppose $\gamma(G)=1$. If $v\in S$ and $C=V(K_1+G)\setminus S$, then $S=V(K_1+G)\setminus C=\{v\}\cup(V(G)\setminus C)$. By Proposition 3, C is a superclique in K_1+G. Hence for $x,y\in C,x\neq y$, there exists $w\in V(G)\setminus C$ such that $w\in N_G(x)\setminus N_G(y)$ or $w\in N_G(y)\setminus N_G(x)$, showing that C is a superclique in G. On the other hand, if $v\notin S$, then $S\subsetneq V(G)$. Let $C=V(K_1+G)\setminus S$. Hence, $S=V(K_1+G)\setminus C=V(G)\setminus C$. By Proposition 3, C is a superclique in K_1+G. Hence, C is also a superclique in G. Since $\gamma(G)=1$, $A_G=\{z\in V(G);deg_G(z)=n-1\}\neq\varnothing$. By Proposition 2, $|C\cap A_G|=1$. Let $z\in C\cap A_G$. Since $d_{K_1+G}(z,v)=1$ and $d_{K_1+G}(v)=n$, none of the elements in S strongly resolves z and v, a contradiction. Hence, $z\in S$. Thus, $S=(V(G)\setminus C)\cup\{z\}$. In addition, since $\{v\}$ is a superclique in K_1+G, $S=$

$V(K_1+G)\setminus\{v\}=V(G)$. Similarly, if $C^*=V(G)\setminus S$, then C^* is a dominated superclique in G.

For the converse, the case when $S=V(G)$ is trivial. Suppose $S=V(K_1+G)\setminus C$, where C is a superclique in G. Let $x,y\notin S, x\neq y$. Then $x,y\in C$ and there exists $w\in V(G)\setminus C$ such that $xy\in E(G)$ and $w\in N_G(x)\setminus N_G(y)$ or $w\in N_G(y)\setminus N_G(x)$. By Remark 5, $x\in I_{K_1+G}[y,w]$ or $y\in I_{K_1+G}[x,w]$, showing that S is a strong resolving dominating set of K_1+G.

Suppose $S=(V(G)\setminus C^*)\cup\{z\in C^*; deg_G(z)=n-1\}$, where C^* is a dominated superclique in G. Let $x,y\notin S, x\neq y$. Then $x,y\in C^*$. By the same argument above, there exists $w\in S$ that strongly resolves x and y. Now, consider the vertices x and v. Since $x\notin S$, then $deg_G(x)<n-1$. Hence, there exists $z\in V(G)$ such that $xz\notin E(G)$. It follows that $v\in I_{K_1+G}[x,z]$. Thus, S is a strong resolving dominating set of K_1+G. □

Corollary 1. Let $P_n=[v_1,v_2,\ldots,v_n]$ and $C_m=[c_1,c_2,\ldots,c_m,c_1]$ where $n,m\geq 3$.

(i) The sets $V(P_n)\setminus\{v_i,v_{i+1}\}$, for $i=2\ldots,n-2$ are the strong resolving dominating sets of $\langle v\rangle+P_n$.

(ii) The sets $V(C_m)\setminus\{c_i,c_{i+1}\}$ and $V(C_m)\setminus\{c_1,c_m\}$, for $i=1,2\ldots,m-1$ are the strong resolving dominating sets of $\langle v\rangle+C_m$.

Corollary 2. Let G be a nontrivial connected graph of order n. Then

(i) for $\gamma(G)=1$, we have $\gamma_{sr}(K_1+G)=n-\omega_S(G)+1$;

(ii) for $\gamma(G)\neq 1$, we have $\gamma_{sr}(K_1+G)=\min\{\gamma_{sr}(G),n-\omega_S(G)\}$.

The next result follows from Proposition 3, Theorem 1 and Theorem 2.

Corollary 3. Let G be nontrivial connected graph with $\text{diam}(G)\leq 2$. Then

(i) for $\gamma(G)=1$, we have $\gamma_{sr}(K_1+G)=sdim(G)+1$;

(ii) for $\gamma(G)\neq 1$, we have $\gamma_{sr}(K_1+G)=\min\{\gamma_{sr}(G),sdim(G)+1\}$.

The following theorem gives a characterization of the strong resolving dominating sets in the join of K_1 and a disconnected graph G.

Theorem 3. Let $K_1=\langle v\rangle$ and G be a disconnected graph whose components are G_i for $i=1,2,\ldots,m$. A proper subset S of $V(K_1+G)$ is a strong resolving dominating set of K_1+G if and only if $S=V(G)$ or $S=V(G)\setminus C_i^*$ or $S=V(K_1+G)\setminus C_i$ where C_i is a superclique in G_i, for $i=1,2,\ldots,m$ and C_i^* is a dominated superclique of G_i.

Proof: Let S be a strong resolving dominating set of K_1+G. Suppose $v\notin S$. Then $S\subsetneq V(G)$. Let $C_i=V(K_1+G)\setminus S$, for $i=1,2,\ldots,m$. Then $S=V(K_1+G)\setminus C_i=V(G)\setminus C_i$. Let $x,y\in C_i, x\neq y$. Since $d_{K_1+G}(w,x)=d_{K_1+G}(w,y)$, for all $w\in V(G)\setminus V(G_i)$, there exists $z\in V(G_i)\setminus C_i$ such that $x\in I_{G_i}[y,z]$ or $y\in I_{G_i}[x,z]$. By Remark 5,

$x \in N_{G_i}(y) \setminus N_{G_i}(z)$ or $y \in N_{G_i}(x) \setminus N_{G_i}(z)$. Thus, C_i is a superclique in G_i. Since $\{v\}$ is a superclique in K_1+G, by Proposition 3, $S = \big(V(K_1+G) \setminus \{v\}\big) = V(G)$. On the other hand, if $v \in S$ and $C_i = V(K_1+G) \setminus S$, for $i = 1, 2, \ldots, m$, then $S = V(K_1+G) \setminus C_i$, where C_i is a superclique in K_1+G, by Proposition 3. Hence, C_i is a superclique in G_i. Similarly, if $C_i^* = V(G) \setminus S$ for $i = 1, 2, \ldots, m$ where C_i^* is a dominated superclique of G_i and since S is dominating, then $V(G_i) \setminus C_i^*$ is a dominating set of G_i.

For the converse, if $S = V(G)$, then we are done. Suppose $S = V(G) \setminus C_i^*$, or $S = V(K_1+G) \setminus C_i$, where C_i and C_i^* are superclique and dominated superclique, respectively, in G_i for $i = 1, 2, \ldots, m$. Then by Theorem 1 and Theorem 2, $V(G_i) \setminus C_i^*$ is a strong resolving dominating set of $K_1 + G_i$. By Remark 4, $S = V(K_1+G) \setminus C_i$ is a strong resolving dominating set of K_1+G. □

Corollary 4. Let G_i be connected graphs of orders n_i and G be a disconnected graph whose components are G_i for $i = 1, 2, \ldots, m$ and $S_i = V(G_i) \setminus C_i$ where C_i is a maximum dominated superclique of G_i. Then

$$\gamma_{sr}(K_1+G) = \sum_{i=1}^{m} n_i - \max\big\{\gamma_{sr}(G_i), \omega_{DS}(G_i) + 1 \big| i = 1, 2, \ldots, m\big\}.$$

In the join of two graphs G and H, the previous results have already considered the case when G or H is trivial. Hence, in the following theorem, a characterization of the strong resolving dominating sets in the join of nontrivial connected graphs G and H is considered.

Theorem 4. Let G and H be nontrivial connected graphs of orders m and n, respectively. A proper subset S of $V(G+H)$ is a strong resolving dominating set of $G+H$ if and only if at least one of the following is satisfied:

(i) $S = V(G+H) \setminus C_G$ where C_G is a superclique in G.

(ii) $S = V(G+H) \setminus C_H$ where C_H is a superclique in H.

(iii) If $\gamma(G) = 1$ and $\gamma(H) = 1$,

$$S = [V(G+H) \setminus (C_G \cup C_H)] \cup \{z \in C_G : \deg_G(z) = m-1\}, \text{ or}$$
$$S = [V(G+H) \setminus (C_G \cup C_H)] \cup \{w \in C_H : \deg_H(w) = n-1\}$$

where C_G and C_H are supercliques in G and H, respectively.

(iv) If $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$,

$$S = [V(G+H) \setminus (C_G \cup C_H)] = (V(G) \setminus C_G) \cup (V(H) \setminus C_H),$$

where C_G and C_H are supercliques in G and H, respectively.

Proof: Let S be a strong resolving dominating set of $G+H$. Since $d_{G+H}(x,y)=1$, for each $x \in V(G)$ and $y \in V(H)$, none of the vertices in $V(G)$ and $V(H)$ strongly resolves any pair of distinct vertices in $V(H)$ and $V(G)$, respectively. Thus, $S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$. If $S \cap V(H) = V(H)$, then $S \neq V(G)$. Let $C_G = V(G) \setminus S$. Hence, $S = V(G+H) \setminus C_G$. Let $u, v \in C_G, u \neq v$. Then there exists $w \in S \cap V(G)$ such that $u \in I_{G+H}[v,w]$ or $v \in I_{G+H}[u,w]$. By Remark 5, $w \in N_G(u) \backslash N_G(v)$ or $w \in N_G(v) \backslash N_G(u)$. Thus, C_G is a superclique in G. Similarly, $S \cap V(G) = V(G)$. On the other hand, if $S \cap V(G) \neq V(G)$, $S \cap V(H) \neq V(H)$, $C_G = V(G) \setminus S$ and $C_H = V(H) \setminus S$, then $S = V(G+H) \setminus (C_G \cup C_H)$. Hence, C_G and C_H are supercliques in G and H, respectively.

Suppose $\gamma(G)=1$ and $\gamma(H)=1$. Then

$$A_G = \{z_G \in V(G); deg_G(z) = m-1\} \neq \varnothing$$

and

$$A_H = \{z_H \in V(H); deg_H(z) = m-1\} \neq \varnothing.$$

By Proposition 2, $|C_G \cap A_G| \leq 1$ and $|C_H \cap A_H| \leq 1$. Hence, we may assume that there exists $z_G \in C_G \cap A_G$ and $z_H \in C_H \cap A_H$. Then none of the vertices in $S \cap V(G)$ and $S \cap V(H)$ strongly resolves z_G and z_H, a contradiction. Thus $z_G \in S$ or $z_H \in S$ so that $S = [V(G+H) \setminus (C_G \cup C_H)] \cup \{z \in C_G : \deg_G(z) = m-1\}$, or $S = [V(G+H) \setminus (C_G \cup C_H)] \cup \{w \in C_H : \deg_H(w) = n-1\}$.

Suppose $\gamma(G) \neq 1$ or $\gamma(H) = 1$. Then $A_G = \varnothing$ or $A_H = \varnothing$. Hence,

$$S = V(G+H) \setminus (C_G \cup C_H) = (V(G) \setminus C_G) \cup (V(G) \setminus C_G).$$

Conversely, suppose S satisfies condition (i). Since C_G is a superclique in G, there exists $w \in (V(G) \setminus C_G) \subseteq S$ such that $u \in I_G[v,w]$ or $v \in I_G[u,w]$, for any $u, v \notin S, u \neq v$, showing that w strongly resolves u, v. A similar argument applies if S satisfies condition (ii).

Suppose S satisfies condition (iii) or (iv). Let $u, v \notin S, u \neq v$. If $u, v \in C_G$ or $u, v \in C_H$, then we are done. Consider the pair $u \in V(G) \setminus S$ and $v \in V(H) \setminus S$. Since C_G is a superclique in G, there exists $z \in (V(G) \setminus C_G) \subseteq S$ such that $z \in N_G(u) \setminus N_G(v)$ or $z \in N_G(v) \setminus N_G(u)$. Hence, $u \in I_G[v,z]$ or $v \in I_G[u,z]$. Thus, S is a strong resolving dominating set of $G+H$. □

Corollary 5. Let G and H be nontrivial connected graphs of orders m and n, respectively. Then

$$\gamma_{sr}(G+H) = \begin{cases} (m - \omega_S(G)) + (n - \omega_S(H)) + 1, & \text{if } \gamma(G) = 1 \text{ and } \gamma(H) = 1 \\ (m - \omega_S(G)) + (n - \omega_S(H)), & \text{if } \gamma(G) \neq 1 \text{ or } \gamma(H) \neq 1. \end{cases}$$

Remark 7. If G is a nontrivial connected graph with $\gamma(G) = 1$, then $diam(G) \leq 2$.

Corollary 6. Let G and H be nontrivial connected graphs with $\gamma(G) = 1$ and $\gamma(H) = 1$. Then $\gamma_{sr}(G+H) = \mathrm{sdim}(G) + \mathrm{sdim}(H) + 1$. In particular,

(i) $\gamma_{sr}(G+H) = 3$ for $G = P_m$ and $H = P_n$ $(m \geq 2, n \geq 2)$;

(ii) $\gamma_{sr}(G+H) = \lceil \frac{n}{2} \rceil + 2$ for $G = P_m$ and $H = C_n$ $(m \geq 2, n \geq 3)$;

(iii) $\gamma_{sr}(G+H) = 4$ for $G = C_m$ and $H = C_n$ $(m = n = 3)$

(iv) $\gamma_{sr}(G+H) = \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil + 1$ for $G = C_m$ and $H = C_n$ $(m, n \geq 4)$

Theorem 5. Let G be a disconnected graph with components $G_1, \ldots, G_n$ and H a disconnected graph with components $H_1, \ldots, H_m$. A proper subset S of $V(G+H)$ is a strong resolving dominating set of $G+H$ if and only if S satisfies any of the following:

(i) $S = S_G \cup V(H)$ where $V(G) \setminus S_G$ is a superclique of G_i for some $i \in \{1, 2, \ldots, n\}$;

(ii) $S = S_H \cup V(G)$ where $V(H) \setminus S_H$ is a superclique of H_j for some $j \in \{1, 2, \ldots, m\}$;

(iii) $S = S_G \cup S_H$, where $V(G) \setminus S_G$ and $V(H) \setminus S_H$ are supercliques of G_i and H_j, for some $i \in \{1, 2, \ldots, n\}$ and some $j \in \{1, 2, \ldots, m\}$.

Proof: Let S be a strong resolving dominating set of $G+H$ and $x \in G_i$ and $y \in G_k$, $i \neq k$. Since $d_{G+H}(x, y) = 2$ and $d_{G+H}(x, h) = d_{G+H}(y, h) = 1$, for all $h \in V(H)$, then $x \in S$ or $y \in S$. Hence, $S \cap V(G) = \varnothing$. Similarly, $S \cap V(H) \neq \varnothing$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Suppose $S \cap V(H) = V(H)$. Then $S_G \subseteq V(G)$. Let $C_G = V(G) \setminus S_G$. Then $S = S_G \cup V(H)$. Let $u, v \notin S_G, u \neq v$. Hence, $u, v \in C_G$. Since S is a strong resolving dominating set of $G+H$, $C_G \subseteq V(G_i)$ for some $i \in \{1, 2, \ldots, n\}$. Then there exists $z \in S \cap V(G_i)$ such that $u \in I_{G+H}[v, z]$ or $v \in I_{G+H}[u, z]$. It follows from Remark 5 that C_G is a superclique of G_i. Similarly, $S \cap V(G) = V(G)$. On the other hand, if $S \cap V(G) \neq V(G)$, $S \cap V(H) \neq V(H)$, $C_G = V(G) \setminus S_G$ and $C_H = V(H) \setminus S_H$, then $S = S_G \cup S_H$. Hence, C_G and C_H are supercliques of G_i and H_j for some $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$.

Conversely, suppose S satisfies condition (i). Let $u, v \notin S, u \neq v$. Then $u, v \in C_G = V(G) \setminus S_G$. Hence, there exists $w \in V(G_i) \setminus C_G$ such that $w \in N_{G+H}(u) \setminus N_{G+H}(v)$ or $w \in N_{G+H}(v) \setminus N_{G+H}(u)$. By Remark 5, $u \in I_{G+H}[v, w]$ or $v \in I_{G+H}[u, w]$. Thus, S is a strong resolving dominating set of $G+H$. Similarly, the same conclusion holds if S satisfies condition (ii).

Suppose S satisfies condition (iii). Let $u, v \notin S, u \neq v$. If $u, v \in C_G$ or $u, v \in C_H$, then we are done. Assume $u \in C_G$ and $v \in C_H$. Since $d_{G+H}(u, u') = 2$ for $u' \in G_k, k \neq i$ and $d_{G+H}(v, v') = 2$ for $v' \in H_p, p \neq j, v \in I_{G+H}[u, u']$ or $u \in I_{G+H}[v, v']$. Thus, S is a strong resolving dominating set of $G+H$. □

4. On Strong Resolving Domination in the Corona of Graphs

The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining every vertex of the ith copy of H to the ith vertex of G. For $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v, v \in V(G)$.

Remark 8. For the coronas $P_n \circ K_1$ and $C_n \circ K_1$, it can be verified easily that

$$\gamma_{sr}(P_n \circ K_1) = \gamma_{sr}(C_n \circ K_1) = n, \ \forall n \geq 3.$$

Theorem 6. Let G be a nontrivial connected graph and H a connected graph. A proper subset S of $V(G \circ H)$ is a strong resolving dominating set of $G \circ H$ if and only if one of the following holds:

(i) $S = A \cup \big(\cup_{u \in V(G)} V(H^u)\big)$ where $A \subseteq V(G)$;

(ii) $S = A \cup \big(\cup_{u \in V(G) \setminus \{v\}} V(H^u)\big) \cup B_v$ for a unique $v \in V(G)$, where $A \subseteq V(G) \setminus \{v\}$ and B_v is a strong resolving dominating set of H^v if $\gamma(H) = 1$ or B_v is a strong resolving dominating set of $\langle v \rangle + H^v$ if $\gamma(H) \neq 1$;

(iii) $S = A \cup \big(\cup_{u \in V(G) \setminus \{v\}} V(H^u)\big) \cup B_v$ for a unique $v \in V(G)$ where $v \in A \subseteq V(G)$ and B_v is a strong resolving set of H^v if $\gamma(H) = 1$ and B_v is a strong resolving set of $\langle v \rangle + H^v$ if $\gamma(H) \neq 1$.

Proof: Suppose S is a strong resolving dominating set of $G \circ H$. Let $A = S \cap V(G)$ and $B_v = S \cap V(H^v)$, where $v \in V(G)$. Consider the following cases:
Case 1. $S \cap V(H^v) = V(H^v)$

Then $B_v = H^v$. Thus, $S = A \cup \big(\cup_{u \in V(G)} V(H^u)\big)$.
Case 2. $S \cap V(H^v) \neq V(H^v)$

Let $u, v \in V(G), u \neq v$ such that $S \cap V(H^u) \neq V(H^u)$ and $S \cap V(H^v) \neq V(H^v)$. Pick $p_u \in V(H^u) \setminus S$ and $p_v \in V(H^v) \setminus S$. Then, none of the vertices in S strongly resolves p_u and p_v, a contradiction. Thus, the vertex $v \in V(G)$ such that $S \cap V(H^v) \neq V(H^v)$ must be unique. Hence, $S = A \cup \big(\cup_{u \in V(G) \setminus \{v\}} V(H^u)\big) \cup B_v$.

Subcase 2.1 $v \in S$

Let $C_v = V(H^v) \setminus B_v$. Hence, $B_v = V(H^v) \setminus C_v$. Then it can be verified that C_v is a superclique in H^v. If $\gamma(H) \neq 1$, by Theorem 1, B_v is a strong resolving set of $\{v\} + H^v$. If $\gamma(H) = 1$, then by Remark 7 and Theorem 2, B_v is a strong resolving set of H^v.

Subcase 2.2 $v \notin S$

Since S is a dominating set of $G \circ H$, B_v is a dominating set of H^v. By similar argument in the proof of subcase 2.1, B_v is a strong resolving dominating set of H^v if $\gamma(H) = 1$ or B_v is a strong resolving dominating set of $\langle v \rangle + H^v$ if $\gamma(H) \neq 1$.

Conversely, suppose (i), (ii) and (iii) hold. Consider the following cases:
Case 1. $p, q \in V(G) \setminus A$

Let $p, q \in V(G \circ H) \setminus S$ where $p \neq q$ and $p = v$ or $q = v$, but not both, then $p \in I_{G \circ H}[q, z]$ or $q \in I_{G \circ H}[p, z]$ for some $z \in B_v$. On the other hand, if $p \neq v$ and $q \neq v$, then $q \in I_{G \circ H}[p, w]$ for some $w \in V(H^q) \subset S$ or $p \in I_{G \circ H}[q, r]$ for some $r \in V(H^p) \setminus B_v$.
Case 2. $p, q \in V(H^v) \setminus B_v$

Since B_v is a strong resolving set of H^v, there exists $t \in B_v \subset S$ that strongly resolves p and q.
Case 3. $p \in V(G) \setminus (A \cup \{v\})$ and $q \in V(H^v) \setminus B_v$

Since $p \neq v$, then $V(H^p) \subset S$ and $p \in I_{G \circ H}[q, z]$ for all $z \in V(H^p)$.

Case 4. $p = v$, $q \in V(H^v) \setminus B_v$

Let $t \in N_G(v)$. Then $V(H^t) \subset S$ and $p \in I_{G \circ H}[q, u]$, for some $u \in V(H^t)$.

Cases 1 to 4 imply that S is a strong resolving dominating set of $G \circ H$ and $(i), (ii), (iii)$ imply that S is a dominating set of $G \circ H$. Accordingly, S is a strong resolving dominating set of $G \circ H$. □

Corollary 7. Let G and H be connected graphs of orders m and n, respectively

$$\gamma_{sr}(G \circ H) = \begin{cases} (m-1)n + \gamma_{sr}(H), & \text{if } \gamma(H) = 1 \\ (m-1)n + \gamma_{sr}(K_1 + H), & \text{if } \gamma(H) \neq 1 \end{cases}$$

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References

[1] C. Berge. *Theorie des graphes et ses applications.* Metheun and Wiley, London and New York, 1962.

[2] E. Cockayne and S. Hedetniemi. Towards a theory of domination in graphs. *Networks*, 7(3):247–261, 1977.

[3] A. Cuivillas and Jr. S. Canoy. Restrained double domination in the join and corona of graphs. *International Journal of Math. Analysis*, 8(27):1339–1347, 2014.

[4] F. Harary. *Graph Theory.* Addison-Wesley Publishing Company, USA, 1969.

[5] O. Oellermann and J. Peter-Fransen. The Strong Metric Dimension of Graphs and Digraphs. *Discrete Applied Mathematics*, 155(3):356–364, 2007.

[6] P. Slater. Dominating and reference sets in a graph. *Journal of Mathematics and Physical Science*, 22(4):445–455, 1988.