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Asymptotic solutions of scalar integro-differential equations with partial derivatives and with fast oscillating coefficients

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Abstract. In the paper, ideas of the Lomov regularization method are generalized to the Cauchy problem for a singularly perturbed partial integro-differential equation in the case when the integral term contains a rapidly varying kernel. Regularization of the problem is carried out, the normal and unique solvability of general iterative problems is proved.

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1. Introduction

In the paper, we consider the Cauchy problem for the integro-differential equation with partial derivatives:

$$\begin{aligned} &L_\varepsilon y(x,t,\varepsilon) \equiv \varepsilon \tfrac{\partial y}{\partial x} = a(x)y + \int\limits_{x_0}^{x} K(x,t,s)y(s,t,\varepsilon)ds + h(x,t)+ \\ &+\varepsilon g(x) cos \tfrac{\beta(x)}{\varepsilon} y,\ y(x_0,t,\varepsilon) = y^0(t) \quad (\,(x,t) \in [x_0,X] \times [0,T]\,), \end{aligned} \tag{1}$$

where $\beta'(x) > 0, g(x), a(x)$ is a scalar functions, $y^0(t)$ constant, $\varepsilon > 0$ is a small parameter. The problem of constructing a regularized asymptotic solution [1] of the problem (1) is posed. Earlier, in [2], [3], [4], [5], [6], [7], systems for ordinary integro-differential equations were mainly considered. In this paper we consider an partial integro-differential equations. Construction of asymptotic solutions for singularly perturbed integro-differential equations with partial derivatives in the case when integral operators change rapidly was first investigated in the works [8], [9], [10]. Construction of asymptotical solutions for

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ordinary integro-differential equations with fast oscillating coefficients from the position of the regularization method are considered in [11].

Denote by $\lambda_1(x) = -a(x)$, $\beta'(x)$ is a frequency of fast oscillating cosine. In the following, functions $\lambda_2(x) = -i\beta'(x)$, $\lambda_3(x) = +i\beta'(x)$ will be called *the spectrum of a fast oscillating coefficient.*

We assume that the conditions are fulfilled:

(i) $K(x,t,s) \in C^{\infty}\{x_0 < x < s < X,\, 0 < t < T\}, h(x,t) \in C^{\infty}([x_0, X] \times [0,T]), a(x), g(x), \beta(x) \in C^{\infty}[x_0, X]$,

(ii) $\lambda_1(x) \neq \lambda_j(x), \quad j = 2,3, \quad \lambda_i(x) \neq 0, \ (\forall x \in [x_0, X]), \ i = 1,2,3;$

(iii) $Re\lambda_1(x) \leq 0, \quad (\forall x \in [x_0, X]);$

(iv) for $\forall x \in [x_0, X]$ and $n_2 \neq n_3$ inequalities

$$\begin{array}{l} n_2\lambda_2(x) + n_3\lambda_3(x) \neq \lambda_1(x), \\ \lambda_1(x) + n_2\lambda_2(x) + n_3\lambda_3(x) \neq \lambda_1(x), \ \ (\forall x \in [x_0, X]) \end{array}$$

for all multi-indices $n = (n_2, n_3)$ with $|n| \equiv n_2 + n_3 \geq 1$ (n_2 and n_3 are non-negative integers) are holds.

We will develop an algorithm for constructing a regularized [1] asymptotic solution of problem (1).

2. Regularization of the problem

Denote by $\sigma_j = \sigma_j(\varepsilon)$ independent of magnitude $\sigma_1 = e^{-\frac{i}{\varepsilon}\beta(t_0)}, \sigma_2 = e^{+\frac{i}{\varepsilon}\beta(t_0)}$, and rewrite system (1) as

$$\begin{array}{l} \varepsilon\frac{\partial y}{\partial x} = a(x)y + \varepsilon\frac{g(x)}{2}\left(e^{-\frac{i}{\varepsilon}\int\limits_{t_0}^{t}\beta'(\theta)d\theta}\sigma_1 + +e^{+\frac{i}{\varepsilon}\int\limits_{t_0}^{t}\beta'(\theta)d\theta}\sigma_2\right)y + \\ +\int\limits_{x_0}^{x}K(x,t,s)y(s,t,\varepsilon)ds + h(x,t), \ \ y(x_0,t,\varepsilon) = y^0. \end{array} \qquad (2)$$

Introduce the regularized variables:

$$\tau_j = \frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_j(\theta)d\theta \equiv \frac{\psi_j(x)}{\varepsilon}, \quad j = \overline{1,3}$$

and instead of problem (2), consider the problem

$$\begin{array}{c} \varepsilon\frac{\partial\tilde{y}}{\partial x} + \sum\limits_{j=1}^{3}\lambda_j(x)\frac{\partial\tilde{y}}{\partial\tau_j} - a(x)\tilde{y} - \int\limits_{x_0}^{x}K(x,t,s)\tilde{y}(s,t,\frac{\psi(s)}{\varepsilon},\varepsilon)ds - \\ -\varepsilon\frac{g(x)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2)\tilde{y} = h(x,t), \quad \tilde{y}(x_0,t,0,\varepsilon) = y^0, \end{array} \qquad (3)$$

for the function $\tilde{y} = \tilde{y}(x,t,\tau,\varepsilon)$ where is indicated: $\psi = (\psi_1, \psi_2, \psi_3)$. It is clear that if $\tilde{y} = \tilde{y}(x,t,\tau,\varepsilon)$ is a solution of the problem (3), then the function is $\tilde{y} = \tilde{y}(x,t,\frac{\psi(x)}{\varepsilon},\varepsilon)$ an exact solution to problem (2), therefore, problem (3) is extended with respect to problem (2). However, it cannot be considered fully regularized, since it does not regularize the integral

$$J\tilde{y} = \int_{x_0}^{x} K(x,t,s)\tilde{y}(s,t,\psi(s,\varepsilon),\varepsilon)ds.$$

Definition. A class M_ε is said to be asymptotically invariant (with $\varepsilon \to +0$) with respect to an operator P_0 if the following conditions are fulfilled:

1) $M_\varepsilon \subset D(P_0)$ for each fixed $\varepsilon > 0$;

2) the image $P_0\mu(x,t,\varepsilon)$ of any element $\mu(x,t,\varepsilon) \in M_\varepsilon$ decomposes in a power series

$$P_0\mu(x,t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \mu_n(x,t,\varepsilon)(\varepsilon \to +0,\ \ \mu_n(x,t,\varepsilon) \in M_\varepsilon,\ \ n = 0,1,...),$$

convergent asymptotically for $\varepsilon \to +0$) (uniformly with $\in [t_0,T]$).

From this definition it can be seen that the class M_ε depends on the space U, in which the operator P_0 is defined. In our case $P_0 = J$. For the space U we take the space of vector functions $y\,(x,t,\tau)$, represented by sums

$$\begin{array}{l} y(x,t,\tau,\sigma) = \sum\limits_{i=1}^{3} y_i(x,t,\sigma)e^{\tau_i} + \sum\limits_{2\leq|m|\leq N_y}^{*} y^m(x,t,\sigma)e^{(m,\tau)} + \\ +y_0(x,t,\sigma) + \sum\limits_{1\leq|m|\leq N_y}^{*} y^{e_1+m}(x,t,\sigma)e^{(e_1+m,\tau)}, \quad y_i(x,t,\sigma), \\ y^m(x,t,\sigma), y^{e_1+m}(x,t,\sigma) \in C^\infty\left([x_0,X]\times[0,T]\right), \\ 1 \leq |m| \equiv m_2+m_3 \leq N_y,\, i=\overline{0,3},\ \ m=(0,m_2,m_3). \end{array} \tag{4}$$

where is denoted: $(m,\lambda(x)) \equiv m_2\lambda_2(x) + m_3\lambda_3(x)$, $(e_1+m,\lambda(x)) \equiv \lambda_1(x) + m_2\lambda_2(x) + m_3\lambda_3(x)$; an asterisk $*$ above the sum sign indicates that the summation for $|m| \geq 1$ it occurs only over multi-indices $m = (0,m_2,m_3)$ with $m_2 \neq m_3$, $e_1 = (1,0,0)$, $\sigma = (\sigma_1,\sigma_2)$.

Note that here the degree N_y of the polynomial $y\,(x,t,\tau)$, relative to the exponentials e^{τ_j} depends on the element y. In addition, the elements of space U depend on bounded in $\varepsilon > 0$ terms of constants $\sigma_1 = \sigma_1\,(\varepsilon)$ and $\sigma_2 = \sigma_2\,(\varepsilon)$ and which do not affect the development of the algorithm described below, therefore, in the record of element (4) of this space U, we omit the dependence on $\sigma = (\sigma_1,\sigma_2)$ for brevity. We show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant with respect to the operator J.

The image of the integral operator J on an arbitrary element $y\,(x,t,\tau)$, of the space U has the form

$$Jy(x,t,\tau) = \int_{x_0}^{x} K(x,t,s)y_0(s,t)ds + \sum_{i=1}^{3}\int_{x_0}^{x} K(x,t,s)y_i(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds +$$

$$+\sum_{2\leq|m|\leq N_y}^{*}\int_{x_0}^{x} K(x,t,s)y^m(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(m,\lambda(\theta))d\theta}ds +$$

$$+\sum_{1\leq|m|\leq N_y}^{*}\int_{x_0}^{x}K(x,t,s)y^{e_1+m}(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds.$$

Apply the operation of integration by parts to the first term.

$$J_i(x,t,\varepsilon)=\int_{x_0}^{x}K(x,t,s)y_i(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds=\varepsilon\int_{x_0}^{x}\frac{K(x,t,s)y_i(s,t)}{\lambda_i(s)}de^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}=$$

$$=\varepsilon\left[\frac{K(x,t,s)y_i(s,t)}{\lambda_i(s)}e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}\Bigg|_{s=x_0}^{s=x}-\int_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y_i(s,t)}{\lambda_i(\theta)}\right)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds\right]=$$

$$=\varepsilon\left[\frac{K(x,t,x)y_i(x,t)}{\lambda_i(x)}e^{\frac{1}{\varepsilon}\int_{x_0}^{x}\lambda_i(\theta)d\theta}-\frac{K(x,t,x_0)y_i(x_0,t)}{\lambda_i(x_0)}\right]-$$

$$-\varepsilon\int_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y_i(s,t)}{\lambda_i(s)}\right)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds\,.$$

Continuing this process, we obtain the series

$$J_i(x,t,\varepsilon)=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_i^{\nu}\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x}e^{\frac{1}{\varepsilon}\int_{x_0}^{x}\lambda_i(\theta))\,d\theta}-\right.$$

$$\left.-\left(I_i^{\nu}\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x_0}\right],$$

where $I_i^0=\frac{1}{\lambda_i(s)}\cdot, I_i^{\nu}=\frac{1}{\lambda_i(s)}I_i^{\nu-1}(\nu\geq 1,\, i=\overline{1,3})$.

Applying the integration operation in parts to integrals

$$J_m(x,t,\varepsilon)=\int_{x_0}^{x}K(x,t,s)y^m(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(m,\lambda(\theta))d\theta}ds,$$

$$J_{e_1+m}(x,t,\varepsilon)=\int_{x_0}^{x}K(x,t,s)y^{e_1+m}(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds,$$

we note that for all multi-indices $m=(0,m_2,m_3)$, $m_2\neq m_3$, inequalities

$$(m,\lambda(x))\equiv m_2\lambda_2(x)+m_3\lambda_3(x)\neq 0\ \ \forall x\in[x_0,X],\ m_2+m_3\geq 2$$

are satisfied. In addition, for the same multi-indices we have

$$(e_1+m,\lambda(x))\neq 0\forall x\in[x_0,X],\ m_2\neq m_3, |m|=m_2+m_3\geq 1.$$

Indeed, if $(e_1+m,\lambda(x))=0$ for some $x\in[x_0,X]$ and $m_2\neq m_3,\ m_2+m_3\geq 1$, then $m_2\lambda_2(x)+m_3\lambda_3(x)=-\lambda_1(x)),\, m_2+m_3\geq 1$, which contradicts condition (iv). Therefore, integration by parts in integrals $J_m\left(t,\varepsilon\right), J_{e_1+m}\left(t,\varepsilon\right)$ is possible. Performing it, we will have:

$$J_m(x,t,\varepsilon)=\int\limits_{t_0}^{x}K(x,t,s)y^m(s,t)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}ds=\varepsilon\int\limits_{x_0}^{x}\frac{K(x,t,s)y^m(s,t)}{(m,\lambda(s))}de^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}=$$

$$=\varepsilon\left[\frac{K(x,t,x)y^m(x,t)}{(m,\lambda(x))}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(m,\lambda(\theta))d\theta}-\frac{K(x,t,x_0)y^m(x_0,t)}{(m,\lambda(x_0))}\right]-$$

$$-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y^m(s,t)}{(m,\lambda(s))}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}ds=$$

$$=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_m^{\nu}\left(K(x,t,s)y^m(s,t)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(m,\lambda(\theta))d\theta}-\right.$$

$$\left.-\left(I_m^{\nu}\left(K(x,t,s)y^m(s,t)\right)\right)_{s=t_0}\right],$$

where $I_m^0=\frac{1}{(m,\lambda(s))}\cdot,\ \ I_m^{\nu}=\frac{1}{(m,\lambda(s))}\frac{\partial}{\partial s}I_m^{\nu-1}(\nu\geq 1,|m|\geq 2)$,

$$J_{e_1+m}(x,t,\varepsilon)=\int\limits_{x_0}^{x}K(x,t,s)y^{e_1+m}(s,t)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds=$$

$$=\varepsilon\int\limits_{x_0}^{s}\frac{K(x,t,s)y^{e_1+m}(s,t)}{(e_1+m,\lambda(s))}de^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}=$$

$$=\varepsilon\left[\frac{K(x,t,x)y^{e_1+m}\left(x,t\right)}{(e_1+m,\lambda(x))}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(e_1+m,\lambda(\theta))d\theta}-\frac{K(x,t,x_0)y^{e_1+m}\left(x_0,t\right)}{(e_1+m,\lambda(x_0))}\right]-$$

$$-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K\left(t,s\right)y^{e_1+m}(s,t)}{\left(e_1+m,\lambda\left(s\right)\right)}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds=$$

$$=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_{e_1+m}^{\nu}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(e_1+m,\lambda(\theta))d\theta}-\right.$$

$$\left.-\left(I_{e_1+m}^{\nu}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=t_0}\right],$$

where $I_{e_1+m}^0=\frac{1}{(e_1+m,\lambda(s))}\cdot,\ \ I_{e_1+m}^{\nu}=\frac{1}{(e_1+m,\lambda(s))}\frac{\partial}{\partial s}I_{e_1+m}^{\nu-1}(\nu\geq 1,\ |m|\geq 1,$

Therefore, the image of the operator J on the element (5) of the space U is represented as a series

$$Jy(x,t,\tau)=\int\limits_{x_0}^{x}K(x,t,s)y_0(s,t)ds+$$

$$+\sum_{i=1}^{3}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_i^{\nu}\left(K(x,t,s)y_i(s,t)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_i(\theta))\,d\theta}-\right.$$

$$\left.-\left(I_i^{\nu}\left(K(x,t,s)y_i(s,t)\right)\right)_{s=t_0}\right]+$$

$$+\sum_{2\le|m|\le N_Y}^{*}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_m^{\nu}\left(K(x,t,s)y^m(s,t)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(m,\lambda(\theta))\,d\theta}-\right.$$

$$\left.-\left(I_m^{\nu}\left(K(x,t,s)y^m(s,t)\right)\right)_{s=t_0}\right]+$$

$$+\sum_{1\le|m|\le N_Y}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[\left(I_{e_1+m}^{\nu}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=t}\times$$

$$\left.\times e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(e_1+m,\lambda(\theta))d\theta}-\left(I_{e_1+m}^{\nu}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=t_0}\right].$$

It is easy to show (see, for example, [12], pp. 291-294) that this series converges asymptotically for $\varepsilon\to+0$ (uniformly in $(x,t)\in[x_0,X]\times[0,T]$). This means that the class M_ε is asymptotically invariant (for $\varepsilon\to+0$) with respect to the operator J.

We introduce operators $R_\nu:\ U\to U$, acting on each element $y(x,t,\tau)\in U$ of the form (5) according to the law:

$$R_0y(x,t,\tau)=\int\limits_{x_0}^{x}K(x,t,s)y_0(s,t)ds,\tag{6_0}$$

$$R_1y(x,t,\tau)=\sum_{i=1}^{3}\left[\left(I_i^{0}\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x}e^{\tau_i}-(\left(I_i^{0}\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x_0}\right]+$$

$$+\sum_{1\le|m|\le N_y}^{*}[\left(I_m^{0}\left(K(x,t,s)y^m(s,t)\right)\right)_{s=x}e^{(m,\tau)}-\left(I_m^{0}\left(K(x,t,s)y^m(s,t)\right)\right)_{s=x_0}]+$$

$$+\sum_{1\le|m|\le N_y}^{*}\left[\left(I_{e_1+m}^{0}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x}e^{(e_1+m,\tau)}-\right.\tag{6_1}$$

$$\left.-\left(I_{e_1+m}^{0}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x_0}\right],$$

Now let $\tilde{y}(x,t,\tau,\varepsilon)$ be an arbitrary continuous function on $(x,t,\tau)\in[x_0,X]\times[0,T]\times\{\tau: Re\,\tau_j, j=\overline{1,3}\}$, with asymptotic expansion

$$\tilde{y}(x,t,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k y_k(x,t,\tau),\quad y_k(x,t,\tau)\in U, \tag{7}$$

converging as $\varepsilon\to+0$ (uniformly in $(x,t,\tau)\in[x_0,X]\times[0,T]\times\{\tau: Re\,\tau_j, j=\overline{1,3}\}$). Then the image $J\tilde{y}\,(x,t,\tau,\varepsilon)$ of this function is decomposed into an asymptotic series

$$J\tilde{y}(x,t,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k Jy_k(x,t,\tau)=\sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}y_s(x,t,\tau)|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing an extension of an operator J on series of the form (7):

$$\tilde{J}\tilde{y}\equiv\tilde{J}\left(\sum_{k=0}^{\infty}\varepsilon^k y_k(x,t,\tau)\right)=\sum_{r=0}^{\infty}\varepsilon^r\left(\sum_{k=0}^{r}R_{r-k}y_k(x,t,\tau)\right).$$

Although the operator $\tilde{J}$ is formally defined, its utility is obvious, since in practice it is usual to construct the N-th approximation of the asymptotic solution of the problem (2), in which impose only N-th partial sums of the series (7), which have not a formal, but a true meaning. Now you can write a problem that is completely regularized with respect to the original problem (2):

$$\begin{gathered}L_\varepsilon\tilde{y}(x,t,\tau,\varepsilon)\equiv\varepsilon\tfrac{\partial\tilde{y}}{\partial x}+\sum_{j=1}^{3}\lambda_j(x)\tfrac{\partial\tilde{y}}{\partial\tau_j}-a(x)\tilde{y}-\tilde{J}\tilde{y}-\varepsilon\tfrac{g(x)}{2}(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2)\tilde{y}=\\=h(x,t),\quad\tilde{y}(x_0,t,0,\varepsilon)=y^0,\ \ ((x,t)\in[x_0,X]\times[0,T]).\end{gathered}\tag{8}$$

3. Solvability of iterative problems

Substituting the series (7) into (8) and equating the coefficients of the same powers of ε, we obtain the following iterative problems:

$$Ly_0\equiv\sum_{j=1}^{3}\lambda_j(x)\tfrac{\partial y_0}{\partial\tau_j}-a(x)y_0-R_0y_0=h(x,t),\ y_0(x_0,t,0)=y^0; \tag{9_0}$$

$$Ly_1=-\frac{\partial y_0}{\partial x}+\frac{g(x)}{2}(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2)y_0+R_1y_0,\quad y_1(x_0,t,0)=0; \tag{9_1}$$

$$Ly_2=-\frac{\partial y_1}{\partial x}+\frac{g(x)}{2}(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2)y_1+R_1y_1+R_2y_0,\ y_2(x_0,t,0)=0; \tag{9_2}$$

....

$$Ly_k=-\tfrac{\partial y_{k-1}}{\partial x}+\tfrac{g(x)}{2}(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2)y_{k-1}+R_ky_0+R_1y_{k-1},y_k(x_0,t,0)=0,\ k\geq1. \tag{9_k}$$

Each iterative problem (9_k) has the form

$$Ly \equiv \sum_{j=1}^{3} \lambda_j(x) \frac{\partial y}{\partial \tau_j} - a(x)y - R_0 y = H(x,t,\tau), \quad y(x_0,t,0) = y_*, \tag{10}$$

where $H(x,t,\tau) \in U$, is the known vector function of space U, y_* is the known constant vector of the complex space C, and the operator R_0 has the form (see (6_0))

$$R_0 y(x,t,\tau) \equiv R_0 \left[y_0(x,t) + \sum_{i=1}^{3} y_i(x,t) e^{\tau_i} + \sum_{2\leq|m|\leq N_y}^{*} y^m(x,t) e^{(m,\tau)} + \right.$$

$$\left. + \sum_{1\leq|m|\leq N_y}^{*} y^{e_1+m}(x,t) e^{(e_1+m,\tau)} \right] \triangleq \int_{x_0}^{x} K(x,t,s) y_0(s,t) ds.$$

We introduce scalar (for each $x \in [x_0, X]$) product in space U:

$$< u, w > \equiv < u_0(x,t) + \sum_{i=1}^{3} u_i(x,t) e^{\tau_i} + \sum_{2\leq|m|\leq N_y}^{*} u^m(x,t) e^{(m,\tau)} +$$

$$+ \sum_{1\leq|m|\leq N_y}^{*} u^{e_1+m}(x,t) e^{(e_1+m,\tau)}, w_0(x,t) + \sum_{i=1}^{3} w_i(x,t) e^{\tau_i} +$$

$$+ \sum_{2\leq|m|\leq N_w}^{*} w^m(x,t) e^{(m,\tau)} + \sum_{1\leq|m|\leq N_w}^{*} w^{e_1+m}(x,t) e^{(e_1+m,\tau)} > \triangleq$$

$$\triangleq (u_0(x,t), w_0(x,t)) + \sum_{i=1}^{3} (u_i(x,t), w_i(x,t)) + \sum_{2\leq|m|\leq min(N_y,N_w)}^{*} (u^m(x,t), w^m(x,t)) +$$

$$+ \sum_{1\leq|m|\leq min(N_y,N_w)}^{*} \left(u^{e_1+m}(x,t),\, w^{e_1+m}(x,t) \right),$$

where we denote by $(*, *)$ the usual scalar product in the complex space C. Let us prove the following statement.

Theorem 1. Let conditions (i)-(ii), (iv) be fulfilled and the right-hand side $H(x,t,\tau)$ of system (10) belongs to the space U. Then the system (10) is solvable in U, if and only if

$$H_1(x,t,\tau) \equiv 0, \, \forall x \in [x_0, X]\,. \tag{11}$$

Proof. We will determine the solution of system (10) as an element (5) of the space $U:$

$$y(x,t,\tau) = y_0(x,t) + \sum_{i=1}^{3} y_i(x,t) e^{\tau_i} + \sum_{2\leq|m|\leq N_y}^{*} y^m(x,t) e^{(m,\tau)} +$$

$$+\sum_{1\leq|m|\leq N_y}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)} \equiv y_0(x,t)+\sum_{i=1}^{3} y_i(x,t)e^{\tau_i}+ \tag{12}$$

$$+\sum_{2\leq|m|\leq N_y}^{*} y^{m}(x,t)e^{(m,\tau)}+\sum_{2\leq|m^1|\leq N_y}^{*} y^{m^1}(x,t)e^{(m^k,\tau)},$$

where for convenience introduced multi-indices $m^1=e_1+m\equiv(1,m_2,m_3)$, m_2 and m_3 are non-negative integer numbers. Substituting (12) into system (10), we will have

$$\sum_{i=1}^{3}\left[\lambda_i(x)-a(x)\right]y_i(x,t)e^{\tau_i}+\sum_{2\leq|m|\leq N_y}^{*}\left[(m,\lambda(x))-a(x)\right]y^m(x,t)e^{(m,\tau)}+$$

$$+\sum_{2\leq|m^1|\leq N_y}^{*}\left[\left(m^1,\lambda(x)\right)-a(x)\right]y^{m^1}(x,t)e^{(m^1,\tau)}-$$

$$-a(x)y_0(x,t)-\int\limits_{x_0}^{x}K(x,t,s)y_0(s,t)ds=H_0(x,t)+$$

$$+\sum_{i=1}^{3}H_i(x,t)e^{\tau_i}+\sum_{2\leq|m|\leq N_y}^{*}H^m(x,t)e^{(m,\tau)}+\sum_{2\leq|m^1|\leq N_y}^{*}H^{m^1}(x,t)e^{(m^1,\tau)}.$$

Equating here the free terms and coefficients separately for identical exponents, we obtain the following systems of equations:

$$-a(x)y_0(x,t)-\int\limits_{x_0}^{x}K(x,t,s)y_0(s,t)ds=H_0(x,t), \tag{13}$$

$$\left[\lambda_i(x)-a(x)\right]y_i(x,t)=H_i(x,t),\, i=\overline{1,4}, \tag{13_i}$$

$$\left[(m,\lambda(x))-a(x)\right]y^m(x,t)=H^m(x,t),\, m_2\neq m_3,\, 2\leq|m|\leq N_y, \tag{13_m}$$

$$\left[\left(m^1,\lambda(x)\right)-a(x)\right]z^{m^1}(x,t)=H^{m^1}(x,t), m_2\neq m_3, 2\leq\left|m^1\right|\leq N_y. \tag{14}$$

The equation (13) can be written as

$$y_0(x,t)=\int\limits_{x_0}^{x}\left(-a^{-1}(x)K(x,t,s)\right)y_0(s,t)ds-a^{-1}(x)H_0(x,t). \tag{13_0}$$

Due to the smoothness of the kernel $-a^{-1}(x)K(x,t,s)$ and heterogeneity $-a^{-1}(x)H_0(x,t)$, this Volterra integral equation has a unique solution $z_0(x,t)\in C^\infty\left([x_0,X]\times[0,T]\right)$. The equations ($13_2$) and ($13_3$) also have unique solutions

$$z_i(x,t)=\left[\lambda_1(x)-a(x)\right]^{-1}H_i(x,t)\in C^\infty\left([x_0,X]\times[0,T]\right), i=2,3.$$

Equation (13_1) are solvable in space $C^\infty([x_0, X]\times[0,T])$ if and only if there are identities

$$H_1(x,t)\equiv 0 \quad \forall x\in[x_0,X],$$

It is not difficult to see that these identities coincide with identities (11).

Further, since $(m,\lambda(x))\equiv m_2\lambda_2(x)+m_3\lambda_3(x)\neq\lambda_1(x)$, $|m|=m_2+m_3\geq 2$ (see condition (iv)) the absence of resonance), the equation system (13_m) has a unique solution

$$z^m(x,t)=\left[(m,\lambda(x))-a(x)\right]^{-1}H^m(x,t),\, 2\leq|m|\leq N_y\in C^\infty([x_0,X]\times[0,T]).$$

We now consider equation (14). Let $\left(m^1,\lambda(x)\right)=\lambda_1(x)$, $\left|m^1\right|\geq 2$. Then

$$\lambda_1(x)+m_2\lambda_2(x)+m_3\lambda_3(x)=\lambda_1(x)\Leftrightarrow$$

$$\Leftrightarrow\; m_2\lambda_2(x)+m_3\lambda_3(x)=0\Leftrightarrow\; m_2\neq m_3,\, m_2+m_3\geq 1,$$

which cannot be (see definition of class U). Unique solution of equation (18) for $\left|m^1\right|\geq 2$ in the class $C^\infty([x_0,X]\times[0,T])$:

$$z^{m^1}(x,t)=\left[\left(m^1,\lambda(x)\right)-a(x)\right]^{-1}H^{m^1}(x,t),\, 2\leq\left|m^1\right|\leq N_y.$$

Thus, condition (11) is necessary and sufficient for the solvability of equation (10) in the space U. The theorem is proved.

Remark. If identity (11) holds, then under conditions (i)-(ii) and (iv), equation (10) has the following solution in the space U :

$$y(x,t,\tau)=y_0(x,t)+\alpha_1(x,t)e^{\tau_1}+\sum_{i=2}^{3}\left[\lambda_i(x)-a(x)\right]^{-1}H_i(x,t)e^{\tau_i}+$$
$$+\sum_{2\leq|m|\leq N_y}^{*}\left[(m,\lambda(x))-a(x)\right]^{-1}H^m(x,t)e^{(m,\tau)}+ \qquad (14)$$
$$+\sum_{1\leq|m|\leq N_y}^{*}\left[(e_1+m,\lambda(x))-a(x)\right]^{-1}H^{e_1+m}(x,t)e^{(e_1+m,\tau)},$$

where $\alpha_1(x,t)\in C^\infty([x_0,X]\times[0,T])$ are arbitrary function, $y_0(x,t)$ is the solution of an integral equation (13_0), $m\equiv(0,m_2,m_3)\,, m_2\neq m_3$, $|m|=m_2+m_3\geq 1$.

4. The unique solvability of the general iterative problem in the space U. Residual term theorem

Let us proceed to the description of the conditions for the unique solvability of equation (10) in space U. Along with problem (10), we consider the equatiom

$$Ly(x.t,\tau)=-\frac{\partial y}{\partial x}+\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)y+Q(x,t,\tau), \qquad (15)$$

where $y = y(x,t,\tau)$ is the solution (14) of the equation (10), $Q(x,t,\tau) \in U$ is the well-known function of the space U. The right part of this equation:

$$G(x,t,\tau) \equiv -\frac{\partial y}{\partial x} + \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right) y + Q(x,t,\tau) =$$

$$= -\frac{\partial}{\partial x}\left[y_0(x,t) + \sum_{i=1}^{3} y_i(x,t)e^{\tau_i} + \sum_{2\le|m|\le N_y}^{*} y^m(x,t)e^{(m,\tau)} + \right.$$

$$\left. + \sum_{1\le|m|\le N_y}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)} \right] +$$

$$+\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[y_0(x,t) + \sum_{i=1}^{3} y_i(x,t)e^{\tau_i} + \sum_{2\le|m|\le N_y}^{*} y^m(x,t)e^{(m,\tau)} + \right.$$

$$\left. + \sum_{1\le|m|\le N_y}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)} \right] + Q(x,t,\tau),$$

may not belong to space U, if $y = y(x,t,\tau) \in U$. Indeed, taking into account the form (14) of the function $y = y(x,t,\tau) \in U$, we will have

$$Z(x,t,\tau) \equiv G(x,t,\tau) + \frac{\partial y}{\partial x} - \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[y_0(x,t) + \sum_{i=1}^{3} y_i(x,t)e^{\tau_i} + \right.$$

$$\left. + \sum_{2\le|m|\le N_y}^{*} y^m(x,t)e^{(m,\tau)} + \sum_{1\le|m|\le N_y}^{*} z^{e_1+m}(x,t)e^{(e_1+m,\tau)} \right] =$$

$$= \frac{g(x)}{2} y_0(x,t)\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right) + \sum_{i=2}^{3} \frac{g(x)}{2} y_i(x,t)\left(e^{\tau_i+\tau_2}\sigma_1 + e^{\tau_i+\tau_3}\sigma_2\right) +$$

$$+\frac{g(x)}{2} y_1(x,t)\left(e^{\tau_1+\tau_2}\sigma_1 + e^{\tau_1+\tau_3}\sigma_2\right) + \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[\sum_{2\le|m|\le N_y}^{*} y^m(x,t)e^{(m,\tau)} + \right.$$

$$\left. + \sum_{1\le|m|\le N_y}^{*} z^{e_1+m}(x,t)e^{(e_1+m,\tau)} \right] + Q(x,t,\tau).$$

Here are terms with exponents

$$e^{\tau_3+\tau_2} = e^{(m,\tau)}|_{m=(0,1,1)},$$

$$e^{\tau_2+(m,\tau)}\ (\text{if } m_2+1=m_3)\,,\ e^{\tau_3+(m,\tau)}\ (\text{if } m_3+1=m_2)\,, \qquad (*)$$

$$e^{\tau_2+(e_1+m,\tau)} \text{ (if } m_2+1=m_3) \, m_3+1=m_2,$$

do not belong to space U, since in multi-index $m=(0,m_2,m_3)$ of the space U must be $m_2 \neq m_3$, $m_2+m_3 \geq 1$. Then, according to the well-known theory (see, [1] , p. 234), we embed these terms in the space U according to the following rule (see $(*)$):

$$\widehat{e^{\tau_2+\tau_3}} = e^0 = 1, \; \widehat{e^{\tau_2+(m,\tau)}} = e^0 = 1 \left(m_2+1=m_3, m_2 \neq m_3\right),$$

$$\widehat{e^{\tau_3+(m,\tau)}} = e^0 = 1 \left(m_3+1=m_2, m_2 \neq m_3\right),$$

$$\widehat{e^{\tau_2+(e_1+m,\tau)}} = e^{\tau_1} \left(m_2+1=m_3, m_2 \neq m_3\right). \qquad (**)$$

In $Z(x,t,\tau)$ need of embedding only the terms

$$M(x,t,\tau) \equiv \sum_{i=2}^{3} \frac{g(x)}{2} y_i(x,t) \left(e^{\tau_i+\tau_2}\sigma_1 + e^{\tau_i+\tau_3}\sigma_2\right) + \frac{g(x)}{2} y_1(x,t) \left(e^{\tau_1+\tau_2}\sigma_1 + e^{\tau_1+\tau_3}\sigma_2\right),$$

$$S(x,t,\tau) \equiv \frac{g(x)}{2} \left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right) [\sum_{2\leq|m|\leq N_y}^{*} y^m(x,t) e^{(m,\tau)} + \sum_{1\leq|m|\leq N_y}^{*} y^{e_1+m}(x,t) e^{(e_1+m,\tau)}].$$

We describe this embedding in more detail, taking into account formulas $(**)$:

$$M(x,t,\tau) \equiv \frac{g(x)}{2} y_1(x,t) \left(e^{\tau_1+\tau_2}\sigma_1 + e^{\tau_1+\tau_3}\sigma_2\right) + \sum_{i=2}^{3} \frac{g(x)}{2} y_i(x,t) \left(e^{\tau_i+\tau_2}\sigma_1 + e^{\tau_i+\tau_3}\sigma_2\right) =$$

$$= \frac{g(x)}{2} \left[y_1(x,t) e^{\tau_1+\tau_2}\sigma_1 + y_1(x,t) e^{\tau_1+\tau_3}\sigma_2 + y_2(x,t) e^{2\tau_2}\sigma_1 + y_2(x,t)\sigma_2 + \right.$$
$$\left. + y_3(x,t)\sigma_1 + y_3(x,t) e^{2\tau_3}\sigma_2 \right] \quad \Rightarrow$$

$$\Rightarrow \quad \widehat{M}(x,t,\tau) = \frac{g(x)}{2} \left[y_1(x,t) e^{\tau_1+\tau_2}\sigma_1 + y_1(x,t) e^{\tau_1+\tau_3}\sigma_2 + y_2(x,t) e^{2\tau_2}\sigma_1 + \right.$$
$$\left. + y_2(x,t)\sigma_2 + y_3(x,t)\sigma_1 + y_3(x,t) e^{2\tau_3}\sigma_2 \right],$$

(note that in $\widehat{M}(x,t,\tau)$ there are no members containing e^{τ_1}, measurement exponents $|m|=1)$:

$$S(x,t,\tau) \equiv \frac{g(x)}{2} \left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right) \left[\sum_{2\leq|m|\leq N_y}^{*} y^m(x,t) e^{(m,\tau)} + \sum_{1\leq|m|\leq N_y}^{*} y^{e_1+m}(x,t) e^{(e_1+m,\tau)} \right] =$$

$$= \frac{g(x)}{2} \left[\sum_{2\leq|m|\leq N_y}^{*} y^m(x,t) \left(e^{\tau_2+(m,\tau)}\sigma_1 + e^{\tau_3+(m,\tau)}\sigma_2\right) + \right.$$

$$\left. + \sum_{1\leq|m|\leq N_y}^{*} y^{e_1+m}(x,t) \left(e^{(e_1+m,\tau)+\tau_2}\sigma_1 + e^{(e_1+m,\tau)+\tau_3}\sigma_2\right) \right. \quad \Rightarrow$$

$$\Rightarrow \widehat{S}(x,t,\tau) = \frac{g(x)}{2}\Big[\sum_{\substack{2 \le |m| \le N_y,\\ m_2+1=m_3}} y^m(x,t)\sigma_1 + \sum_{\substack{2 \le |m| \le N_y,\\ m_3+1=m_2}} y^m(x,t)\sigma_2 +$$

$$+ \sum^{*}_{\substack{2 \le |m| \le N_y,\\ m_2+1\neq m_3, m_3+1\neq m_2}} y^m(x,t)e^{(m,\tau)} +$$

$$+ \left[\sum_{\substack{1 \le |m| \le N_y,\\ m_2+1=m_3}} y^{e_1+m}(x,t)\sigma_1 + \sum_{\substack{1 \le |m| \le N_y,\\ m_3+1=m_2}} y^{e_1+m}(x,t)\sigma_2 \right] e^{\tau_1} +$$

$$+ \sum^{*}_{\substack{1 \le |m| \le N_y,\\ m_2+1\neq m_3, m_3+1\neq m_2}} y^{e_1+m}(x,t)e^{(e_1+m,\tau)},$$

After embedding, the right-hand side of system (15) will look like

$$\widehat{G}(x,t,\tau) = -\frac{\partial}{\partial x}\left[y_0(x,t) + \sum_{i=1}^{3} y_i(x,t)e^{\tau_i} + \sum^{*}_{2\le|m|\le N_y} y^m(x,t)e^{(m,\tau)} \right] -$$

$$-\frac{\partial}{\partial x}\left[\sum^{*}_{1\le|m|\le N_y} y^{e_1+m}(x,t)e^{(e_1+m,\tau)} \right] + \widehat{M}(x,t,\tau) + \widehat{S}(x,t,\tau) + Q(x,t,\tau),$$

moreover, in $\widehat{S}(x,t,\tau)$ the coefficients at e^{τ_1} do not depend on $z_1(x,t)$. As indicated in [1], the embedding $G(x,t,\tau) \to \widehat{G}(x,t,\tau)$ will not affect the accuracy of the construction of asymptotic solutions of problem (2), since $G(x,t,\tau) \to \widehat{G}(x,t,\tau)$.

Theorem 2. Let conditions (i)-(ii), (iv) be fulfilled and the right-hand side $H(x,t,\tau) \in U$ of equation (10) satisfy condition (11). Then problem (10) under additional conditions

$$\widehat{G}(x,t,\tau) \equiv 0 \,\forall t \in [x_0, X]\,, \tag{16}$$

where $Q(x,t,\tau)$ is the known vector function of space U, is uniquely solvable in U.

Proof. Since the right-hand side of equation (10) satisfies condition (11), this equation has a solution in space U in the form (14), where $\alpha_1(x,t) \in C^\infty([x_0,X]\times[0,T])$ are arbitrary function so far. Submit (14) to the initial condition $y(x_0,t,0) = y^*$. We get $\alpha_1(x_0,t) = y_*$, where denoted

$$y_* = y^* + a^{-1}(x_0)H_0(x_0,t) - \sum_{i=2}^{3}\left[\lambda_i(x_0) - a(x_0)\right]^{-1} H_i(x_0,t) -$$

$$-\sum_{2\leq|m|\leq N_y}^{*}\left[(m,\lambda(x_0))-a(x_0)\right]^{-1}H^m(x_0,t)-$$

$$-\sum_{1\leq|m^k|\leq N_y}^{*}\left[\left(m^k,\lambda(x_0)\right)-a(x_0)\right]^{-1}H^{m^k}(x_0,t).$$

where do we find the values $\alpha_1(x_0,t)=y_*$. Then condition (16) takes the form

$$-\frac{\partial}{\partial x}\alpha_1(x,t)e^{\tau_1}+$$

$$+\left[\sum_{\substack{1\leq|m|\leq N_y,\\ m_2+1=m_3}} y^{e_1+m}(x,t)\sigma_1+\sum_{\substack{1\leq|m|\leq N_y,\\ m_3+1=m_2}} y^{e_1+m}(x,t)\sigma_2\right]e^{\tau_1}+$$

$$+Q_1(x,t)e^{\tau_1}\equiv 0\quad \forall(x,t)\in[x_0,X]\times[0,T],\,.$$

We obtain linear ordinary differential equations with respect to the function $\alpha_1(x,t)$, involved in the solution (14) of equation (10). Attaching to them the initial conditions $\alpha_1(t_0)=y_*$ computed earlier, we find uniquely the function $\alpha_1(x_0,t)=y_*$ and, therefore, we construct solution (14) in the space in a unique way. The theorem 2 is proved.

Applying Theorems 1 and 2 to iterative problems (9_k) (in this case, the right-hand sides $H^{(k)}(x,t,\tau)$ of these problems are embedded in the space U, i.e. $H^{(k)}(x,t,\tau)$ we replace with $\hat{H}^{(k)}(x,t,\tau)\in U$), we find uniquely their solutions in space U and construct series (7). Justasin [1], we prove the following statement.

Theorem 3. Suppose that conditions (i)-(ii), (iv) are satisfied for problem (2). Then, when $\varepsilon\in(0,\varepsilon_0](\varepsilon_0>0$ is sufficiently small), problem (2) has a unique solution $y(x,t,\varepsilon)\in C^1([x_0,X]\times[0,T])$, in this case, the estimate

$$||y(x,t,\varepsilon)-y_{\varepsilon N}(x,t)||_{C[x_0,X]\times[0,T]}\leq c_N\varepsilon^{N+1},$$

holds true, where $z_{\varepsilon N}(x,t)$ is the restriction (for $\tau=\frac{\psi(t)}{\varepsilon}$) of the N - partial sum of series (7) (with coefficients $y_k(x,t,\tau)\in U$, satisfying the iteration problems (9_k)), and the constant $c_N>0$ does not depend on $\varepsilon\in(0,\varepsilon_0]$.

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