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Applications of Lacunary Sequences to develop Fuzzy Sequence Spaces for Ideal Convergence and Orlicz Function

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Abstract. In the present paper, we introduce and study ideal convergence of some fuzzy sequence spaces via lacunary sequence, infinite matrix and Orlicz function. We study some topological and algebraic properties of these spaces. We also make an effort to show that these spaces are normal as well as monotone. Further, it is very interesting to show that if I is not maximal ideal then these spaces are not symmetric.

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1. Introduction and preliminaries

The concept of ordinary convergence of a sequence of fuzzy numbers was introduced by Matloka [18] and proved some basic theorems for sequences of fuzzy numbers. Later on Nanda [28] introduced sequences of fuzzy numbers and studied that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Recently, Nuray

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and Savaş [30] studied statistical convergence and statistically Cauchy for sequence of fuzzy numbers. They proved that a sequence of fuzzy numbers is statistically convergent if and only if it is statistically Cauchy. Initially the idea of I-convergence was introduced by Kostyrko et al. [15]. A lot of developments have been made in this area, one may refer to the articles (see [1–4, 10, 11, 16, 19, 23, 37]).

Let X be a non empty set. Then a family of sets $I \subseteq 2^X$ (power set of X) is said to be an ideal if I is additive i.e $U_1, U_2 \in I \Rightarrow U_1 \cup U_2 \in \mathcal{I}$ and $U_1 \in I, U_2 \subseteq U_1 \Rightarrow U_2 \in I$. A non empty family of sets $G \subseteq 2^X$ is said to be filter on X if and only if $\Phi \notin G$, for $U_1, U_2 \in G$ we have $U_1 \cap U_2 \in G$ and for each $U_1 \in G$ and $U_1 \subseteq U_2$ implies $U_2 \in G$. An ideal $I \subseteq 2^X$ is called non trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

A fuzzy number u is a fuzzy set [42] on the real axis, i.e., a mapping $u : \mathbb{R} \to [0, 1]$ which satisfies the following conditions:

(i) u is normal, i.e., there exist an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, i.e., for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1, u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$;

(iii) u is upper semi-continuous;

(iv) the closure of the set $\mathrm{supp}(u)$ is compact, where $\mathrm{supp}(u) = \{x \in \mathbb{R} : u(x) > 0\}$ and it is denoted by $[u]^0$.

Let $L(\mathbb{R})$ denotes the set of all fuzzy numbers. The α-level set of a fuzzy real number u, for $0 < \alpha \leq 1$ denoted by u^α is defined as $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$, for $\alpha = 0$ it is the closure of the strong 0 cut (i.e. closure of the set $\{t \in \mathbb{R} : u(t) > 0\}$).
For each $r \in \mathbb{R}$, $\overline{r} \in L(\mathbb{R})$ is defined by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r; \\ 0, & \text{if } t \neq 0. \end{cases}$$

Define a map $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$$\overline{d}(x, y) = \sup_{\alpha \in [0,1]} \{\max\{|u_1^\alpha - v_1^\alpha|, |u_2^\alpha - v_2^\alpha|\}\},$$

where $u^\alpha = [u_1^\alpha, u_2^\alpha]$ and $v^\alpha = [v_1^\alpha, v_2^\alpha]$. In this case, $(L(\mathbb{R}), \overline{d})$ is a complete metric space. The additive identity and multiplicative identity in $L(\mathbb{R})$ are denoted by $\overline{0}$ and $\overline{1}$, respectively.

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is convex, continuous and non-decreasing function which also satisfy $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called the modulus function and characterized by Nakano [27] and followed by Ruckle [33] and others. An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists $R > 0$ such that $M(2u) \leq RM(u), u \geq 0$. Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

An increasing non-negative integer sequence $\theta = (i_r)$ with $i_0 = 0$ and $h_r = (i_r - i_{r-1}) \to \infty$ as $r \to \infty$ is known as lacunary sequence. The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be denoted by q_r. Freedman et al. [7] defined the space N_θ in the following way:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0 \ \text{ for some } \ L\Big\}.$$

Fridy and Orhan [8] defined and studied the idea of lacunary statistical for sequence of real number. Nuray [29] and Mursaleen and Mohiuddine [25] defined this notion, respectively, for sequences of fuzzy numbers and in the setting of intuitionistic fuzzy normed space. Most recently, Mohiuddine and Alamri [20] defined the notion of weighted lacunary equi-statistical convergence and, as an application, proved some approximation theorems.

In [14] Kızmaz introduced the notion of difference sequence spaces and studied $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ which has been recently used to define statistical convergence [12, 21]. Further this notion was generalized by Et and Çolak [6] by introducing the spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Later on, another type of generalization of the difference sequence spaces is due to Tripathy and Esi [39] who studied the spaces $\ell_\infty(\Delta_\nu)$, $c(\Delta_\nu)$ and $c_0(\Delta_\nu)$. Recently, Esi et al. [5] and Tripathy et al. [40] have introduced a new type of generalized difference operators and unified those as follows: Let ν, m be non-negative integers, then for Z a given sequence space, we have

$$Z(\Delta_\nu^m) = \{x = (x_k) \in w : (\Delta_\nu^m x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞ where $\Delta_\nu^m x = (\Delta_\nu^m x_k) = (\Delta_\nu^{m-1} x_k - \Delta_\nu^{m-1} x_{k+1})$ and $\Delta_\nu^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_\nu^m x_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} x_{k+\nu i}.$$

Taking $\nu = 1$, we get the spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Çolak [6]. Taking $m = \nu = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [14]. For more details about sequence spaces (see [9, 31, 32, 34, 36, 38, 41]) and references therein.

Let λ and η be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from λ into η if for every sequence $x = (x_k)_{k=0}^{\infty} \in \lambda$, the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$, the A-transform of x, is in η, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1}$$

By (λ, η), we denote the class of all matrices A such that $A : \lambda \to \eta$. Thus, $A \in (\lambda, \eta)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$.

The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) : Ax \in \lambda\}. \tag{2}$$

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors (see [35]).

Kumar and Kumar [16] defined the notion of ideal (or, I-) convergence for sequence of fuzzy numbers and recently studied by Mursaleen and Mohiuddine [26] in probabilistic normed spaces (see also [22]).

Definition 1. *A sequence $X = (X_k)$ of fuzzy numbers is said to be I-convergent to a fuzzy number X_0, if for every $\varepsilon > 0$ such that*

$$\{k \in \mathbb{N} : \overline{d}(X_k, X_0) \geq \varepsilon\} \in I.$$

The fuzzy number X_0 is called I-limit of the sequence (X_k) of fuzzy numbers and we write $I\text{-}\lim X_k = X_0$.

Definition 2. *A sequence $X = (X_k)$ of fuzzy numbers is said to be I-bounded if there exists $M > 0$ such that*

$$\{k \in \mathbb{N} : \overline{d}(X_k, \overline{0}) > M\} \in I.$$

Definition 3. *Let $\theta = (k_r)$ be lacunary sequence. Then a sequence (X_k) of fuzzy numbers is said to be lacunary I-convergent if for every $\varepsilon > 0$ such that $\Big\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \overline{d}(X_k, X) \geq \varepsilon\Big\} \in I$. We write $I_\theta\text{-}\lim X_k = X$.*

Definition 4. *Let E_F be denote the sequence space of fuzzy numbers. Then E_F is said to be solid (or normal) if $(Y_k) \in E_F$ whenever $(X_k) \in E_F$ and $\overline{d}(Y_k, \overline{0}) \leq \overline{d}(X_k, \overline{0})$ for all $k \in \mathbb{N}$.*

Example 1. *(i) If we take $I = I_F = \{A \subseteq \mathbb{N} : A$ is a finite subset $\}$. Then I_F is a nontrivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the usual convergence.*
(ii) If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. where $\delta(A)$ denote the asymptotic density of the set A. Then I_δ is a non-trival admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the statistical convergence.

Lemma 1. *[24] If $\overline{d}$ is a translation invariant metric. Then*
(i) $\overline{d}(X+Y,\overline{0}) \leq \overline{d}(X,\overline{0}) + \overline{d}(Y,\overline{0})$,
(ii) $\overline{d}(\lambda X,\overline{0}) \leq |\lambda|\overline{d}(X,\overline{0}),\ |\lambda| > 1$.

Lemma 2. *A sequence space E_F is normal implies E_F is monotone. (For the crisp set case, one may refer to Kamthan and Gupta [13]).*

Lemma 3. *[15] If $I \subset 2^{\mathbb{N}}$ is a maximal ideal then for each $A \in \mathbb{N}$, we have either $A \in I$ or $\mathbb{N} \setminus A \in I$.*

2. Some fuzzy sequence spaces

Throughout the paper w^F denote the class of all fuzzy real-valued sequences. By $\mathbb{N}$ and $\mathbb{R}$ we denote the set of natural and real numbers respectively. Let I be an admissible ideal of $\mathbb{N}$ and $\theta = (i_r)$ be lacunary sequence. Suppose $p = (p_k)$ is a bounded sequence of positive real numbers, $u = (u_k)$ be a sequence of nonzero, nonnegative real numbers, $A = (a_{nk})$ an infinite matrix and $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions. In this paper, we define the following sequence spaces as follows:

$$w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]$$

$$= \Bigg\{(x_k) \in w^F : \forall \varepsilon > 0, \Bigg\{n,r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\Bigg\} \in I, \text{ for some } \rho > 0, s \geq 0 \text{ and } X_0 \in L(\mathbb{R})\Bigg\},$$

$$w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]_0$$

$$= \Bigg\{(x_k) \in w^F : \forall \varepsilon > 0, \Bigg\{n,r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\Bigg\} \in I, \text{ for some } \rho > 0 \text{ and } s \geq 0\Bigg\},$$

$$w_\theta^{F}[A,\mathcal{M},p,u,\Delta_v^m]_\infty$$

$$= \Bigg\{(x_k) \in w^F : \sup_{n,r}\frac{1}{h_r}\sum_{k\in I_r} a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0 \text{ and } s \geq 0\Bigg\},$$

and

$$w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]_\infty$$
$$= \Bigg\{(x_k) \in w^F : \exists\, K > 0 \text{ such that } \Bigg\{n, r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} a_{nk}\Bigg[k^{-s} M_k\Bigg(\frac{\overline{d}(u_k\Delta_v^m X_k, X_0)}{\rho}\Bigg)\Bigg]^{p_k}$$
$$\geq K\Bigg\} \in I, \text{ for some } \rho > 0 \text{ and } s \geq 0\Bigg\}.$$

Example 2. *Let* $X_k(l) = \overline{1}$ *for* $k = 2^q,\ \ q = 1,2,3.....$

$$otherwise,\quad X_k(l) = \begin{cases} \frac{k}{3}(l-2)+1 & \text{for } l \in \left[\frac{2k-3}{2}, 2\right], \\ -\frac{k}{3}(l-2)+1 & \text{for } l \in \left[2, \frac{2k+3}{2}\right]. \end{cases}$$

For instance take $m = v = 1$*, then the* $\alpha-$*level sets of* (X_k) *and* (ΔX_k) *are*

$$[X_k]^\alpha = \begin{cases} [1,1] & k = 2^q, \\ [2 - \frac{3}{k}(1-\alpha), 2 + \frac{3}{k}(1-\alpha)] & otherwise. \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} [-1 - \frac{3}{k}(1-\alpha), -1 + \frac{3}{k}(1-\alpha)] & k = 2^q, \\ [1 - \frac{3}{k}(1-\alpha), 1 + \frac{3}{k}(1-\alpha)] & k+1 = 2^q. \\ [(\frac{3}{k} + \frac{3}{k+1})(\alpha - 1), (\frac{3}{k} + \frac{3}{k+1})(1-\alpha)] & otherwise. \end{cases}$$

Let $A = (C,1)$*, the Cesàro matrix,* $\mathcal{M}(x) = x$*,* $s = 0$*,* $u = (u_k) = 1$*,* $p = (p_k) = 1$*, for all* $k \in \mathbb{N}$*,* $\rho = 1$ *and* $\theta = 2^r$*, we have*

$$\sup_n \sum_{k\in I_r} a_{nk}\Bigg[M_k\Bigg(\frac{\overline{d}(u_k\Delta_v^m X_k, \overline{0})}{\rho}\Bigg)\Bigg]^{p_k} < \infty$$

Thus, $(X_k) \in w_\theta^F[A,\mathcal{M},p,u,\Delta_v^m]_\infty$ *but* (X_k) *is not an Ideal convergent.*

Let us consider a few special cases of the above sequence spaces:

(i) If $M_k(x) = x$ for all $k \in \mathbb{N}$, then we have $w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m] = w_\theta^{I(F)}[A,p,u,\Delta_v^m]$, $w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]_0 = w_\theta^{I(F)}[A,p,u,\Delta_v^m]_0$, $w_\theta^F[A,\mathcal{M},p,u,\Delta_v^m]_\infty = w_\theta^F[A,p,u,\Delta_v^m]_\infty$ and $w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]_\infty = w_\theta^{I(F)}[A,p,u,\Delta_v^m]_\infty$.

(ii) If $p = (p_k) = 1$, for all k, then we have $w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m] = w_\theta^{I(F)}[A,\mathcal{M},u,\Delta_v^m]$, $w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]_0 = w_\theta^{I(F)}[A,\mathcal{M},u,\Delta_v^m]_0$, $w_\theta^F[A,\mathcal{M},p,u,\Delta_v^m]_\infty = w_\theta^F[A,\mathcal{M},u,\Delta_v^m]_\infty$ and $w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]_\infty = w_\theta^{I(F)}[A,\mathcal{M},u,\Delta_v^m]_\infty$.

(iii) If we take $A = (C, 1)$, i.e., the Cesàro matrix, then the above classes of sequences are denoted by $w_\theta^{I(F)}[w, \mathcal{M}, p, u, \Delta_v^m]$, $w_\theta^{I(F)}[w, \mathcal{M}, p, u, \Delta_v^m]_0$, $w_\theta^F[w, \mathcal{M}, p, u, \Delta_v^m]_\infty$ and
$w_\theta^{I(F)}[w, \mathcal{M}, p, u, \Delta_v^m]_\infty$ respectively.

(iv) If we take $A = (a_{nk})$ a de la Vallée-Poussin mean, i.e.,

$$a_{nk} = \begin{cases} \frac{1}{\lambda_n}, & \text{if } k \in I_n = [n - \lambda_n + 1, n]; \\ 0, & \text{otherwise.} \end{cases}$$

where (λ_n) is a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, then the above classes of sequences are denoted by $w_\lambda^{I(F)}[\mathcal{M}, p, u, \Delta_v^m]$,
$w_\lambda^{I(F)}[\mathcal{M}, p, u, \Delta_v^m]_0$, $w_\lambda^F[\mathcal{M}, p, u, \Delta_v^m]_\infty$ and $w_\lambda^{I(F)}[\mathcal{M}, p, u, \Delta_v^m]_\infty$ respectively.

(v) If $I = I_F$ then we obtain

$$w_\theta^F[A, \mathcal{M}, p, u, \Delta_v^m] = \Bigg\{(x_k) \in w^F : \lim_{n,r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k} = 0, \text{ for some } \rho > 0 \text{ and } s \geq 0, \quad X_0 \in L(\mathbb{R})\Bigg\},$$

$$w_\theta^F[A, \mathcal{M}, p, u, \Delta_v^m]_0 = \Bigg\{(x_k) \in w^F : \lim_{n,r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k} = 0, \text{ for some } \rho > 0 \text{ and } s \geq 0\Bigg\},$$

$$w_\theta^F[A, \mathcal{M}, p, u, \Delta_v^m]_\infty = \Bigg\{(x_k) \in w^F : \lim_{n,r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0 \text{ and } s \geq 0\Bigg\}.$$

(vi) If $I = I_\delta$ is an admissible ideal of $\mathbb{N}$, then

$$w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$$

$$= \left\{(x_k) \in w^F : \forall \varepsilon > 0, \left\{n, r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}\right.$$

$$\left. \in I_\delta, \ \text{ for some } \rho > 0, s \geq 0 \ \text{ and } X_0 \in L(\mathbb{R})\right\},$$

$$w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$$

$$= \left\{(x_k) \in w^F : \forall \varepsilon > 0, \left\{n, r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}\right.$$

$$\left. \in I_\delta, \ \text{ for some } \rho > 0 \text{ and } s \geq 0\right\},$$

and

$$w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty$$

$$= \left\{(x_k) \in w^F : \exists\, K > 0 \text{ such that } \left\{n, r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k}\right.\right.$$

$$\left.\left. \geq K\right\} \in I_\delta, \ \text{ for some } \rho > 0 \text{ and } s \geq 0\right\}.$$

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$ and let $D = \max\{1, 2^{H-1}\}$. Then, for the factorable sequences (a_k) and (b_k) in the complex plane, we have

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}). \tag{3}$$

Also $|a_k|^{p_k} \leq \max\{1, |a|^H\}$ for all $a \in \mathbb{C}$.

The main purpose of this paper is to introduced and study some lacunary I-convergent sequence spaces of fuzzy numbers by using an infinite matrix and a sequence of Orlicz functions in more general setting. We also make an effort to study some properties like linearity, paranorm, solidity and some interesting inclusion relations between the spaces $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$, $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$, $w_\theta^F[A, \mathcal{M}, p, u, \Delta_v^m]_\infty$ and $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty$.

3. Main Results

In the current section we study some topological properties and some inclusion relations between the sequence spaces which we have defined above.

Theorem 1. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the spaces $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$, $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ and $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. We shall prove the result for the space $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_n^m]_0$ only and others can be proved in the similar way. Let $X = (X_k)$ and $Y = (Y_k)$ be two elements in $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_n^m]_0$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$A_{\frac{\varepsilon}{2}} = \left\{ n, r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \in I$$

and

$$B_{\frac{\varepsilon}{2}} = \left\{ n, r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m Y_k, \overline{0})}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \in I.$$

Let α and β be two scalars. Then by using the inequality (3) and continuity of the function $\mathcal{M} = (M_k)$, we have

$$\begin{aligned}
&\tfrac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(\alpha u_k \Delta_v^m X_k + \beta u_k \Delta_v^m Y_k, \overline{0})}{|\alpha|\rho_1 + |\beta|\rho_2} \right) \right]^{p_k} \\
&\leq D \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho_1} \right) \right]^{p_k} \\
&+ D \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m Y_k, \overline{0})}{\rho_2} \right) \right]^{p_k} \\
&\leq DK \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho_1} \right) \right]^{p_k} \\
&+ DK \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m Y_k, \overline{0})}{\rho_2} \right) \right]^{p_k},
\end{aligned}$$

where $K = \max \left\{ 1, \left(\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} \right)^H, \left(\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} \right)^H \right\}$.

From the above relation we obtain the following:

$$\left\{n, r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(\alpha u_k \Delta_v^m X_k + \beta u_k \Delta_v^m Y_k, \overline{0})}{|\alpha|\rho_1 + |\beta|\rho_2}\right)\right]^{p_k} \geq \varepsilon\right\}$$

$$\subseteq \left\{n, r \in \mathbb{N} : DK \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho_1}\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\}$$

$$\cup \left\{n, r \in \mathbb{N} : DK \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m Y_k, \overline{0})}{\rho_2}\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\} \in I.$$

This completes the proof.

Theorem 2. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the spaces $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$, $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ and $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty$ are paranormed spaces with the paranorm g_Δ defined by*

$$g_\Delta(X) = \inf\left\{(\rho)^{\frac{p_n}{H}} : \left(\frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, \text{ for some } \rho > 0 \right.$$

$$\left. \text{and } s \geq 0\ , n = 1, 2, \ldots.\ \ r \in \mathbb{N}\right\}$$

where $H = \max\{1, \sup_k p_k\}$.

Proof. Clearly, $g_\Delta(-X) = g_\Delta(X)$ and $g_\Delta(\theta) = 0$. Let $X = (X_k)$ and $Y = (Y_k)$ be two elements in $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$. Then for every $\rho > 0$ we write

$$A_1 = \left\{\rho > 0 : \left(\frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}$$

and

$$A_2 = \left\{\rho > 0 : \left(\frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m Y_k, \overline{0})}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}.$$

Let $\rho_1 \in A_1$ and $\rho_2 \in A_2$. If $\rho = \rho_1 + \rho_2$, then we get the following

$$\left(\frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m (X_k + Y_k), \overline{0}))}{\rho}\right)\right]\right)$$

$$\leq \frac{\rho_1}{\rho_1 + \rho_2} \left(\frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho_1}\right)\right]\right)$$

$$+ \quad \frac{\rho_2}{\rho_1+\rho_2}\left(\frac{1}{h_r}\sum_{k\in I_r} a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m Y_k,\overline{0})}{\rho_2}\right)\right]\right).$$

Thus, we have

$$\frac{1}{h_r}\sum_{k\in I_r} a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m (X_k+Y_k),\overline{0})}{\rho}\right)\right]^{p_k}\leq 1$$

and

$$\begin{aligned} g_\Delta(X+Y) &= \inf\{(\rho_1+\rho_2)^{\frac{p_n}{H}} : \rho_1\in A_1, \rho_2\in A_2\} \\ &\leq \inf\{(\rho_1)^{\frac{p_n}{H}} : \rho_1\in A_1\}+\inf\{(\rho_2)^{\frac{p_n}{H}} : \rho_2\in A_2\} \\ &= g_\Delta(X)+g_\Delta(Y). \end{aligned}$$

Let $t_k^m\to t$, where $t_k^m,\ t\in\mathbb{C}$, and let $g_\Delta(X_k^m-X_k)\to 0$ as $m\to\infty$. To prove that $g_\Delta(t_k^m X_k^m - tX_k)\to 0$ as $m\to\infty$. Let $t_k\to t$, where $t_k,\ t\in\mathbb{C}$, and $g_\Delta(X_k^m-X_k)\to 0$ as $m\to\infty$. We have

$$A_3=\left\{\rho_k>0 : \frac{1}{h_r}\sum_{k\in I_r} a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,\overline{0})}{\rho_k}\right)\right]^{p_k}\leq 1\right\}$$

and

$$A_4=\left\{\rho_k'>0 : \frac{1}{h_r}\sum_{k\in I_r} a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m Y_k,\overline{0})}{\rho_k'}\right)\right]^{p_k}\leq 1\right\}.$$

If $\rho_k\in A_3$ and $\rho_k'\in A_4$ then by inequality (3) and continuity of the function $\mathcal{M}=(M_k)$, we have that

$$\begin{aligned} & k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m (t^m X_k^m - tX,\overline{0}))}{|t^m-t|\rho_k+|t|\rho_k'}\right) \\ &\leq k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m (t^m X_k^m - tX_k),\overline{0})}{|t^m-t|\rho_k+|t|\rho_k'}\right)+k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m (tX_k - tX,\overline{0}))}{|t^m-t|\rho_k+|t|\rho_k'}\right) \\ &\leq \frac{|t^m-t|\rho_k}{|t^m-t|\rho_k+|t|\rho_k'}k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k^m,\overline{0})}{\rho_k}\right) \\ &+ \frac{|t|\rho_k'}{|t^m-t|\rho_k+|t|\rho_k'}k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m (X_k^m-X_k),\overline{0})}{\rho_k'}\right). \end{aligned}$$

From the above inequality it follows that

$$\frac{1}{h_r}\sum_{k\in I_r} a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m (t^m X_k^m - tX),\overline{0})}{|t^m-t|\rho_k+|t|\rho_k'}\right)\right]^{p_k}\leq 1$$

and consequently,

$$\begin{aligned} g_\Delta(t_k^m X_k + tX) &= \inf\{(|t_k^m - t|\rho_k + |t|\rho_k')\}^{\frac{p_n}{H}} : \rho_k \in A_3, \rho_k' \in A_4\} \\ &\leq |t_k^m - t|\rho_k^{\frac{p_n}{H}} \inf\{(\rho_k)^{\frac{p_n}{H}} : \rho_k \in A_3\} + |t|\rho_k' \inf\{(\rho_k')^{\frac{p_n}{H}} : \rho_k' \in A_4\} \\ &\leq \max\{|t|, |t|^{\frac{p_n}{H}}\} g_\Delta(X_k^m - X_k). \end{aligned}$$

Note that $g_\Delta(X_k^m) \leq g_\Delta(X^m) + g_\Delta(X_k^m - X^m)$, for all $k \in \mathbb{N}$. Hence, by our assumption the right hand tends to 0 as $m \to \infty$. This completes the proof.

Theorem 3. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers,*

(i) Let $0 < \inf p_k \leq p_k \leq 1$. Then $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m] \subseteq w_\theta^{I(F)}[A, \mathcal{M}, u, \Delta_v^m]$, $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0 \subseteq w_\theta^{I(F)}[A, \mathcal{M}, u, \Delta_v^m]_0$.

(ii) Let $1 \leq p_k \leq \sup p_k < \infty$. Then $w_\theta^{I(F)}[A, \mathcal{M}, u, \Delta_v^m] \subseteq w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$, $w_\theta^{I(F)}[A, \mathcal{M}, u, \Delta_v^m]_0 \subseteq w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$.

Proof. (i) Let $X = (X_k)$ be an element in $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Since $0 < \inf p_k \leq p_k \leq 1$ we have

$$\frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right] \leq \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right]^{p_k}.$$

Therefore,

$$\left\{ n, r \in \mathbb{N} : \tfrac{1}{h_r} \textstyle\sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right] \geq \varepsilon \right\} \subseteq \left\{ n, r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I.$$

The other part can be proved in the same way.
(ii) Let $X = (X_k)$ be an element in $w_\theta^{I(F)}[A, \mathcal{M}, u, \Delta_v^m]$. Since $1 \leq p_k \leq \sup p_k < \infty$. Then for each $0 < \varepsilon < 1$ there exists a positive integer n_0 such that

$$\frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right] \leq \varepsilon < 1 \text{ for all } n \geq n_0.$$

This implies that

$$\frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right]^{p_k} \leq \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right].$$

Therefore, we have

$$\left\{n,r\in\mathbb{N}:\tfrac{1}{h_r}\sum_{k\in I_r}a_{nk}\left[k^{-s}M_k\left(\tfrac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}$$

$$\subseteq\left\{n,r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)\right]\geq\varepsilon\right\}\in I.$$

The other part can be proved in the similar way. This completes the proof.

Theorem 4. *Let $X=(X_k)$ be a sequence of Fuzzy numbers, $\mathcal{M}=(M_k)$ be a sequence of Orlicz functions, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then*

$$w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]_0\subset w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]\subset w_\theta^F[A,\mathcal{M},p,u,\Delta_v^m]_\infty.$$

Proof. The inclusion $w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]_0\subset w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]$ is obvious. Let $X=(X_k)\in w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]$. Then there is some fuzzy number X_0, such that

$$\frac{1}{h_r}\sum_{k\in I_r}a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)\right]^{p_k}\geq\varepsilon.$$

Now, by inequality (3), we have

$$\frac{1}{h_r}\sum_{k\in I_r}a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,\overline{0})}{\rho}\right)\right]^{p_k}\leq D\frac{1}{h_r}\sum_{k\in I_r}a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)\right]^{p_k}$$
$$+D\frac{1}{h_r}\sum_{k\in I_r}a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(X_0,\overline{0})}{\rho}\right)\right]^{p_k}.$$

This implies that $X=(X_k)\in w_\theta^F[A,\mathcal{M},p,u,\Delta_v^m]_\infty$. This completes the proof.

Theorem 5. *Let $\mathcal{M}=(M_k)$ and $\mathcal{S}=(S_k)$ be a sequence of Orlicz functions. Then*

$$w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]\cap w_\theta^{I(F)}[A,\mathcal{S},p,u,\Delta_v^m]\subset w_\theta^{I(F)}[A,\mathcal{M}+\mathcal{S},p,u,\Delta_v^m].$$

Proof. Let $X=(X_k)\in w_\theta^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]\cap w_\theta^{I(F)}[A,\mathcal{S},p,u,\Delta_v^m]$ using the inequality (3), we have

$$\tfrac{1}{h_r}\sum_{k\in I_r}a_{nk}\left[k^{-s}(M_k+S_k)\left(\tfrac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)\right]^{p_k}$$

$$=\ \frac{1}{h_r}\sum_{k\in I_r}a_{nk}\left[k^{-s}M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)+\ k^{-s}S_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)\right]^{p_k}$$

$$\leq \quad D \left\{ \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right]^{p_k} + \frac{1}{h_r} \sum_{k \in I_r}^{\infty} a_{nk} \left[k^{-s} S_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right]^{p_k} \right\}.$$

Thus, $X = (X_k) \in w_\theta^{I(F)}[A, \mathcal{M} + \mathcal{S}, p, u, \Delta_v^m]$. This completes the proof.

Theorem 6. *The sequence spaces* $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ *and* $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty$ *are normal as well as monotone.*

Proof. We give the proof of the theorem for $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ only. Let $X = (X_k) \in w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ and $Y = (Y_k)$ be such that $\overline{d}(Y_k, \overline{0}) \leq \overline{d}(X_k, \overline{0})$ for all $k \in \mathbb{N}$. Then for given $\varepsilon > 0$ we have

$$B = \left\{ n, r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I,$$

again the set

$$B_1 = \left\{ n, r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} a_{nk} \left[k^{-s} M_k \left(\frac{\overline{d}(u_k \Delta_v^m Y_k, \overline{0})}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \subseteq B.$$

Hence, $B_1 \in I$ and so $Y = (Y_k) \in w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$. Thus, the space $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ is normal. Also, from the Lemma 2, it follows that $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ is monotone. This completes the proof.

Theorem 7. *If I is not maximal ideal then the space* $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ *is neither normal nor monotone.*

Example 3. *Let us consider a sequence of fuzzy numbers*

$$X_k(l) = \begin{cases} \frac{1+l}{2} & -1 \leq l \leq 1, \\ \frac{3-l}{2} & 1 \leq l \leq 3, \\ 0 & otherwise. \end{cases}$$

If $m = 0$, then $\Delta_v^m X_k = 1$. Let $A = (C, 1)$, the Cesàro matrix, $\mathcal{M}(x) = x$, $u = (u_k) = 1$, $s = 0$, $p = (p_k) = 1$, for all $k \in \mathbb{N}$, $\rho = 1$ and $\theta = 2^r$ then we have $(X_k) \in w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Since I is not maximal by Lemma 3, their exist a subset K of $\mathbb{N}$ such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let us define sequence $Y = (Y_k)$ by

$$Y_k = \begin{cases} X_k & k \in K \\ 0 & otherwise. \end{cases}$$

Then, (Y_k) belongs to the canonical pre image of the k-step spaces of $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. But $Y_k \notin w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Hence, $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ is not monotone. Therefore, by Lemma 2, $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ is not normal.

Theorem 8. *If I is neither maximal nor $I = I_F$ then the spaces $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ and $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ are not symmetric.*

Example 4. *Let us consider a sequence of fuzzy numbers*

$$X_k(l) = \begin{cases} l - 2k + 1 & l \in [2k-1, 2k], \\ -l + 2k + 1 & l \in [2k, 2k+1], \\ 0 & otherwise. \end{cases}$$

If $m = 1, v = 1$, then $\Delta_v^m X_k = \Delta X_k$. Let $A = (C, 1)$, the Cesàro matrix, $\mathcal{M}(x) = x^2$, $u = (u_k) = 1$, $s = 0$, $I = I_\delta$, $p = (p_k) = 1$, for all $k \in \mathbb{N}$ and $\theta = 2^r$. Thus, we have $(X_k) \in w^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. But the rearrangement $Y = (Y_k)$ of the sequence space (X_k) is defined as

$$Y_k = \{X_1, X_4, X_2, X_9, X_3, X_{16}, X_5, X_{25}, X_6, ...\}$$

This implies that $(Y_k) \in w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Hence, $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ is not symmetric. Similarly, $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ is not symmetric.

Theorem 9. *The spaces $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ and $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ are not convergent free in general.*

Example 5. *Let us consider a sequence of fuzzy numbers*

$$X_k(l) = \begin{cases} \frac{1+l}{2} & -1 \le l \le 1, \\ \frac{3-l}{2} & 1 \le l \le 3, \\ 0 & otherwise. \end{cases}$$

If $m = 0$, then $\Delta_v^m X_k = 1$. Let $A = (C, 1)$, the Cesàro matrix, $\mathcal{M}(x) = x$, $u = (u_k) = 1$, $s = 0$, $p = (p_k) = 1$, for all $k \in \mathbb{N}$ and $\rho = 1$ then we have $(X_k) \in w^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Let $Y_k(l) = \frac{1}{k}$ for all $k \in \mathbb{N}$. Then $(Y_k) \in w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. But $X_k = 0$ does not imply $Y_k = 0$. Hence, $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ is not convergent free. Similarly, $w_\theta^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ is not convergent free.

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