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A Certain Class of Relatively Equi-Statistical Fuzzy Approximation Theorems

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Abstract. The aim of this paper is to introduce the notions of relatively deferred Nörlund uniform statistical convergence as well as relatively deferred Nörlund point-wise statistical convergence through the difference operator of fractional order of fuzzy-number-valued sequence of functions, and a type of convergence which lies between aforesaid notions, namely, relatively deferred Nörlund equi-statistical convergence. Also, we investigate the inclusion relations among these aforesaid notions. As an application point of view, we establish a fuzzy approximation (Korovkin-type) theorem by using our new notion of relatively deferred Nörlund equi-statistical convergence and intimate that this result is a non-trivial generalization of several well-established fuzzy Korovkin-type theorems which were presented in earlier works. Moreover, we estimate the fuzzy rate of the relatively deferred Nörlund equi-statistical convergence involving a non-zero scale function by using the fuzzy modulus of continuity.

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1. Introduction and Preliminaries

Moore [31] was the first who presented the notion of of uniform convergence of sequence of functions associated with a scale function, and later Chittenden [12] studied this concept. Recalling this concept, a sequence of functions (f_n) defined over $[a, b]$ converges relatively uniformly to a limit function $f(x)$, if there exists a scale function $\sigma(x)$ $(\neq 0)$ defined over $[a, b]$ and for every $\epsilon > 0$, there exists a positive integer n_ϵ such that

$$\left|\frac{f_n(x) - f(x)}{\sigma(x)}\right| \leqq \epsilon \quad (\forall\ n > n_\epsilon)$$

holds true (uniformly) for all $x \in [a, b] \subseteq \mathbb{R}$. The importance of relatively uniform convergence over the usual uniform convergence is discussed in the following example.

Example 1. *For all $n \in \mathbb{N}$, consider $f_n : [0, 1] \to \mathbb{R}$ defined by*

$$f_n(x) = \begin{cases} \frac{nx}{1+n^2x^2} & (0 < x \leqq 1) \\ 0 & (x = 0). \end{cases}$$

The sequence (f_n) of functions is not classically uniformly convergent on $[0, 1]$; but convergent uniformly to $f = 0$ relative to a scale function

$$\sigma(x) = \begin{cases} \frac{1}{x} & (0 < x \leqq 1) \\ 1 & (x = 0) \end{cases}$$

on $[0, 1]$. Here, we write $f_n \rightrightarrows f = 0 \quad ([0, 1]; \sigma)$.

Thus, uniform convergence can be viewed as a special case of relative uniform convergence (or, convergent uniformly relative to a non-zero scale function). Recently, Demirci and Orhan [16] defined the notion of relatively uniform statistical convergence of a sequence of functions which is based on the natural density of a set as follows:

Let $E \subset \mathbb{R}$ be compact and (f_n) be a sequence of functions defined on E. The sequence (f_n) is relatively uniform statistical convergent to the limit function f defined on E, if there exists a non-zero scale function $\sigma(x) \quad (\sigma(x) > 0)$ over E provided, for each $\epsilon > 0$,

$$\left\{k : k \leqq n \quad \text{and} \quad \sup_{x\in E}\left|\frac{f_k(x) - f(x)}{\sigma(x)}\right| \geqq \epsilon\right\}$$

has natural (asymptotic) density zero; equivalently, one writes

$$\lim_{n\to\infty}\left|\left\{k : k \leqq n \quad \text{and} \quad \sup_{x\in E}\left|\frac{f_k(x) - f(x)}{\sigma(x)}\right| \geqq \epsilon\right\}\right| = 0.$$

Here, we write stat $\quad f_n(x) \rightrightarrows f(E; \sigma)$.

The rapid growth of sequence spaces have been accompanied by the recent works of many researchers in the field of statistical convergence and it has got tremendous importance over conventional convergence. The basic concept of statistical convergence was initially studied in the year 1951 by Fast [17] and Steinhaus [41] independently. Recently, the approximation of functions by linear operators (positive) based on statistical convergence has become an energetic area of research. In the current years, the use of statistical convergence in approximation theory has enabled the researchers to achieve more powerful outcomes than that of the classical aspects of convergence. In this context, we refer to the recent works [9], [10], [15], [14], [13], [20], [21], [24], [22], [23], [25], [33], [34], [35], [36], [37], [38] and [40].

We use the symbol ω to denote the space of all real valued sequences. Also, the usual notations ℓ_∞, c and c_0 will be used to denote the classes of bounded linear spaces, convergent sequences and null sequences respectively. Moreover, it is also known that any subspace of ω is a sequence space. Note that all these spaces are Banach spaces under the sup-norm $(x_k)_{k\in\mathbb{N}}$, which a is sequence of real or complex terms, given by $\|x\|_\infty = \sup_k |x_k|$.

In the year 1981, Kızmaz [27] gave the preliminary notion of space of difference sequence and subsequently, the difference sequence of order r ($r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) is defined as follows:

$$\lambda(\Delta^r) = \{x = (x_k) : \Delta^r(x) \in \lambda, \quad \lambda \in (\ell_\infty, c_0, c)\};$$
$$\Delta^0 x = (x_k); \quad \Delta^r x = (\Delta^{r-1}x_k - \Delta^{r-1}x_{k+1})$$

and

$$\Delta^r x_k = \sum_{i=0}^{r} (-1)^i \binom{r}{i} x_{k+i}).$$

The norm for difference sequence of order r is given by

$$\|x\|_{\Delta^r} = \sum_{i=1}^{r} |x_i| + \sup_k |\Delta^r x_k|.$$

Note that the difference spaces obtained from $\lambda(\Delta^r)$ are Banach spaces under the above norm.

The difference operator of fractional-order was initially used by Chapman [11] and subsequently, many researchers used it with different settings (see [4], [8]). Recently, Baliarsingh [6] has studied a fractional-order difference sequence involving the Euler-Gamma function as follows:

$$\Delta^\alpha(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k-i}$$

for $k \in \mathbb{N}$. Moreover, in [7], Baliarsingh has introduced certain new fractional-order difference sequence spaces. Assume that α, β and γ are real numbers and also assume that h is a positive constant. For any $(x_k) \in \omega$, the generalized difference sequence with a view of fractional-order difference operator $\Delta_h^{\alpha,\beta,\gamma} : \omega \to \omega$ is defined by

$$(\Delta_h^{\alpha,\beta,\gamma} x_k) = \sum_{i=0}^{\infty} \frac{(-\alpha)_i - (\beta)_i}{i!(-\gamma)_i h^{\alpha+\beta-\gamma}} x_{k-i} \quad (k \in \mathbb{N}). \tag{1}$$

We now recall the sequence of fractional-order backward difference operator (see [7]) which is required for the present investigation. Suppose that $f(x)$ is a fractional order differentiable function and also suppose that $h \to 0$. Consider the sequence $\{f_h(x)\}$ which is associated to $f(x)$ given by $f_h(x) = (f(x - ih))_{i \in \mathbb{N}_0}$. Next, in view of $\Delta_{h,x}^{\alpha,\beta,\gamma}$, the sequence spaces $\Delta_{h,x}^{\alpha,\beta,\gamma}(f_h(x))$ is given by

$$\Delta_{h,x}^{\alpha,\beta,\gamma} f(x) = \sum_{i=0}^{\infty} \frac{(-\alpha)_i(-\beta)_i}{i!(-\gamma)_i h^{\alpha+\beta-\gamma}} f(x - ih). \tag{2}$$

In the last equality, $(\alpha)_k$ denotes the Pochhammer symbol of a real number α and is given by the formula

$$(\alpha)_k = \begin{cases} 1 & (\alpha = 0 \text{ or } k = 0) \\ \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)(\alpha+2)...(\alpha+k-1) & (k \in \mathbb{N}). \end{cases}$$

Also, without loss of generality, the summation considered in the equality (2) converges for all $\gamma > \alpha + \beta$ (see [18]).

Moreover, since in all the cases it is not possible to calculate either the simple limits or the statistical limits with exact precision. So different approaches such as fuzzy logic, fuzzy set theory, set valued analysis, interval analysis, etc. were introduced by various researchers to model several mathematical structures. In this context, we recall some fuzzy approximation (Korovkin-type) theorems recently studied by Anastassiou [2], Anastassiou and Duman [3], Karaisa and Kadak [26], and Mohiuddine et al. ([29] and [30]). The main objective of the proposed work is to establish a fuzzy approximation (Korovkin-type) theorem by using relatively deferred Nörlund equi-statistical convergence based on $\Delta_h^{\alpha,\beta,\gamma}$ and further to estimate its statistical fuzzy rates with the help of the fuzzy modulus of continuity.

2. Some Basic Definitions

Consider a fuzzy number valued function $X : \mathbb{R} \to [0, 1]$ which is upper semi-continuous, normal and convex. Also, $\sup(X)$, the closure of this set is compact, where $\sup(X) = \overline{\{x \in \mathbb{R} : X(x) > 0\}}$. Let $\mathbb{R}_{\mathcal{F}}$ denotes the set of fuzzy numbers and let $[X]^0 = \overline{\{x \in \mathbb{R} : X(x) > 0\}}$

be the closure of the set. Also, recall that the closed and bounded interval of $\mathbb{R}$ (see [19]) denoted by $[X]^r$ is given as $[X]^r = \{x \in \mathbb{R} : X(x) \geqq r\}$ $(r \in (0,1])$.

Now, we define the fuzzy sum and the fuzzy product as follows: Let $u, v \in \mathbb{R}_{\mathcal{F}}$ and suppose that $\lambda \in \mathbb{R}$. For $r \in (0,1]$ the fuzzy sum and the fuzzy product respectively denoted by $u \oplus v$ and $u \odot v$ are given by $[u \oplus v]^r = [u]^r + [v]^r$ and $[\lambda \odot v]^r = \lambda[u]^r$. Next, consider the interval $[u]^r$ of the form $[u_-^r, u_+^r]$, where $u_-^r \leqq u_+^r$ and $u_-^r, u_+^r \in \mathbb{R}$ for $0 \leqq r \leqq 1$. Also, for $u, v \in \mathbb{R}_{\mathcal{F}}$, define

$$u \preceq v \Longleftrightarrow u_-^r \leqq v_-^r \quad \text{and} \quad u_+^r \leqq v_+^r \quad \forall\, r \in [0,1].$$

Further, the metric $\mathcal{D}$ is such that $\mathcal{D} : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ be defined as

$$\mathcal{D}(u,v) = \sup_{0 \leqq r \leqq 1} \max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}.$$

Note that, $(\mathbb{R}_{\mathcal{F}}, \mathcal{D})$ is a metric space which is complete (see [42]).

Consider two fuzzy valued functions f and g such that $f, g : [a,b] \to \mathbb{R}$, the distance between two functions f and g denoted as $\mathcal{D}^*(f,g)$ is given by

$$\mathcal{D}^*(f,g) = \sup_{a \leqq x \leqq b} \sup_{0 \leqq r \leqq 1} \max\{|f_-^r - g_-^r|, |f_+^r - g_+^r|\}.$$

The fuzzy counterpart of the idea statistical convergence for sequence $(x_n)_{n \in \mathbb{N}}$ of fuzzy numbers based on $\mathcal{D}$ was introduced by Nuray and Savaş [32]. Recall that, $(x_n)_{n \in \mathbb{N}}$ is statistically convergent to a fuzzy number ℓ, in symbols, one writes stat $\mathcal{D}(x_n, \ell) = 0$, if for each $\epsilon > 0$,

$$d(K_\epsilon) = \lim_{n \to \infty} \frac{|K_\epsilon|}{n} = \lim_{n \to \infty} \frac{|\{k : k \leqq n \quad \text{and} \quad \mathcal{D}(x_n, \ell) \geqq \epsilon\}|}{n} = 0,$$

where $|\cdot|$ denotes the cardinality of the set.

Next, before presenting the notion of the deferred Nörlund mean under the generalized differentiable function $f(x)$ with fractional order, we recall the regularity condition (see Agnew [1]) as follows: Let us assume two non-negative sequences (a_n) and (b_n) of integers such that it satisfies (i) $a_n < b_n$ and (ii) $\lim_{n \to \infty} b_n = +\infty$ $(n \in \mathbb{N})$. We remark that these conditions are also used to define the proposed deferred weighted mean.

We now assume that (s_n) and (t_n) be the sequences of non-negative real numbers such that

$$S_n = \sum_{m=a_n+1}^{b_n} s_m \quad \text{and} \quad T_n = \sum_{m=a_n+1}^{b_n} t_m.$$

The convolution of the above sequences can be presented as (see [39]),

$$R_{a_n+1}^{b_n} = (S * T)_{b_n} = \sum_{v=a_n+1}^{b_n} s_v t_{b_n-v}.$$

Now, for defining the sequence (f_n) of functions via deferred Nörlund mean depends on $\Delta_{h,x}^{\alpha,\beta,\gamma}$, we first set

$$\Lambda_n = \frac{1}{R_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} s_{b_n-m} t_m \left(\Delta_{h,x}^{\alpha,\beta,\gamma} f_m(x)\right). \tag{3}$$

We say that (f_n) is deferred Nörlond summable to a number ℓ under the operator $\Delta_{h,x}^{\alpha,\beta,\gamma}$, provided $\lim_{n\to\infty} \Lambda_n = \ell$.

3. Relatively Deferred Weighted Equi-statistical Convergence with Associated Inclusion Relations

In 2018, Srivastava et al. [39] used the concept of deferred Nörlund mean to introduce the ideas of point-wise and uniform statistical convergence as well as equi-statistical convergence while these notions in classical sense were defined and studied by Balcerzak et al. [5] and in view of weighted lacunary sequence by Mohiuddine and Alamri [28]. Analogous to these definitions, here we present the definitions of relatively deferred Nörlund point-wise and uniform statistical convergence and relatively deferred Nörlund equi-statistical convergence of a FNVS (fuzzy number valued sequence) of functions as follows:

Definition 1. *Assume that (a_n) and (b_n) are sequences of integers and $E \subset \mathbb{R}$ is compact. Also assume that (f_m) and f are fuzzy number valued sequence (in short, FNVS) of functions and fuzzy number valued function (in short, FNVF) respectively, both defined on E. Then, the sequence (f_m) is*

(D1) relatively weighted point-wise statistically convergent to limit of FNVF f under the fractional-order difference operator $\Delta_{h,x}^{\alpha,\beta,\gamma}$, denoted by

$$f_m \longrightarrow f \ (RW\text{-}St_{pw}(\Delta_{h,x}^{\alpha,\beta,\gamma})) \quad or \quad St_{pw}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma}) \lim f_m = f \ (E;\sigma),$$

if there is a non-zero scale function $\sigma(x)$ on E such that, for each $\epsilon > 0$ and for every $x \in E$,

$$\lim_{n\to\infty} \frac{1}{R_{a_n+1}^{b_n}} \left| \left\{ m : m \leqq R_{a_n+1}^{b_n} \quad and \quad s_{b_n-m} t_m \frac{\mathcal{D}\left(\Delta_{h,x}^{\alpha,\beta,\gamma} f_m(x), f(x)\right)}{|\sigma(x)|} \geqq \epsilon \right\} \right| = 0.$$

(D2) relatively weighted equi-statistically convergent to the limit of FNVF f under the fractional-order difference operator $\Delta_{h,x}^{\alpha,\beta,\gamma}$; here, we write

$$f_m \longrightarrow f \ (RW\text{-}St_{equi}(\Delta_{h,x}^{\alpha,\beta,\gamma})) \quad or \quad St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma}) \lim f_m = f \ (E;\sigma),$$

if there exists $\sigma(x)$ $(\sigma(x)>0)$ on E such that, for every $\epsilon>0$ and for every $x\in E$,

$$\lim_{n\to\infty}\frac{\Omega_m(x;\epsilon\sigma)}{R_{a_n+1}^{b_n}}=0,$$

relatively uniformly with regards to $x\in E$, that is,

$$\lim_{n\to\infty}\frac{\|\Omega_m(x;\epsilon\sigma)\|_{\mathcal{C}_{\mathcal{F}}(E)}}{R_{a_n+1}^{b_n}}=0,$$

where

$$\Omega_m(x;\epsilon\sigma)=\left|\left\{m:m\leqq R_{a_n+1}^{b_n}\quad and\quad \sup_{x\in E}s_{b_n-m}t_m\frac{\mathcal{D}\left(\Delta_{h,x}^{\alpha,\beta,\gamma}f_m(x),f(x)\right)}{|\sigma(x)|}\geqq\epsilon\right\}\right|.$$

(D3) *relatively weighted uniformly statistically convergent to the limit of FNVF f under the fractional-order difference operator $\Delta_{h,x}^{\alpha,\beta,\gamma}$; here, we write*

$$f_m\longrightarrow f\ (RW\text{-}St_{uf}(\Delta_{h,x}^{\alpha,\beta,\gamma}))\quad or\quad St_{uf}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\lim f_m=f\ (E;\sigma),$$

if there is a scale function $\sigma(x)$ $(\sigma(x)>0)$ on E such that, for each $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{R_{a_n+1}^{b_n}}\left|\left\{m:m\leqq R_{a_n+1}^{b_n}\quad and\quad \sup_{x\in E}s_{b_n-m}t_m\frac{D\left(\Delta_{h,x}^{\alpha,\beta,\gamma}f_m(x),f(x)\right)}{|\sigma(x)|}\geqq\epsilon\right\}\right|=0.$$

As a consequence of the above definitions, we present the following inclusion relations in the similar lines of our earlier work (see [39]).

Lemma 1. *The implications,*

$$\begin{aligned}f_m\longrightarrow f(RW\text{-}St_{uf}(\Delta_{h,x}^{\alpha,\beta,\gamma}))&\Longrightarrow f_m\longrightarrow f(RW\text{-}St_{equi}(\Delta_{h,x}^{\alpha,\beta,\gamma}))\\&\Longrightarrow f_m\longrightarrow f(RW\text{-}St_{pw}(\Delta_{h,x}^{\alpha,\beta,\gamma}))\end{aligned}\qquad(4)$$

are fairly true. Moreover, reverse of the implications of (4) are not necessarily true, that is, the above inclusions are strict.

4. A Fuzzy Korovkin-type Approximation Theorem

Based upon the proposed methods, in the present section we wish to investigate a fuzzy approximation theorem (Korovkin-type) via relatively deferred Nörlund equi-statistical convergence.

Let f be a fuzzy number valued function, and Suppose that $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$. Recall that, f is fuzzy continuous at a point $x_0 \in [a, b]$, if $\mathcal{D}(x_n, x_0) < \epsilon$ $(n \to \infty)$ whenever $x_n \to x_0$. Furthermore, if f is fuzzy continuous at each point $x \in [a, b]$, then it is also fuzzy continuous in the interval $[a, b]$. Let $\mathcal{C}_{\mathcal{F}}[a, b]$ be denoted as the set of all fuzzy functions (continuous) defined in the interval $[a, b]$ (note that, $\mathcal{C}_{\mathcal{F}}[a, b]$ is a scalar, but not usually a vector space).

Note that, $\mathfrak{T} : \mathcal{C}_{\mathcal{F}}[a, b] \to \mathcal{C}_{\mathcal{F}}[a, b]$ is fuzzy linear operator, if for each $\mu_1, \mu_2 \in \mathbb{R}$ and $f_1, f_2 \in \mathcal{C}_{\mathcal{F}}[a, b]$,

$$\mathfrak{T}(\mu_1 \odot f_1 \oplus \mu_2 \odot f_1; x) = \mu_1 \odot \mathfrak{T}(f_1) \oplus \mu_2 \odot \mathfrak{T}(f_2).$$

Moreover, a fuzzy linear operator $\mathfrak{T}$ is a positive fuzzy linear operator, if $\mathfrak{T}(f_1; x) \preceq \mathfrak{T}(f_2; x)$ such that $f_1, f_2 \in \mathcal{C}_{\mathcal{F}}[a, b]$ and for all $x \in [a, b]$ with $f_1(x) \preceq f_2(x)$.

In the year 2005, Anastassiou [2] has proved a classical version of fuzzy Korovkin-type theorem. Subsequently, Anastassiou and Duman [3] established the statistical versions of a fuzzy approximation theorem based on the fuzzy positive linear operator. Here in this paper, we extend the result of Anastassiou and Duman [3] by using relatively Nörlund equi-statistical convergence and accordingly established the following theorem for the same set of functions. Throughout the paper we choose the test functions $f_1 = x^i$, where $i = 0, 1, 2$.

Theorem 1. *Let (a_n) and (b_n) be sequences of integers (non-negative) and let $\mathfrak{T}_m : \mathcal{C}_{\mathcal{F}}[a, b] \to \mathcal{C}_{\mathcal{F}}[a, b]$ $(m \in \mathbb{N})$ be the fuzzy sequence of positive linear operators. Suppose that $\{\mathfrak{T}^*_m\}_{n\in\mathbb{N}}$ be the corresponding sequence of positive linear operators from $\mathcal{C}[a, b]$ into itself such that*

$$\{\mathfrak{T}_m(f; x)\}^r_{\pm} = \mathfrak{T}^*_m(f^r_{\pm}; x) \tag{5}$$

for all $x \in [a, b]$, $r \in [0, 1]$, $n \in \mathbb{N}$ and $f \in \mathcal{C}_{\mathcal{F}}[a, b]$. Further, assume that

$$\mathfrak{T}^*_m(f_i) \longrightarrow f_i \ (RW\text{-}S_{equi}(\Delta^{\alpha,\beta,\gamma}_{h,x}))\ (E; \sigma_i), \tag{6}$$

where $f_i(x) = x^i$ $(i = 0, 1, 2)$ and $E = [a, b]$. Then, for all $f \in \mathcal{C}_{\mathcal{F}}[a, b]$,

$$St^{rw}_{equi}(\Delta^{\alpha,\beta,\gamma}_{h,x}) \lim \mathcal{D}^* (\mathfrak{T}_m(f), f) = 0 \ (E; \sigma). \tag{7}$$

Here $\sigma(x) = \max\{|\sigma_i(x)| : |\sigma_i(x)| > 0, i = 0, 1, 2\}$.

Proof. Suppose $f \in \mathcal{C}_{\mathcal{F}}[a, b]$, $x \in [a, b]$ and $r \in [0, 1]$. Moreover, since $f^r_{\pm}(x) \in \mathcal{C}[a, b]$ (by the hypothesis), so we have for every $\epsilon > 0$, there exists $\delta > 0$, such that

$$|f^r_{\pm}(y) - f^r_{\pm}(x)| < \epsilon \quad \text{whenever} \quad |y - x| < \delta \tag{8}$$

for all $x, y \in [a, b]$. Since f is fuzzy bounded, we have $|f^r_{\pm}(x)| \leq \mathcal{K}^r_{\pm}$ $(a < x < b)$. Therefore $|f^r_{\pm}(y) - f^r_{\pm}(x)| \leq 2\mathcal{K}^r_{\pm}$ $(a < x, y < b)$. Let us choose $\theta(y, x) = (y - x)^2$. Then, we clearly get

$$|f^r_{\pm}(y) - f^r_{\pm}(x)| < \epsilon + \frac{2\mathcal{K}^r_{\pm}}{\delta^2}\theta(y, x)$$

which yields

$$-\epsilon - \frac{2\mathcal{K}^r_\pm}{\delta^2}\theta(y,x) < \big(f^r_\pm(y) - f^r_\pm(x)\big) < \epsilon + \frac{2\mathcal{K}^r_\pm}{\delta^2}\theta(y,x). \tag{9}$$

Next, as the operator $\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m$ is linear and monotone, so by applying $\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x)$ in (9), we obtain

$$\begin{aligned}\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x)\left(-\epsilon - \frac{2\mathcal{K}^r_\pm}{\delta^2}\theta(y,x)\right) &< \Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x)\left(f^r_\pm(y) - f^r_\pm(x)\right)\\ &< \Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x)\left(\epsilon + \frac{2\mathcal{K}^r_\pm}{\delta^2}\theta(y,x)\right). \end{aligned}\tag{10}$$

Furthermore, x is supposed to be fixed and $f^r_\pm(x)$ being a constant number, we thus get

$$\begin{aligned}-\epsilon\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x) - \frac{2\mathcal{K}^r_\pm}{\delta^2}\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(\theta,x) &< \Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(f^r_\pm,x) - f^r_\pm(x)\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x)\\ &< \epsilon\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x) + \frac{2\mathcal{K}^r_\pm}{\delta^2}\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(\theta,x), \end{aligned}\tag{11}$$

and moreover in association with the identity (below)

$$\begin{aligned}\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*{}_m(f^r_\pm,x) - f^r_\pm(x) &= [\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(f^r_\pm,x) - f^r_\pm(x)\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x)]\\ &\quad + f^r_\pm(x)[\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x) - 1] \end{aligned}\tag{12}$$

yields

$$\begin{aligned}\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(f^r_\pm,x) - f^r_\pm(x) &< \epsilon\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x)\\ &\quad + \frac{2\mathcal{K}^r_\pm}{\delta^2}\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(\theta,x) + f^r_\pm(x)[\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^r_m(1,x) - 1]. \end{aligned}\tag{13}$$

Furthermore, computing $\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(\theta,x)$ as,

$$\begin{aligned}\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(\theta,x) &= \Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(y^2 - 2xy + x^2,x)\\ &= \Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(y^2,x) - 2x\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(y,x) + x^2\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x)\\ &= [\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(y^2,x) - x^2] - 2x[\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(y,x) - x]\\ &\quad + x^2[\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x) - 1]\end{aligned}$$

and using (13), we get

$$\begin{aligned}\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(f^r_\pm,x) - f^r_\pm(x) &< \epsilon\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x) + \frac{2\mathcal{K}^r_\pm}{\delta^2}\{[\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(y^2,x) - x^2]\\ &\quad - 2x[\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(y,x) - x] + x^2[\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x) - 1]\}\\ &\quad + f^r_\pm(x)[\Delta^{\alpha,\beta,\gamma}_{h,x}\mathfrak{T}^*_m(1,x) - 1]\end{aligned}$$

$$
\begin{aligned}
&= \epsilon[\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1,x)-1]+\epsilon+\frac{2\mathcal{K}_{\pm}^r}{\delta^2}\{[\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(y^2,x)-x^2] \\
&\quad -2x[\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^r(y,x)-x]+x^2[\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1,x)-1]\} \\
&+ f_{\pm}^r(x)[\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1,x)-1].
\end{aligned}
$$

We certainly write

$$
\begin{aligned}
|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(f_{\pm}^r,x)-f_{\pm}^r(x)| \leqq \epsilon + \left(\epsilon+\frac{2\mathcal{K}_{\pm}^r c^2}{\delta^2}+\mathcal{K}_{\pm}^r\right) \\
|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1,x)-1| + \frac{4\mathcal{K}_{\pm}^r c}{\delta^2}|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^r(y,x)-x| \\
+\frac{2\mathcal{K}_{\pm}^r}{\delta^2}|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^r(y^2,x)-x^2|,
\end{aligned}
$$

where $c=\max\{|a|,|b|\}$. Consequently, we obtain

$$
\begin{aligned}
|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(f_{\pm}^r,x)-f_{\pm}^r(x)| \leqq \epsilon + \mathcal{M}_{\pm}^r(\epsilon)\Big(|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1,x)-1| \\
+|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(y,x)-x|+|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(y^2,x)-x^2|\Big), \qquad (14)
\end{aligned}
$$

where

$$
\mathcal{M}_{\pm}^r(\epsilon)=\max\left(\epsilon+\frac{2\mathcal{K}_{\pm}^r c^2}{\delta^2}+\mathcal{K}_{\pm}^r,\ \frac{4\mathcal{K}_{\pm}^r c}{\delta^2},\ \frac{2\mathcal{K}_{\pm}^r}{\delta^2}\right).
$$

Now it clearly follows from (5) that,

$$
\begin{aligned}
\mathcal{D}^*(\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m(f),f) &= \sup_{x\in E}\mathcal{D}\left(\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m(f;x),f(x)\right) \\
&= \sup_{x\in E}\sup_{r\in[0,1]}\max\left\{\left|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(f_-^r;x)-f_-^r\right|,\left|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(f_+^r;x)-f_+^r(x)\right|\right\}.
\end{aligned}
$$

Considering (14) with the last equality, one can easily write

$$
\begin{aligned}
\frac{\mathcal{D}^*(\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m(f),f)}{|\sigma(x)|} \leqq \sup_{x\in E}\frac{\epsilon}{\sigma(x)}+\mathcal{M}(\epsilon)\Bigg(\sup_{x\in E}\left|\frac{\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1,x)-1}{\sigma_0(x)}\right| \\
+\sup_{x\in E}\left|\frac{\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(y,x)-x}{\sigma_1(x)}\right|+\sup_{x\in E}\left|\frac{\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(y^2,x)-x^2}{\sigma_2(x)}\right|\Bigg),
\end{aligned}
$$

where

$$
\mathcal{M}(\epsilon)=\sup_{r\in[0,1]}\max\left\{\mathcal{M}_-^r(\epsilon),\mathcal{M}_+^r(\epsilon)\right\} \quad \text{and}
$$

$$
\sigma(x)=\max\{|\sigma_i(x)|:|\sigma_i(x)|>0, i=0,1,2\}.
$$

Therefore,

$$
s_{b_n-m}t_m\frac{\mathcal{D}^*(\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m(f),f)}{|\sigma(x)|} \leqq s_{b_n-m}t_m\sup_{x\in E}\frac{\epsilon}{\sigma(x)}
$$

$$+\mathcal{M}(\epsilon)\Bigg(s_{b_n-m}t_m\sup_{x\in E}\left|\frac{\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1,x)-1}{\sigma_0(x)}\right|$$

$$+s_{b_n-m}t_m\sup_{x\in E}\left|\frac{\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(y,x)-x}{\sigma_1(x)}\right|$$

$$+s_{b_n-m}t_m\sup_{x\in E}\left|\frac{\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(y^2,x)-x^2}{\sigma_2(x)}\right|\Bigg). \tag{15}$$

Next, for given $\kappa > 0$, choose $\epsilon > 0$ such that $s_{b_n-m}t_m\sup_{x\in E}\frac{\epsilon}{\sigma(x)} < \kappa$. Then, we can write

$$\Theta_m(x;\epsilon\sigma) = \left|\left\{m : m \leqq R_{a_n+1}^{b_n} \quad \text{and} \quad s_{b_n-m}t_m\left(\frac{\mathcal{D}^*\left(\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m(f),f\right)}{|\sigma(x)|}\right) \geqq \epsilon'\right\}\right|$$

and

$$\Theta_{i,m}(x,\epsilon\sigma) = \Bigg|\Bigg\{m : m \leqq R_{a_n+1}^{b_n} \quad \text{and}$$
$$s_{b_n-m}t_m\left(\frac{\mathcal{D}^*\left(\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^* f_i(x), f_i(x)\right)}{|\sigma_i(x)|}\right) \geqq \frac{\epsilon' - \dfrac{\epsilon}{\sigma(x)}}{3\mathcal{M}_{\pm}^r}\Bigg\}\Bigg|,$$

we easily obtain from (15) that

$$\Theta_m(x,\epsilon\sigma(x)) \leqq \sum_{i=0}^{2}\Theta_{i,m}(x,\epsilon\sigma(x)).$$

Thus, we fairly have

$$\frac{\|\Theta_m(x,\epsilon\sigma(x))\|}{R_{a_n+1}^{b_n}} \leqq \sum_{i=0}^{2}\frac{\|\Theta_{i,m}(x,\epsilon\sigma(x))\|}{R_{a_n+1}^{b_n}}. \tag{16}$$

Consequently, by Definition 1(D_2) and under the above assumption for the implication in (6), the right-hand side of (16) seems tend to zero as $n \to \infty$. We thus get

$$\lim_{n\to\infty}\frac{\|\Theta_m(x,\epsilon\sigma)\|}{R_{a_n+1}^{b_n}} = 0 \quad (\epsilon > 0).$$

Hence, the implication in (7) is fairly true. This completes the proof of the Theorem.

5. Fuzzy Rate of Relatively Eqiu-statistical Convergence

We intend to investigate here the fuzzy rate of the relatively equi-statistical convergence of a sequence of fuzzy positive linear operators defined from $\mathcal{C}_{\mathcal{F}}(E)$ into itself based on the fuzzy modulus of continuity.

Definition 2. *Let (a_n) and (b_n) be sequences of non-negative integers, and also let (u_n) be a positive non-increasing sequence. A fuzzy number valued sequence (f_n) of functions is relatively Λ_n-equi-statistical convergent to a fuzzy number valued function f on E with rate $o(u_n)$, if for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{\Upsilon_n(x;\epsilon\sigma)}{u_n R_{a_n+1}^{b_n}} = 0$$

uniformly relatively with respect to $x \in E$ or, otherwise if

$$\lim_{n\to\infty} \frac{\|\Upsilon_n(x;\epsilon\sigma)\|_{\mathcal{C}_{\mathcal{F}}[0,1]}}{u_n R_{a_n+1}^{b_n}} = 0,$$

where

$$\Upsilon_n(x,\epsilon\sigma) = \left| \left\{ m : m \leqq R_{a_n+1}^{b_n} \quad \text{and} \quad s_{b_n-m} t_m \frac{\mathcal{D}^*\left(\Delta_{h,x}^{\alpha,\beta,\gamma} f_m(x), f(x)\right)}{|\sigma(x)|} \geqq \epsilon \right\} \right|.$$

Here, we write

$$St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\mathcal{D}^*\left(f_n(x), f(x)\right) = o(u_n) \quad \text{on } (E;\sigma).$$

We now need to prove the following Lemma.

Lemma 2. *Let (u_n) and (v_n) be two positive non-increasing sequences. Suppose the fuzzy valued sequence of functions (f_n) and $(g_n) \in \mathcal{C}_{\mathcal{F}}(E)$ satisfy the conditions:*

$$St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\mathcal{D}^*\left(f_n(x), f(x)\right) = o(u_n) \quad \text{on } (E;\sigma).$$

and

$$St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\mathcal{D}^*\left(g_n(x), g(x)\right) = o(v_n) \quad \text{on } (E;\sigma_1),$$

where $\sigma_0 > 0$ and $\sigma_1 > 0$; then all the following assertions are true:

(i) $St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\mathcal{D}^*\left(f_n(x)+g_n(x), f(x)+g(x)\right) = o(w_n)$ *on* $(E;\max\{\sigma_0,\sigma_1\})$;

(ii) $St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\mathcal{D}^*\left(f_n(x), f(x)\right)\mathcal{D}^*\left(g_n(x), g(x)\right) = o(u_n v_n)$ *on* $(E;\{\sigma_0,\sigma_1\})$*;*

(iii) $St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\mu\mathcal{D}^*\left(f_n(x), f(x)\right) = o(u_n)$ *on* $(E;\sigma_0)$, *for any scalar* μ*;*

(iv) $St^{rw}_{equi}(\Delta^{\alpha,\beta,\gamma}_{h,x})\left\{\mathcal{D}^*\left(f_n(x),f(x)\right)\right\}^{\frac{1}{2}}=o(u_n)$ *on* $(E;\sqrt{|\sigma_0(x)|})$,

where $w_n=\max\{u_n,v_n\}$.

Proof. For proving the assertion (i) of Lemma 2, we consider the following sets for which $\epsilon>0$ and $x\in E$:

$$\mathfrak{A}_n(x,\epsilon\sigma)=\left|\left\{m : m \leqq R^{b_n}_{a_n+1} \quad \text{and} \quad s_{b_n-m}t_m\frac{\mathcal{D}^*\left(\left[\Delta^{\alpha,\beta,\gamma}_{h,x}f_m+\Delta^{[r]}_{p,q}g_m\right](x),(f+g)(x)\right)}{|\sigma(x)|}\geqq\epsilon\right\}\right|,$$

$$\mathfrak{A}_{0,n}(x,\epsilon\sigma)=\left|\left\{m : m\leqq R^{b_n}_{a_n+1} \quad \text{and} \quad s_{b_n-m}t_m\frac{\mathcal{D}^*\left(\Delta^{\alpha,\beta,\gamma}_{h,x}f_m(x),f(x)\right)}{|\sigma_0(x)|}\geqq\frac{\epsilon}{2}\right\}\right|$$

and

$$\mathfrak{A}_{1,n}(x,\epsilon\sigma)=\left|\left\{m : m\leqq R^{b_n}_{a_n+1} \quad \text{and} \quad s_{b_n-m}t_m\frac{\mathcal{D}^*\left(\Delta^{\alpha,\beta,\gamma}_{h,x}g_m(x),g(x)\right)}{|\sigma_1(x)|}\geqq\frac{\epsilon}{2}\right\}\right|,$$

where

$$\sigma(x)=\max\{|\sigma_i(x)| : i=0,1\}.$$

Clearly, we have

$$\mathfrak{A}_n(x,\epsilon\sigma)\subseteq\mathfrak{A}_{0,n}(x,\epsilon\sigma)\cup\mathfrak{A}_{1,n}(x,\epsilon\sigma).$$

Moreover, since

$$w_n=\max\{u_n,v_n\}, \tag{17}$$

by using the assertion (7) of Theorem 1, we obtain

$$\frac{\|\mathfrak{A}_n(x,\epsilon\sigma)\|_{\mathcal{C}_{\mathcal{F}}(E)}}{w_nR^{b_n}_{a_n+1}}\leqq\frac{\|\mathfrak{A}_{0,n}(x,\epsilon\sigma)\|_{\mathcal{C}_{\mathcal{F}}(E)}}{u_nR^{b_n}_{a_n+1}}+\frac{\|\mathfrak{A}_{1,n}(x,\epsilon\sigma)\|_{\mathcal{C}_{\mathcal{F}}(E)}}{v_nR^{b_n}_{a_n+1}}. \tag{18}$$

Also, by using the assertion (6) of Theorem 1, we obtain

$$\frac{\|\mathfrak{A}_n(x,\epsilon\sigma)\|_{\mathcal{C}_{\mathcal{F}}(E)}}{w_nR^{b_n}_{a_n+1}}=0. \tag{19}$$

Thus, assertion (i) of this Lemma is proved.

Next, since all other assertions (ii) - (iv) of Lemma 2 are similar as in the assertion (i), so these can be proved along similar lines to complete the proof of the Lemma 2.

The fuzzy modulus of continuity of f is such that $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ was studied by [2], It is defined by

$$\omega^{\mathcal{F}}(f, \delta) = \sup_{x,y \in [a,b]} \{\mathcal{D}^*(f(y), f(x)) : |y - x| \leqq \delta \quad (0 < \delta \leq a - b)\}. \tag{20}$$

We now introduce a theorem to obtain the fuzzy rates of relatively deferred Nörlund equi-statistical convergence based on difference sequence of functions under the support of the fuzzy modulus of continuity.

Theorem 2. *Let (a_n) and (b_n) be sequences of integers and let $\mathfrak{T}_m : \mathcal{C}_{\mathcal{F}}[a, b] \to \mathcal{C}_{\mathcal{F}}[a, b]$ $(m \in \mathbb{N})$ be a sequence of fuzzy positive linear operators. Suppose that $\{\mathfrak{T}_n^*\}_{n \in \mathbb{N}}$ be the corresponding sequence of positive linear operators from $\mathcal{C}[a, b]$ into itself such that (5) holds. Assume further that (u_n) and (v_n) be two positive non-increasing sequences and suppose that the operators $\{\mathfrak{T}_m^*\}_{n \in \mathbb{N}}$ satisfy the conditions:*

(i) $St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\mathfrak{T}_m^*(1, x) - 1 = o(u_n)$ *on* $(E; \sigma_0)$,

(ii) $St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\omega^{\mathcal{F}}(f, \delta_n) = o(v_n)$ *on* $(E; \sigma_1)$,

where

$$\delta_n(x) = \left\{\mathfrak{L}_m^*(\theta^2; x)\right\}^{\frac{1}{2}} \quad \text{and} \quad \theta(y) = (y - x),$$

then for each $f \in \mathcal{C}_{\mathcal{F}}(E)$, the assertion as below holds true:

$$St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\mathcal{D}^*\left(\mathfrak{T}_m(f), f\right) = o(w_n) \quad \text{on } (E; \sigma), \tag{21}$$

where (w_n) defined by (17) and

$$\sigma(x) = \max\{|\sigma_0(x)|, |\sigma_1(x)|, |\sigma_0(x)\sigma_1(x)| : \sigma_i(x) > 0 \quad (i = 0, 1)\}.$$

Proof. Suppose $E \subset \mathbb{R}$ be compact and let $f \in \mathcal{C}_{\mathcal{F}}(E)$, $x \in E$. Then, it is obvious that,

$$\mathcal{D}^*\left(\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m(f, x); f\right) \leqq \mathcal{N}|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1; x) - 1| + \Bigg(\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1; x) + \sqrt{\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1; x)}\Bigg)\omega^{\mathcal{F}}(f, \delta_n),$$

where

$$\mathcal{N} = \|f\|_{\mathcal{C}_{\mathcal{F}}(E)}.$$

Which yields

$$\frac{\mathcal{D}^*\left(\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m(f),f\right)}{\sigma(x)} \leqq \mathcal{N}\left(\frac{\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1;x)-1}{|\sigma_0(x)|}\right)+2\frac{\omega^{\mathcal{F}}(f,\delta_n)}{\sigma_1(x)}$$
$$+\frac{\omega^{\mathcal{F}}(f,\delta_n)}{\sigma_1(x)}\left(\frac{\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1;x)-1}{|\sigma_0(x)|}\right)$$
$$+\frac{\omega^{\mathcal{F}}(f,\delta_n)}{\sigma_1(x)}\sqrt{\left(\frac{\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1;x)-1}{|\sigma_0(x)|}\right)}. \tag{22}$$

Finally, for the conditions (i) and (ii) (of Theorem 2) along with Lemma 2, the last inequality (22) helps us to achieve the assertion (21). Therefore, proof of Theorem 2 is completed.

6. Concluding Remarks and Observations

In last section of our investigation, we present various further remarks and observations relating the different outcomes which we have proved here.

Remark 1. *If we put $a_n = 0$, $b_n = n$, $s_n = t_n = 1$ and $\alpha = \beta = \gamma = 0$ in our Theorem 1, then we get the statistical versions of the fuzzy Korovkin-type approximation theorem, which was demonstrated earlier by Anastassiou and Duman (see [3]).*

Remark 2. *Suppose in Theorem 2, we substitute the conditions (i) and (ii) by the following condition:*

$$St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\mathcal{D}^*(\mathfrak{T}_m^*(f_i,x)-f_i)=o(u_{n_i}) \quad on\ (E;\sigma_i). \tag{23}$$

Then, since

$$\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(\theta^2;x)=\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(y^2;x)-2x\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(y;x)+x^2\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(1;x),$$

we can write

$$\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(\theta^2;x)\leqq \kappa\sum_{i=0}^{2}|\Delta_{h,x}^{\alpha,\beta,\gamma}\mathfrak{T}_m^*(f_i;x)-f_i(x)|, \tag{24}$$

where

$$\kappa = 1+2\|f_1\|_{\mathcal{C}_{\mathcal{F}}(E)}+\|f_2\|_{\mathcal{C}_{\mathcal{F}}(E)}.$$

Clearly, from (23), (24) and Lemma 2, it follows that

$$St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\delta_n = St_{equi}^{rw}(\Delta_{h,x}^{\alpha,\beta,\gamma})\sqrt{\mathfrak{T}_m^*(\theta^2)}=o(u_n) \quad on\ (E;\sigma), \tag{25}$$

where

$$o(u_n) = \max\{u_{n_0}, u_{n_1}, u_{n_2}\}.$$

Thus, definitely, we obtain

$$St^{rw}_{equi}(\Delta^{\alpha,\beta,\gamma}_{h,x})\omega^{\mathcal{F}}(f,\delta) = o(u_n) \quad on\ (E;\sigma).$$

By applying (25) in Theorem 2, we instantly see that, $\forall\ f \in \mathcal{C}_{\mathcal{F}}(E)$*,*

$$St^{rw}_{equi}(\Delta^{\alpha,\beta,\gamma}_{h,x})\mathcal{D}^*\left(\mathfrak{T}_m(f), f\right) = o(u_n) \quad on\ (E;\sigma). \tag{26}$$

Therefore, instead of conditions (i) and (ii) of Theorem 2, if we use the condition (23), then we certainly find the fuzzy rates of the relatively $St^{rw}_{equi}(\Delta^{\alpha,\beta,\gamma}_{h,x})$*-equi-statistical convergence for the sequence* $(\mathfrak{T}^*_m)$ *of fuzzy positive linear operators in our Theorem 1.*

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