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# Adjunction and Localization in the Category $A$-Alg of $A$-Algebras

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*In memory of the dead of COVID-19*

**Abstract.** In our paper [3] we built the functor $\widehat{Ext}^n_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ in the category $A$-$Alg$. The purpose of this paper is to show that if $A$ is a ring not necessary commutative, $S$ a central multiplicatively closed subset of $A$ and $\mathscr{B}$ an $(A$-$A)$-bialgebra, then

$$Tor_n^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B}) : Alg\text{-}S^{-1}A \leftrightarrows S^{-1}A\text{-}Mod^o : \widehat{Ext}^n_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o$$

is an adjunction.

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**Key Words and Phrases**: Adjunction, Localization, Algebra, Functor

## 1. Introduction

In this paper, $A$ is assumed unitary, associative and not necessarily commutative. $\mathscr{A}$ and $\mathscr{B}$ are algebras assumed unitary, associative and not necessarily commutative as a ring and unital as an $A$-module. In general, the action of the functor $\widehat{Ext}^n_A(\text{-}, \mathscr{B})$ on an $A$-module $M$(resp. $A$-algebra $\mathscr{A}$) is not an algebra. In our paper [3] we built the functors $\widehat{Ext}^n_A(\text{-}, \mathscr{B}) : Alg\text{-}A \to B\text{-}Alg$ and $S^{-1}() : A\text{-}Alg \to S^{-1}(A)\text{-}Alg$. The notion of adjunction allows to see if two categories are equivalent. This notion of adjunction makes it possible to preserve some properties from one category to another, such as monomorphisms. The purpose of this paper is to show that $Tor_n^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B}) : Alg\text{-}S^{-1}A \to S^{-1}A\text{-}Mod^o$ and $\widehat{Ext}^n_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o : S^{-1}A\text{-}Mod^o \to Alg\text{-}S^{-1}A$ are adjoint functors, where $S$ is a central multiplicatively closed subset of $A$ and $\mathscr{B}$ a $(A$-$A)$-bialgebra but before, we show that the functors $\text{-} \otimes_A S^{-1}(A) : Alg\text{-}A \to A\text{-}Mod^o$ and $Hom_A(\text{-}, S^{-1}(A)) : A\text{-}Mod^o \to Alg\text{-}A$ are adjoint, then we show that the functors $S^{-1}() : A\text{-}Alg \to S^{-1}(A)\text{-}Alg$ and $\text{-} \otimes_A S^{-1}(A) :$

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$Alg\text{-}A \to A\text{-}Mod^o$ are naturally isomorphic and we deduce that $S^{-1}() : Alg\text{-}A \leftrightarrows A\text{-}Mod^o : Hom_A(\text{-}, S^{-1}(A))^o$ is an adjunction. This paper is divided into three parts. In the first part entitled "preliminary results" we recall some basic results. In the second part, we show that the functors $S^{-1}() : Alg\text{-}A \to A\text{-}Mod^o$ and $Hom_A(\text{-}, S^{-1}(A))^o : A\text{-}Mod^o \to Alg\text{-}A$ are adjoint and in the third part we show the main results of this paper (see Theorem 6).

## 2. Preliminary Results

**Proposition 1.** *Let $A$ and $\mathscr{A}$ be two rings, and $\theta : A \longrightarrow \mathscr{A}$ be a ring morphism. Then $\mathscr{A}$ has a structure of left (resp. right) $A$-module as the same way:*

$$\bullet : A \times \mathscr{A} \longrightarrow \mathscr{A}$$
$$(a, x) \longmapsto a \bullet x = \theta(a)x$$
$$(\textit{resp. } * : \mathscr{A} \times A \longrightarrow \mathscr{A}$$
$$(x, a) \longmapsto x * a = x\theta(a))$$

*Proof.* Easy.

In all that follows $\bullet$ (resp. $*$) designates the external law of the left (resp. right) $A$-module $\mathscr{A}$ relatively to $\theta$.

**Definition 1.** *Let $A$ and $\mathscr{A}$ be two rings, and $\theta : A \longrightarrow \mathscr{A}$ be a ring morphism. Then $(\mathscr{A}, +, \times, \bullet)$ (resp. $(\mathscr{A}, +, \times, *)$) is called left (resp. right) $A$-algebra relatively to $\theta$.*

This definition shows that to provide $\mathscr{A}$ with a structure of left (resp. right) $A$-algebra it suffices to have a ring morphism of $A$ into $\mathscr{A}$.

**Definition 2.** *[1] Let $A$ and $\mathscr{A}$ be two rings, and $\theta : A \longrightarrow \mathscr{A}$ be a ring morphism. If $Im(\theta) \subseteq Z(\mathscr{A})$, then $\mathscr{A}$ is called an $A$-algebra relatively to $\theta$.*

**Definition 3.** *Let $A$ be a ring, a subset $S$ of $A$ is called multiplicative if $1_A \in S$ and $S$ is stable by multiplication i.e for all $x$, $t \in S$, $xt \in S$.*

**Definition 4.** *Let $A$ be a ring and $S$ a multiplicative subset of $A$. We say that $S$ is closed if for all $s$, $s' \in A$ such that $ss' \in S \Rightarrow s \in S$ and $s' \in S$.*

**Definition 5.** *Let $S$ be a multiplicatively closed subset of a ring $A$. We say that $S$ satisfies the left Ore conditions if:*

*(i)* $\forall a \in A$, $\forall s \in S$ $\exists t \in S$ *and* $b \in A$ *such that* $ta = bs$

*(ii)* $\forall a \in A$, $\forall$ $s \in S$ *such that* $as = 0$, *then it exist* $t \in S$ *such that* $ta = 0$.

**Theorem 1.** *Let $A$ be a ring and $S$ a multiplicatively closed subset of $A$ satisfying the left Ore conditions. The binary relation defined in $S \times M$ by*

$$(s, m)\mathcal{R}(s', m') \Longleftrightarrow \exists x, y \in S : \begin{cases} xm = ym' \\ xs = ys' \end{cases}$$

*is an equivalence relation.*

*Proof.* See [4], [2] and [5].

**Theorem 2.** *Let $A$ be a ring not necessary commutative and $S$ a multiplicatively closed subset of $A$ satisfying the left Ore condition, then $S^{-1}A$ is a ring by the two following operations:*

- $\frac{a}{t} + \frac{b}{s} = \frac{xa+yb}{ys}$ *where* $x, y \in S : xt = ys$
- $\frac{a}{t} \times \frac{b}{s} = \frac{zb}{wt}$ *where* $(w, z) \in S \times A : wa = zs.$

*Proof.* see [4].

**Theorem 3.** *Let $\mathscr{A}$ be a left (resp. right) $A$-algebra and $S$ a central multiplicatively closed subset of $A$. Then $S^{-1}(\mathscr{A}) \in Ob(S^{-1}A\text{-}Alg)$ (resp. $S^{-1}\mathscr{A} \in Ob(Alg\text{-}S^{-1}A)$).*

*Proof.* See [3].

**Definition 6.** *Let $F, G : \mathscr{C} \to \mathscr{D}$ be two covariant functors. A natural transformation $\theta$ from $F$ to $G$ is an assignment to every object $X$ of $\mathscr{C}$ of a morphism $\theta_X \in Hom_{\mathscr{D}}(F(X), G(X))$ such that for any morphism $f \in Hom_{\mathscr{C}}(X, Y)$, the following diagram commutes in $\mathscr{D}$*

$$\begin{array}{ccc} F(X) & \xrightarrow{\theta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\theta_Y]{} & G(Y) \end{array} \quad \Leftrightarrow \quad G(f)\circ\theta_X = \theta_Y \circ F(f).$$

*If $\theta_X$ is an isomorphism for any object $X$ of $\mathscr{C}$, then $\theta$ is called functorial isomorphism .*

**Notation:** We note by $F \cong G$, if there is a functorial isomorphism $\theta : F \to G$.

**Definition 7.** *A pair of functors $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$ is an adjunction if we have for any $(X, Y) \in Ob(\mathscr{C}) \times Ob(\mathscr{D})$ a bijection*

$$\phi_{X,Y} : Hom_{\mathscr{D}}(FX, Y) \longrightarrow Hom_{\mathscr{C}}(X, GY)$$

*such that the following two squares are commutative:*

$$\begin{array}{ccc} Hom_{\mathscr{D}}(FX', Y) & \xrightarrow{\phi_{X',Y}} & Hom_{\mathscr{C}}(X', GY) \\ {\scriptstyle (Ff)^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ Hom_{\mathscr{D}}(FX, Y) & \xrightarrow{\phi_{X,Y}} & Hom_{\mathscr{C}}(X, GY) \end{array} \quad and \quad \begin{array}{ccc} Hom_{\mathscr{D}}(FX, Y) & \xrightarrow{\phi_{X,Y}} & Hom_{\mathscr{C}}(X, GY) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle (Gg)_*} \\ Hom_{\mathscr{D}}(FX, Y') & \xrightarrow{\phi_{X,Y'}} & Hom_{\mathscr{C}}(X, GY') \end{array}$$

*where $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are morphisms.*

**Proposition 2.** *If $F : \mathscr{C} \longrightarrow \mathscr{D}$ has two right (resp. left) adjoint $G$ and $H$, then $G$ and $H$ are naturally isomorphic.*
*Reciprocally, if $F$ is left (resp. right) adjoint to $G$, and $G$ is naturally isomorphic to $H$, then $F$ is also a left adjoint to $H$.*

**Proposition 3.** *Suppose $K_i$ is the $i$-th syzygy of some projective resolution of $\mathscr{B}$. Then $Tor_{n+1}(\text{-}, \mathscr{B})$ and $Tor_{n-i}(\text{-}, K_i)$ are naturally isomorphic functors; also $\widehat{Ext}^{n+1}(\text{-}, \mathscr{B})$ and $\widehat{Ext}^{n-i}(\text{-}, K_i)$ are naturally isomorphic functors.*

*Proof.* See [6, chap. 10] and [7, chap. 5].

## 3. The Adjunction of the Functors $S^{-1}()$ and $Hom_A(\text{-}, \mathscr{B})$ in the Category of $A$-Alg

**Theorem 4.** *Let $A$ be a ring and $S$ a central multiplicatively closed subset of $A$. Then the functors $S^{-1}()$ and $\text{-} \otimes_A S^{-1}(A)$ are naturally isomorphic ($S^{-1}() \cong \text{-} \otimes_A S^{-1}(A)$ ).*

*Proof.* Let $\theta : S^{-1}() \to \text{-} \otimes_A S^{-1}(A)$.
∗ Show that $\theta$ is a natural transformation.
Let $\mathscr{A} \in Ob(A\text{-}Alg)$.
Consider

$$\begin{aligned} \theta_{\mathscr{A}} : \mathscr{A} \times S^{-1}(A) &\longrightarrow S^{-1}(\mathscr{A}) \\ (x, \frac{a}{s}) &\longmapsto \frac{a \cdot x}{s} \end{aligned}$$

We have $\theta_{\mathscr{A}}$ which is $A$-bilinear, so by the universal property of tensor product, there exist a morphism of groups $\overline{\theta}_{\mathscr{A}} : \mathscr{A} \otimes S^{-1}(A) \longrightarrow S^{-1}(\mathscr{A})$ defined by

$$\overline{\theta}_{\mathscr{A}}(x_i \otimes \sum \frac{a_i}{s_i}) = \sum \frac{a_i \cdot x_i}{s_i}.$$

Let $f \in Hom_{A\text{-}Alg}(\mathscr{A}, \mathscr{A}')$, show that the following diagram is commutative

$$\begin{array}{ccc} \mathscr{A} \otimes S^{-1}(A) & \xrightarrow{\overline{\theta}_{\mathscr{A}}} & S^{-1}(\mathscr{A}) \\ \downarrow{\scriptstyle f \otimes 1_{S^{-1}(A)}} & & \downarrow{\scriptstyle S^{-1}(f)} \\ \mathscr{A}' \otimes S^{-1}(A) & \xrightarrow[\overline{\theta}_{\mathscr{A}'}]{} & S^{-1}(\mathscr{A}') \end{array} \quad \Leftrightarrow \quad S^{-1}(f) \circ \overline{\theta}_{\mathscr{A}} = \overline{\theta}_{\mathscr{A}'} \circ (f \otimes 1_{S^{-1}(A)}).$$

Let $x_i \otimes \sum \frac{a_i}{s_i} \in \mathscr{A} \otimes S^{-1}(A)$.
On the one hand we have,

$$\overline{\theta}_{\mathscr{A}'} \circ f \otimes 1_{S^{-1}(A)}(x_i \otimes \sum \frac{a_i}{s_i}) = \overline{\theta}_{\mathscr{A}'}(f(x_i) \otimes \sum \frac{a_i}{s_i}) = \sum \frac{a_i \cdot f(x_i)}{s_i}.$$

On the other hand

$$S^{-1}(f)\circ\overline{\theta}_{\mathscr{A}}(x_i\otimes\sum\frac{a_i}{s_i}) = S^{-1}(f)(\sum\frac{a_i\cdot x_i}{s_i}) = \sum S^{-1}(f)(\frac{a_i\cdot x_i}{s_i}) = \sum\frac{a_i\cdot f(x_i)}{s_i}.$$

So we have, $S^{-1}(f)\circ\overline{\theta}_{\mathscr{A}} = \overline{\theta}_{\mathscr{A}'}\circ(f\otimes 1_{S^{-1}(A)})$ and therefore $\theta$ is a natural transformation.
$*$ Show that for any $\mathscr{A}\in Ob(A\text{-}Alg)$, $\overline{\theta}_{\mathscr{A}}$ is bijective.
• Let $\frac{x}{s}\in S^{-1}(\mathscr{A})$.
We have $\overline{\theta}_{\mathscr{A}}(x\otimes\frac{1}{s}) = \frac{x}{s}\Rightarrow\overline{\theta}_{\mathscr{A}}$ is surjective.
• Let $x_i\otimes\sum\frac{a_i}{s_i}\in\mathscr{A}\otimes S^{-1}(A)$.
We have

$$\sum x_i\otimes\frac{a_i}{s_i} = \sum(\prod s_i)s_ia_ix_i\otimes\frac{1}{\prod s_i}.$$

Pose

$$s=\prod s_i \text{ and } z_i = s^{-1}s_i\in S.$$

So we have

$$\sum x_i\otimes\frac{a_i}{s_i} = \sum z_ia_ix_i\otimes\frac{1}{s} = (\sum z_ia_ix_i)\otimes\frac{1}{s}.$$

So the elements of $S^{-1}(A)\otimes\mathscr{A}$ are written in the form $\frac{1}{s}\otimes Y$, where $Y\in\mathscr{A}$ and $s\in S$.
Let $\frac{1}{s}\otimes Y\in Ker\overline{\theta}_{\mathscr{A}}\Leftrightarrow\overline{\theta}_{\mathscr{A}}(\frac{1}{s}\otimes Y) = 0_{S^{-1}\mathscr{A}}\Rightarrow\frac{Y}{s} = \frac{0_{\mathscr{A}}}{1}\Rightarrow\exists\ s_1,s_2\in S$ such that

$$\begin{cases} s_1Y = 0\\ s_1s = s_2\end{cases}$$

So $\frac{1}{s}\otimes Y = \frac{1}{ss_1}\otimes s_1Y = \frac{1}{ss_1}\otimes 0 = 0\Rightarrow Ker\overline{\theta}_{\mathscr{A}} = \{0_{S^{-1}\mathscr{A}}\}\Rightarrow\overline{\theta}_{\mathscr{A}}$ is injective.
So $\overline{\theta}_{\mathscr{A}}$ is bijective.

$$\text{Therefore } S^{-1}()\cong\text{-}\otimes_A S^{-1}(A).$$

**Theorem 5.** *Let A be a ring and S a central multiplicatively closed subset of A. Then*

$$\text{-}\otimes_A S^{-1}(A) : Alg\text{-}A\leftrightarrows A\text{-}Mod^o : Hom_A(\text{-},S^{-1}(A))^o$$

*is an adjunction.*

*Proof.* $*$ Let $\mathscr{A}\in Ob(Alg\text{-}A)$, $\mathscr{R}\in Ob(A\text{-}Mod^o)$.

$$\begin{aligned} \text{Let } \varphi_{\mathscr{A},\mathscr{R}} : Hom_{A\text{-}Mod^o}(\mathscr{A}\otimes_A S^{-1}(A),\mathscr{R}) &\longrightarrow Hom_{Alg\text{-}A}(\mathscr{A},Hom_A(\mathscr{R},S^{-1}(A))^o)\\ f &\longmapsto \varphi_{\mathscr{A},\mathscr{R}}(f) : \mathscr{A}\longrightarrow Hom_A(S^{-1}(A),\mathscr{R})\\ &\qquad x\longmapsto\varphi_{\mathscr{A},\mathscr{R}}(f)(x) : S^{-1}(A)\longrightarrow\mathscr{R}\\ &\qquad\qquad\frac{y}{s}\longmapsto f(x\otimes\frac{y}{s}).\end{aligned}$$

It is clear that $\varphi_{\mathscr{A},\mathscr{R}}$ is well defined.
Consider the map,

$$\begin{aligned}\psi : Hom_{Alg\text{-}A}(\mathscr{A}, Hom_A(\mathscr{R}, S^{-1}(A))^o) &\longrightarrow Hom_{A\text{-}Mod^o}(\mathscr{A} \otimes_A S^{-1}(A), \mathscr{R})\\ g &\longmapsto \psi(g) : \mathscr{A} \otimes_A S^{-1}(A) \longrightarrow \mathscr{R}\\ & \qquad x \otimes \frac{y}{s} \longmapsto \psi(g)(x \otimes \frac{y}{s}) = g(x)(\frac{y}{s}).\end{aligned}$$

Let $g \in Hom_{Alg\text{-}A}(\mathscr{A}, Hom_A(\mathscr{R}, S^{-1}(A))^o)$, $x \in \mathscr{A}$, $\frac{y}{s} \in S^{-1}(A)$,
we have $\varphi_{\mathscr{A},\mathscr{R}} \circ \psi(g)(x \otimes \frac{y}{s}) = \varphi_{\mathscr{A},\mathscr{R}}(\psi(g)(x \otimes \frac{y}{s})) = \varphi_{\mathscr{A},\mathscr{R}}(g(x)(\frac{y}{s})) = g(x \otimes \frac{y}{s})$.

Hence

$$\varphi_{\mathscr{A},\mathscr{R}} \circ \psi(g) = g, \forall\ g \in Hom_{Alg\text{-}A}(\mathscr{A}, Hom_A(\mathscr{R}, S^{-1}(A))^o).$$

So

$$\varphi_{\mathscr{A},\mathscr{R}} \circ \psi = 1_{Hom_{Alg\text{-}A}(\mathscr{A}, Hom_A(\mathscr{R}, S^{-1}(A))^o)}.$$

Similarly, we show that $\psi \circ \varphi_{\mathscr{A},\mathscr{R}} = 1_{Hom_{Alg\text{-}A}(\mathscr{A} \otimes S^{-1}(A), \mathscr{R})}$.
So

$$\varphi_{\mathscr{A},\mathscr{R}} \text{ is an isomorphism of left } A\text{-algebras.}$$

$*$ It remains to show that $\varphi_{\mathscr{A},\mathscr{R}}$ is natural in $\mathscr{A}$ and in $\mathscr{R}$.
Let $f : \mathscr{A} \longrightarrow \mathscr{A}'$ and $g : \mathscr{R} \longrightarrow \mathscr{R}'$ be two morphisms of left $A$-algebras.
Pose

$$F = \text{-} \otimes_A S^{-1}(A) \text{ and } G = Hom_A(\text{-}, S^{-1}(A))^o.$$

We have $f^* \circ \varphi_{\mathscr{A}',\mathscr{R}}(h)(x)(\frac{y}{s}) = \varphi_{\mathscr{A}',\mathscr{R}}(h) \circ f(x)(\frac{y}{s}) = \varphi_{\mathscr{A}',\mathscr{R}}(h)(f(x))(\frac{y}{s})$
$= h(f(x) \otimes \frac{y}{s})$.

On the other hand we have:
$\varphi_{\mathscr{A},\mathscr{R}} \circ (Ff)^*(h)(x)(\frac{y}{s}) = (Ff)^*(h)(x \otimes \frac{y}{s}) = h(Ff)(x \otimes \frac{y}{s}) = h(f \otimes 1_{S^{-1}(A)})(x \otimes \frac{y}{s})$
$= h(f(x) \otimes \frac{y}{s})$.

So

$f^* \circ \varphi_{\mathscr{A}',\mathscr{R}}(h)(x)(\frac{y}{s}) = \varphi_{\mathscr{A},\mathscr{R}} \circ (Ff)^*(h)(x)(\frac{y}{s})$, $\forall\ h \in Hom_A(\mathscr{A} \otimes S^{-1}(A), \mathscr{R})$, $x \in \mathscr{A}$ and $\frac{y}{s} \in S^{-1}(A)$.

So $f^* \circ \varphi_{\mathscr{A}',\mathscr{R}} = \varphi_{\mathscr{A},\mathscr{R}} \circ (Ff)^* \Rightarrow \varphi_{\mathscr{A},\mathscr{R}}$ is natural in $\mathscr{A}$.
We show in the same way that $\varphi_{\mathscr{A},\mathscr{R}}$ is natural in $\mathscr{R}$.
Therefore the functors - $\otimes_A S^{-1}(A)$ and $Hom_A(\text{-}, S^{-1}(A))^o$ are adjoint.

**Corollary 1.** *Let A be a ring and S a central multiplicatively closed subset of A. Then*

$$S^{-1}() : Alg\text{-}A \leftrightarrows A\text{-}Mod^o : Hom_A(\text{-}, S^{-1}(A))^o$$

*is an adjunction.*

*Proof.* By the theorem 4, we have - $\otimes_A S^{-1}(A)$ is a left adjoint to $Hom_A(\text{-}, S^{-1}(A))^o$ and by the theorem 4 $S^{-1}()$ is isomorphic to $Hom_A(\text{-}, S^{-1}(A))^o$, so the fonctor $S^{-1}()$ is a left adjoint to $Hom_A(\text{-}, S^{-1}(A))^o$.

**Corollary 2.** *Let A be a duo ring, P a prime ideal of A and $S = (A\text{-}P) \cap Z(A)$. Then*

$$S^{-1}() : Alg\text{-}A \leftrightarrows A\text{-}Mod^o : Hom_A(\text{-}, S^{-1}(A))^o$$

*is an adjunction.*

*Proof.* Since $A$ is a duo ring, then $A$-$P$ is a multiplicatively closed subset of $A$, so $S = (A\text{-}P) \cap Z(A)$ is a central multiplicatively closed subset of $A$.
So by the corollary 1 $S^{-1}()$ is a left adjoint to $Hom_A(\text{-}, S^{-1}(A))^o$.

**Corollary 3.** *Let A be a duo ring, P a prime ideal of A, $S_R$ the set of regular elements of A-P and $S = S_R \cap Z(A)$. Then*

$$S^{-1}() : Alg\text{-}A \leftrightarrows A\text{-}Mod^o : Hom_A(\text{-}, S^{-1}(A))^o$$

*is an adjunction.*

*Proof.* Since $A$ is a duo ring, then the set of regular elements, $S_R$, of $A$-$P$ is a multiplicatively closed subset of $A$, so $S = S_R \cap Z(A)$ is a central multiplicatively closed subset of $A$.
So by the corollary 1 $S^{-1}()$ is a left adjoint to $Hom_A(\text{-}, S^{-1}(A))^o$.

**Proposition 4.** *Let A be a ring, $\mathscr{B}$ a (A-A)-bialgebra and S a central multiplicatively closed subset of A. Then*

$$\text{-} \otimes_A S^{-1}(\mathscr{B}) : Alg\text{-}S^{-1}A \leftrightarrows S^{-1}A\text{-}Mod^o : Hom_A(\text{-}, S^{-1}(\mathscr{B}))^o$$

*is an adjunction.*

*Proof.* The proof is similar to that of the previous theorem.

## 4. The Adjunction of the Functors $\widehat{Ext}^n_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ and $Tor_n^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ in the Category $A$-Alg

**Proposition 5.** *Let $\mathscr{B}$ be a $(B$-$A)$-bialgebra. Then the correspondence*

$$\widehat{Ext}^n_B(\text{-}, \mathscr{B}) : B\text{-}Mod \longrightarrow Alg\text{-}A$$

*(i) which has any left $B$-module $M$, we associate the right $A$-algebra $\widehat{Ext}^n_B(M, \mathscr{B})$,*

*(ii) which has any morphism of left $B$-modules $f : M \longrightarrow M'$, we associate $\widehat{Ext}^n_B(f, \mathscr{B}) : \widehat{Ext}^n_B(M', \mathscr{B}) \to \widehat{Ext}^n_B(M, \mathscr{B})$*

*is a contravariant functor.*

*Proof.* $*$ We have $M \in Ob(B\text{-}Mod) \Rightarrow \widehat{Ext}^n_B(M, \mathscr{B}) \in Ob(Alg\text{-}A)$ (see [3]), so the action of $\widehat{Ext}^n_B(\text{-}, \mathscr{B})$ on the objects of $Alg$-$A$ makes sense.

$*$ Let $f : M \to M'$ be a morphism of left $B$-modules. By the comparison theorem we have the following commutative diagram

$$\begin{array}{ccccccccccc}
P_M : \ldots & \longrightarrow & P_n & \longrightarrow & P_{n-1} \cdots & \longrightarrow & P_0 & \longrightarrow & \mathscr{A} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \tilde{f}_n} & & \downarrow{\scriptstyle \tilde{f}_{n-1}} & & \downarrow{\scriptstyle \tilde{f}_0} & & \downarrow{\scriptstyle f} & & \\
P_{M'} : \ldots & \longrightarrow & P'_n & \longrightarrow & P'_{n-1} \cdots & \longrightarrow & P'_0 & \longrightarrow & \mathscr{B} & \longrightarrow & 0
\end{array}$$

By applying the contravariant functor $Hom_B(\text{-}, \mathscr{B})$ we have

$$\begin{array}{ccccc}
Hom_B(P_M, \mathscr{B}) : 0 & \longrightarrow & Hom_B(M, \mathscr{B}) & \longrightarrow & Hom_B(P_0, \mathscr{B}) \cdots \\
\downarrow{\scriptstyle Hom_B(\tilde{f}, \mathscr{B})} & & \downarrow{\scriptstyle Hom_B(f, \mathscr{B})} & & \downarrow{\scriptstyle Hom_B(\tilde{f}_0, \mathscr{B})} \\
Hom_B(P_{M'}, \mathscr{B}) : 0 & \longrightarrow & Hom_B(M', \mathscr{B}) & \longrightarrow & Hom_B(P'_0, \mathscr{B}) \cdots
\end{array}$$

So $Hom_B(\tilde{f}, \mathscr{B}) : Hom_B(P_M, \mathscr{B}) \longrightarrow Hom_B(P_{M'}, \mathscr{B})$ is a morphism of chain complex. We have

$$\begin{aligned}
H_n(Hom_B(\tilde{f}, \mathscr{B})) : H_n(Hom_B(P_M, \mathscr{B})) &\longrightarrow \underline{H_n(Hom_B(P_{M'}, \mathscr{B})})\\
\overline{z_n} &\longmapsto \overline{Hom_B(\tilde{f}_n, \mathscr{B}) z_n}.
\end{aligned}$$

$Hom_B(f, \mathscr{B}) = f^*$ and $Hom_B(\tilde{f}_n, \mathscr{B}) = \tilde{f}_n^*$ are morphisms of right $A$-algebras (see [3]). So $H_n(Hom_B(\tilde{f}, \mathscr{B})) = \widehat{Ext}^n_B(f, \mathscr{B})$ is a morphism of right $A$-algebras, so the action of $\widehat{Ext}^n_B(\text{-}, \mathscr{B})$ on the arrow makes sense.

$*$ We have

$$\begin{aligned}
\widehat{Ext}^n_B(g \circ f, \mathscr{B}) &= H_n(Hom_B(g \tilde{\circ} f, \mathscr{B})) = H_n(Hom_B(\tilde{g} \circ \tilde{f}, \mathscr{B})) \\
&= H_n[Hom_B(\tilde{f}, \mathscr{B}) \circ Hom_B(\tilde{g}, \mathscr{B})] \\
&= H_n(Hom_B(\tilde{f}, \mathscr{B})) \circ H_n(Hom_B(\tilde{g}, \mathscr{B})) \\
&= \widehat{Ext}^n_B(f, \mathscr{B}) \circ \widehat{Ext}^n_B(g, \mathscr{B}).
\end{aligned}$$

$*$ We have $\widehat{Ext}^n_B(1_M, \mathscr{B})(\overline{z_n}) = \overline{Hom_B((\tilde{1}_M)_n, \mathscr{B})(z_n)} = \overline{1_{Hom_B(M,\mathscr{B})}(z_n)} = \overline{z_n}$.
So

$$\widehat{Ext}^n_B(1_M, \mathscr{B}) = 1_{\widehat{Ext}^n_B(M,\mathscr{B})}.$$

Therefore $\widehat{Ext}^n_B(\text{-}, \mathscr{B}) : B\text{-}Mod \longrightarrow Alg\text{-}A$ is a contravariant functor.

**Proposition 6.** *Let $\mathscr{B}$ be a $(B\text{-}A)$-bialgebra.Then the correspondence*

$$\widehat{Ext}^n_B(\text{-}, \mathscr{B})^o : B\text{-}Mod^o \longrightarrow Alg\text{-}A$$

*(i) which has any left $B$-module $M$, we associate the right $A$-algebra $\widehat{Ext}^n_B(\text{-}, \mathscr{B})^o(M) = \widehat{Ext}^n_B(M, \mathscr{B})$,*

*(ii) which has any $f \in Hom_{B\text{-}Mod^o}(M, M')$, we associate $\widehat{Ext}^n_B(\text{-}, \mathscr{B})^o(f) = \widehat{Ext}^n_B(f, \mathscr{B})$.*

*is a covariant functor.*

*Proof.* $*$ Let $M \in Ob(B\text{-}Mod^o)$, we have $\widehat{Ext}^n_B(\text{-}, \mathscr{B})^o(M) = \widehat{Ext}^n_B(M, \mathscr{B}) \in Ob(Alg\text{-}A)$, so the action of $\widehat{Ext}^n_B(\text{-}, \mathscr{B})^o$ on the objects of $B\text{-}Mod^o$ makes sense.
$*$ Let $f \in Hom_{B\text{-}Mod^o}(M, M')$.
Since $Hom_{B\text{-}Mod^o}(M, M') = Hom_{B\text{-}Mod}(M', M)$, then $f \in Hom_{B\text{-}Mod}(M', M)$, so $\widehat{Ext}^n_B(f, \mathscr{B}) \in Hom_{Alg\text{-}A}(\widehat{Ext}^n_B(M, \mathscr{B}), \widehat{Ext}^n_B(M', \mathscr{B}))$ because $\widehat{Ext}^n_B(f, \mathscr{B})$ is a contravariant functor.
And hence

$$\widehat{Ext}^n_B(\text{-}, \mathscr{B})^o(f) = \widehat{Ext}^n_B(f, \mathscr{B}) \in Hom_{Alg\text{-}A}(\widehat{Ext}^n_B(M, \mathscr{B}), \widehat{Ext}^n_B(M', \mathscr{B})).$$

Therefore the action of $\widehat{Ext}^n_B(\text{-}, \mathscr{B})^o$ on the arrow makes sens.
$*$ Let $f \in Hom_{B\text{-}Mod^o}(M, M')$, $g \in Hom_{B\text{-}Mod^o}(M', M'')$.
We have

$$\begin{aligned} \widehat{Ext}^n_B(g \circ f, \mathscr{B})^o = \widehat{Ext}^n_B(g \circ f, \mathscr{B}) &= \widehat{Ext}^n_B(f, \mathscr{B}) \circ \widehat{Ext}^n_B(g, \mathscr{B}) \\ &= \widehat{Ext}^n_B(f, \mathscr{B})^o \circ_{B\text{-}Mod^o} \widehat{Ext}^n_B(g, \mathscr{B})^o \\ &= \widehat{Ext}^n_B(g, \mathscr{B})^o \circ_{B\text{-}Mod} \widehat{Ext}^n_B(f, \mathscr{B})^o. \end{aligned}$$

Therefore $\widehat{Ext}^n_B(\text{-}, \mathscr{B})^o : B\text{-}Mod^o \longrightarrow Alg\text{-}A$ is a covariant functor.

**Theorem 6.** *Let $A$ be a ring, $S$ a central multiplicatively closed subset of $A$ and $\mathscr{B}$ a $(A\text{-}A)$-bialgebra. Then*

$$Tor^{S^{-1}A}_n(\text{-}, S^{-1}\mathscr{B}) : Alg\text{-}S^{-1}A \leftrightarrows S^{-1}A\text{-}Mod^o : \widehat{Ext}^n_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o$$

*is an adjunction.*

*Proof.* $*$ For $n = 0$,
$F = Tor_n^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B}) = \text{-}\otimes_{S^{-1}A}S^{-1}\mathscr{B}$ and $G = \widehat{Ext}^n_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o = Hom_{S^{-1}A}(\text{-}, \mathscr{B})^o$.
By the proposition 4, the functors $\text{-}\otimes_{S^{-1}A}S^{-1}\mathscr{B} = Tor_0^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ and $Hom_{S^{-1}A}(\text{-}, \mathscr{B})^o = \widehat{Ext}^0_{S^{-1}A}(\text{-}, \mathscr{B})^o$ are adjoint.
So for $n = 0$ the property is verified.

$*$ Suppose that the property is true up to the order $n$, i.e. the functors $F = Tor_n^{S^{-1}A}(\text{-}, \mathscr{B})$ and $G = \widehat{Ext}^n_{S^{-1}A}(\text{-}, \mathscr{B})^o$ are adjoint.

$*$ Show that the functors $F = Tor_{n+1}^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ and $G = \widehat{Ext}^{n+1}_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o$ are adjoint.
Let
$P_{S^{-1}\mathscr{B}} : \cdots \longrightarrow S^{-1}P_{n+1} \xrightarrow{d_{n+1}} S^{-1}P_n \xrightarrow{d_n} \cdots \longrightarrow S^{-1}P_2 \xrightarrow{d_2} S^{-1}P_1 \xrightarrow{d_1} S^{-1}P_0 \xrightarrow{\epsilon} S^{-1}\mathscr{B} \longrightarrow 0$ be a projective resolution of $S^{-1}\mathscr{B}$.
Pose

$$K_0 = Ker\epsilon \text{ and } K_n = Kerd_n, \forall\ n \geq 1.$$

By the proposition 3 we have:

$Tor_{n+1}^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ and $Tor_1^{S^{-1}A}(\text{-}, K_{n-1})$ are naturally isomorphic functors.

$\widehat{Ext}^{n+1}_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o$ and $\widehat{Ext}^1_{S^{-1}A}(\text{-}, K_{n-1})^o$ are naturally isomorphic functors.

According to the inductive hypothesis, the functors $Tor_1^{S^{-1}A}(\text{-}, K_{n-1})$ and $\widehat{Ext}^1_{S^{-1}A}(\text{-}, K_{n-1})^o$ are adjoint.

So we have $\widehat{Ext}^1_{S^{-1}A}(\text{-}, K_{n-1})^o$ and $Tor_1^{S^{-1}A}(\text{-}, K_{n-1})$ which adjoint, also $Tor_1^{S^{-1}A}(\text{-}, K_{n-1})$ and $Tor_{n+1}^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ are naturally isomorphic functors, so by the proposition 2 the functors $\widehat{Ext}^1_{S^{-1}A}(\text{-}, K_{n-1})^o$ and $Tor_{n+1}^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ are adjoint.

So we have $Tor_{n+1}^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ and $\widehat{Ext}^1_{S^{-1}A}(\text{-}, K_{n-1})^o$ which are adjoint, also $\widehat{Ext}^1_{S^{-1}A}(\text{-}, K_{n-1})^o$ and $\widehat{Ext}^{n+1}_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o$ are naturally isomorphic functors, so by the proposition 2 the functors $Tor_{n+1}^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ and $\widehat{Ext}^{n+1}_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o$ are adjoint.

Hence, the functors $Tor_n^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ and $\widehat{Ext}^n_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o$ are adjoint for all $n \geq 0$ .

**Corollary 4.** *Let $A$ be a duo ring, $P$ a prime ideal of $A$, $S = (A\text{-}P) \cap Z(A)$ and $\mathscr{B}$ an $(A\text{-}A)$-bialgebra. Then*

$$Tor_n^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B}) : Alg\text{-}S^{-1}A \leftrightarrows S^{-1}A\text{-}Mod^o : \widehat{Ext}^n_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o$$

*is an adjunction.*

*Proof.* Since $A$ is a duo ring, then $A$-$P$ is a multiplicatively closed subset of $A$, so $S = (A\text{-}P) \cap Z(A)$ is a central multiplicatively closed subset of $A$.
So by the previous theorem $Tor_n^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ is a left adjoint to $\widehat{Ext}^n_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o$.

**Corollary 5.** *Let $A$ be a duo ring, $P$ a prime ideal of $A$, $S_R$ the set of regular elements of $A$-$P$, $S = S_R \cap Z(A)$ and $\mathscr{B}$ an $(A\text{-}A)$-bialgebra. Then*

$$Tor_n^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B}) : Alg\text{-}S^{-1}A \leftrightarrows S^{-1}A\text{-}Mod^o : \widehat{Ext}^n_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o$$

*is an adjunction.*

*Proof.* Since $A$ is a duo ring, then the set of regular elements $S_R$ of $A$-$P$ is a multiplicatively closed subset of $A$, so $S = S_R \cap Z(A)$ is a central multiplicatively closed subset of $A$.
So by the previous theorem $Tor_n^{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})$ is a left adjoint to $\widehat{Ext}^n_{S^{-1}A}(\text{-}, S^{-1}\mathscr{B})^o$.

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