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Special Issue Dedicated to Professor Hari M. Srivastava On the Occasion of his 80th Birthday

On various formulas with q-integralsand their applications to q-hypergeometric functions

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Abstract. We present three q-Taylor formulas with q-integral remainder. The two last proofs require a slight rearrangement by a well-known formula. The first formula has been given in different form by Annaby and Mansour. We give concise proofs for q-analogues of Eulerian integral formulas for general q-hypergeometric functions corresponding to Erdélyi, and for two of Srivastavas triple hypergeometric functions and other functions. All proofs are made in a similar style by using q-integration. We find some new formulas for fractional q-integrals including a series expansion. In the same way, the operator formulas by Srivastava and Manocha find a natural generalization.

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Key Words and Phrases: q-Taylor formulas with q-integral remainder, q-hypergeometric function, q-Eulerian integral, fractional q-integral

1. Introduction

1.1. General

The aim of this paper is to continue the investigation of single and multiple q-hypergeometric series in the spirit of our book [9] and our paper [12] on the q-Lauricella functions. The fractional q-integrals and direct computations of q-integrals lead to similar results. We refer to previous papers with respect to convergence regions. By quoting Erdélyi and Feldheim, we have managed to save these hypergeometric formulas from oblivion; our proofs of their formulas are quite similar, although these authors never wrote down their proofs. In the same style, Charles Cailler in 1920 [3], [20, p. 242] and Kampé de Fériet in 1922 [19, p.

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26] published a hypergeometric formula and a fractional integral formula, respectively. We will q-deform the latter formula, and show that it is a special case of an operator formula by Srivastava and Manocha [23, p.289 (18)]. Instead, Koschmieder [20, p. 252], for the first time, published a formula with a double Eulerian integral for a fractional derivative of a double hypergeometric series times power functions. By a standard procedure we are able to prove a q-analogue of Koschmieders formula. In the end, we use a generalization of the q-binomial theorem to find a q-analogue of an operator formula by Srivastava and Manocha [23, p.306].

This paper is organized as follows: In section 1.1 we prove the three q-Taylor formulas by using q-integration by parts. Erdélyi formulas and fractional q-integrals are dicussed in section 1.1. Finally, in section **??** we consider similar, more complicated formulas in the spirit of Srivastava and Manocha [23]. `\section{Three`q-Taylor formulas In a recent paper [2] Annaby and Mansour have found two q-Taylor formulas with q-integral remainder: [9, (8.90)] and a formula similar to (2). We are going to prove (2) and two other q-Taylor formulas which use the two types of q-addition. All proofs use q-integration by parts. To be able to work freely, we consider functions in $\mathbb{C}[[x]]$. We use the following definition.

Definition 1.

$$\mathrm{P}_{n,q}(x,a) \equiv \prod_{m=0}^{n-1}(x+aq^m),\ \ n=1,2,\ldots. \tag{1}$$

A different form of the following theorem, without remainder, occurred in Al-Salam, Verma [1, 2.2] [2, p. 480 4.6].

Theorem 1. *Let $0<|q|<1$ and let f be n times q-differentiable in the closed interval $[a,x]$. Then the following generalization of Jackson's formula holds for $n=1,2,\ldots$:*

$$\begin{aligned} f(x) = \sum_{k=0}^{n-1} \frac{(-1)^k q^{-\binom{k}{2}} \mathrm{P}_{k,q}(a,-x)}{\{k\}_q!} (\mathrm{D}_q^k f)(aq^{-k}) + \\ \int_{t=a}^{x} \frac{(-1)^{n-1} q^{-\binom{n}{2}} \mathrm{P}_{n-1,q}(t,-x)}{\{n-1\}_q!} (\mathrm{D}_q^n f)(tq^{-n})\, d_q(t). \end{aligned} \tag{2}$$

Proof. We use q-integration by parts We start with

$$f(x) = f(a) + \int_{t=a}^{x} \mathrm{D}_q f(t) D_{q,t}(t-x)\, d_q(t), \tag{3}$$

which follows from the definition of q-integral. At the next step we obtain

$$f(x) = f(a) + \left[\mathrm{D}_q f(tq^{-1})(t-x)\right]_{t=a}^{t=x} - \int_{t=a}^{x} q^{-1}(t-x)\mathrm{D}_q^2 f(tq^{-1})\, d_q(t). \tag{4}$$

We proceed with

$$\begin{aligned} f(x) = f(a) + \mathrm{D}_q f(aq^{-1})(x-a) - \left[\mathrm{D}_q^2 f(tq^{-2}) \frac{q^{-1}}{\{2\}_q!}(t^2 - xt(1+q) + qx^2)\right]_{t=a}^{t=x} \\ + \int_{t=a}^{x} \mathrm{D}_q^3 f(tq^{-2}) \frac{q^{-3}\mathrm{P}_{2,q}(t,-x)}{\{2\}_q!}\, d_q(t). \end{aligned} \tag{5}$$

In the third step we obtain

$$
\begin{aligned}
& f(x)=f(a)-\mathrm{D}_q f(aq^{-1})\mathrm{P}_{1,q}(a,-x)+\mathrm{D}_q^2 f(aq^{-2})\frac{1}{\{2\}_q!}\mathrm{P}_{2,q}(a,-x)+ \\
& \left[\mathrm{D}_q^3 f(tq^{-3})\frac{q^{-3}\mathrm{P}_{3,q}(t,-x)}{\{3\}_q!}\right]_{t=a}^{t=x}-\int_{t=a}^{x}\mathrm{D}_q^4 f(tq^{-4})\frac{q^{-6}\mathrm{P}_{3,q}(t,-x)}{\{3\}_q!}\,d_q(t).
\end{aligned}
\tag{6}
$$

We can continue this process forever.

We are going to use the following formula in the last proofs:

Theorem 2. *von Grüson [22, S. 36] 1814, [9, 2.19].*

$$
\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}}u^n=(u;q)_m. \tag{7}
$$

Our next aim is to find q-Taylor expansions with q-integral remainder for formulas corresponding to Nalli–Ward and Jackson respectively. These formulas are (18) and (22). We first prove preliminary lemmata (8) and (13), and then show by (7) that these are equivalent to the formulas we want to prove.

Lemma 1.

$$
\begin{aligned}
& F(x\oplus_q y)=F(x)+\sum_{k=1}^{n-1}\frac{y^k}{\{k\}_q!}(-1)^{k+1}q^{\binom{k}{2}}D_{q,y}^k F(x\oplus_q y)+ \\
& \int_{t=0}^{y}D_{q,t}^n\left[F(x\oplus_q t)\right]\frac{(-t)^{n-1}}{\{n-1\}_q!}q^{\binom{n}{2}}\,d_q(t).
\end{aligned}
\tag{8}
$$

Proof. The proof is very straightforward, at each step we only use integration by parts. We start with

$$
(x\oplus_q y)^m=x^m+\int_{t=0}^{y}D_{q,t}\left[(x\oplus_q t)^m\right]D_{q,t}(t)\,d_q(t), \tag{9}
$$

which follows from the definition of q-integral. At the next step we obtain

$$
(x\oplus_q y)^m=x^m+\left[D_{q,t}\left[(x\oplus_q t)^m\right]t\right]_{t=0}^{t=y}-\int_{t=0}^{y}qtD_{q,t}^2\left[(x\oplus_q t)^m\right]\,d_q(t). \tag{10}
$$

We proceed with

$$
\begin{aligned}
& (x\oplus_q y)^m=x^m+yD_{q,y}((x\oplus_q y)^m)-\left[D_{q,t}^2\left[(x\oplus_q t)^m\right]\frac{qt^2}{\{2\}_q!}\right]_{t=0}^{t=y} \\
& +\int_{t=0}^{y}D_{q,t}^3\left[(x\oplus_q t)^m\right]\frac{q^3t^2}{\{2\}_q!}\,d_q(t).
\end{aligned}
\tag{11}
$$

In the third step we obtain

$$(x \oplus_q y)^m = x^m + yD_{q,y}(x \oplus_q y)^m - D_{q,y}^2 \left[(x \oplus_q y)^m\right] \frac{qy^2}{\{2\}_q!} + \left[D_{q,t}^3 \left[(x \oplus_q t)^m\right] \frac{q^3t^3}{\{3\}_q!}\right]_{t=0}^{t=y} - \int_{t=0}^{y} D_{q,t}^4 \left[(x \oplus_q t)^m\right] \frac{q^6t^3}{\{3\}_q!} \, d_q(t). \tag{12}$$

We can continue this process forever.

Lemma 2.

$$F(x \boxplus_q y) = F(x) + \sum_{k=1}^{n-1} \frac{y^k}{\{k\}_q!}(-1)^{k+1} q^{\binom{k}{2}} D_{q,y}^k F(x \boxplus_q y) + \int_{t=0}^{y} D_{q,t}^n \left[F(x \boxplus_q t)\right] \frac{(-t)^{n-1}}{\{n-1\}_q!} q^{\binom{n}{2}} \, d_q(t). \tag{13}$$

Proof. The proof is almost the same as the previous one. We start with

$$(x \boxplus_q y)^m = x^m + \int_{t=0}^{y} D_{q,t} \left[(x \boxplus_q t)^m\right] D_{q,t}(t) \, d_q(t), \tag{14}$$

which follows from the definition of q-integral. At the next step we obtain

$$(x \boxplus_q y)^m = x^m + \left[D_{q,t} \left[(x \boxplus_q t)^m\right] t\right]_{t=0}^{t=y} - \int_{t=0}^{y} qtD_{q,t}^2 \left[(x \boxplus_q t)^m\right] \, d_q(t). \tag{15}$$

We proceed with

$$(x \boxplus_q y)^m = x^m + yD_{q,y}((x \boxplus_q y)^m) - \left[D_{q,t}^2 \left[(x \boxplus_q t)^m\right] \frac{qt^2}{\{2\}_q!}\right]_{t=0}^{t=y} + \int_{t=0}^{y} D_{q,t}^3 \left[(x \boxplus_q t)^m\right] \frac{q^3t^2}{\{2\}_q!} \, d_q(t). \tag{16}$$

In the third step we obtain

$$(x \boxplus_q y)^m = x^m + yD_{q,y}(x \boxplus_q y)^m - D_{q,y}^2 \left[(x \boxplus_q y)^m\right] \frac{qy^2}{\{2\}_q!} + \left[D_{q,t}^3 \left[(x \boxplus_q t)^m\right] \frac{q^3t^3}{\{3\}_q!}\right]_{t=0}^{t=y} - \int_{t=0}^{y} D_{q,t}^4 \left[(x \boxplus_q t)^m\right] \frac{q^6t^3}{\{3\}_q!} \, d_q(t). \tag{17}$$

We can continue this process forever.

Theorem 3. *Compare with the Nalli–Ward q-Taylor formula [9], which is obtained by letting $n \to \infty$.*

$$F(x \oplus_q y) = \sum_{k=0}^{n-1} \frac{y^k}{\{k\}_q!} D_q^k F(x) + \int_{t=0}^{y} D_{q,t}^n \left[F(x \oplus_q t)\right] \frac{(-t)^{n-1}}{\{n-1\}_q!} q^{\binom{n}{2}} \, d_q(t). \tag{18}$$

Proof. We show that this is equivalent with (8). By putting $F(x) = x^m$ it would suffice to prove that

$$\sum_{n=1}^{m} \frac{y^n}{\{n\}_q!} D_q^n x^m = \sum_{k=1}^{m} \frac{y^k}{\{k\}_q!} (-1)^{k+1} q^{\binom{k}{2}} D_{q,y}^k (x \oplus_q y)^m. \tag{19}$$

This is equivalent to the formula

$$\begin{aligned} &\sum_{n=1}^{m} \frac{y^n}{\{n\}_q!} \{m-n+1\}_{n,q} x^{m-n} = \sum_{k=1}^{m} \frac{y^k}{\{k\}_q!} (-1)^{k+1} q^{\binom{k}{2}} \{m-k+1\}_{k,q} \\ &\sum_{l=0}^{m-k} \frac{\{m-k\}_q!}{\{m-k-l\}_q! \{l\}_q!} x^l y^{m-k-l}. \end{aligned} \tag{20}$$

By equating the corresponding exponents for x and y, and thus putting $n = m - l$ we obtain

$$\frac{\{l+1\}_{m-l,q}}{\{n\}_q!} = \sum_{k=1}^{m-l} \frac{(-1)^{k+1} q^{\binom{k}{2}} \{m-k+1\}_{k,q}}{\{k\}_q!} \frac{\{m-k\}_q!}{\{m-k-l\}_q! \{l\}_q!}. \tag{21}$$

After simplification we see that this is equivalent to (7) for the special case $u = 1$.

Theorem 4. *Compare with the second Jackson q-Taylor formula, which is obtained by letting $n \to \infty$.*

$$\begin{aligned} F(x \boxplus_q y) = &\sum_{k=0}^{n-1} \frac{y^k}{\{k\}_q!} q^{\binom{k}{2}} D_q^k F(x) + \\ &\int_{t=0}^{y} D_{q,t}^n \left[F(x \boxplus_q t)\right] \frac{(-t)^{n-1}}{\{n-1\}_q!} q^{\binom{n}{2}} \, d_q(t). \end{aligned} \tag{22}$$

Proof. We show that this is equivalent with (13). By putting $F(x) = x^m$ it would suffice to prove that

$$\sum_{n=1}^{m} \frac{y^n}{\{n\}_q!} q^{\binom{n}{2}} D_q^n x^m = \sum_{k=1}^{m} \frac{y^k}{\{k\}_q!} (-1)^{k+1} q^{\binom{k}{2}} D_{q,y}^k (x \boxplus_q y)^m. \tag{23}$$

This is equivalent to the formula

$$\begin{aligned} &\sum_{n=1}^{m} \frac{y^n}{\{n\}_q!} \{m-n+1\}_{n,q} q^{\binom{n}{2}} x^{m-n} = \sum_{k=1}^{m} \frac{y^k}{\{k\}_q!} (-1)^{k+1} \{m-k+1\}_{k,q} \\ &\sum_{l=0}^{m-k} \frac{\{m-k\}_q!}{\{m-k-l\}_q! \{l\}_q!} x^l y^{m-k-l} \\ &\mathrm{QE}\left(k^2 - k + k(m-k-l) + \frac{(m-k-l)^2 - (m-k-l)}{2}\right). \end{aligned} \tag{24}$$

By equating the corresponding exponents for x and y, and thus putting $n = m - l$ we obtain

$$\begin{aligned}&\frac{\{l+1\}_{m-l,q}}{\{n\}_q!}q^{\frac{m^2+l^2-2ml+l-m}{2}} = \sum_{k=1}^{m-l}\frac{(-1)^{k+1}\{m-k+1\}_{k,q}}{\{k\}_q!}\frac{\{m-k\}_q!}{\{m-k-l\}_q!\{l\}_q!}\\&\mathrm{QE}\left(k^2-k+k(m-k-l)+\frac{(m^2+k^2+l^2-2mk-2ml+2kl)-(m-k-l)}{2}\right).\end{aligned}\tag{25}$$

After simplification we see that this is equivalent to (7) for the special case $u = 1$.

\section{Erd\'elyiformulas and fractional q-integrals In this section we have collected several q-Euler integral expressions for the function ${}_2\phi_1(\alpha,\beta;\gamma|q;z)$ and related formulas. Each of these q-Euler integral formulas have a prefactor Γ_q function. The restrictions for the parameters in these prefactors are the same as in the original formula, i.e. Re(parameters) > 0. For the notation, see our book [9].

Theorem 5. *A q-analogue of Erdélyi [7, (2.6) p. 270]. Assume that $\vec{\alpha}$, $\vec{\mu}$ and $\vec{s}$ are vectors of length m and $\vec{\beta}$ and $\vec{\gamma}$ are vectors of length $p+1-m$ and p, respectively, where $p+1>m$. Then we have the q-Euler integral representation*

$$\begin{aligned}&{}_{p+1}\phi_p\left[\begin{array}{c}\vec{\alpha},\vec{\beta}\\ \vec{\gamma}\end{array}\middle|q;x\right]\\&=\Gamma_q\left[\begin{array}{c}\vec{\mu}\\ \vec{\alpha},\mu\,\vec{-}\,\alpha\end{array}\right]\int_{\vec{s}=\vec{0}}^{\vec{1}}\vec{s}^{\,\alpha\vec{-}1}(q\vec{s};q)_{\mu-\vec{\alpha}-1}\;{}_{p+1}\phi_p\left[\begin{array}{c}\vec{\mu},\vec{\beta}\\ \vec{\gamma}\end{array}\middle|q;x\vec{s}\right]d_q(\vec{s}).\end{aligned}\tag{26}$$

Proof. We compute the right hand side:

$$\begin{aligned}\mathrm{RHS}&\overset{\text{by}[9,6.54]}{=}\Gamma_q\left[\begin{array}{c}\vec{\mu}\\ \vec{\alpha},\mu\,\vec{-}\,\alpha\end{array}\right]\sum_{n=0}^{\infty}\sum_{\vec{k}=\vec{0}}^{\vec{\infty}}\frac{\langle\vec{\mu},\vec{\beta};q\rangle_n x^n}{\langle 1,\vec{\gamma};q\rangle_n}(1-q)^m q^{\overline{k(\alpha+n)}}\langle 1\,\vec{+}\,k;q\rangle_{\mu-\vec{\alpha}-1}\\&\overset{\text{by}[9,6.8,6.10]}{=}\Gamma_q\left[\begin{array}{c}\vec{\mu}\\ \vec{\alpha},\mu\,\vec{-}\,\alpha\end{array}\right]\sum_{n=0}^{\infty}\frac{\langle\vec{\mu},\vec{\beta};q\rangle_n x^n}{\langle 1,\vec{\gamma};q\rangle_n}(1-q)^m\sum_{\vec{k}=\vec{0}}^{\vec{\infty}}q^{\overline{k(\alpha+n)}}\frac{\langle\mu\,\vec{-}\,\alpha;q\rangle_{\vec{k}}\langle\vec{1};q\rangle_{\vec{\infty}}}{\langle\vec{1};q\rangle_{\vec{k}}\langle\mu\,\vec{-}\,\alpha;q\rangle_{\vec{\infty}}}\\&\overset{\text{by}[9,7.27]}{=}\Gamma_q\left[\begin{array}{c}\vec{\mu}\\ \vec{\alpha},\mu\,\vec{-}\,\alpha\end{array}\right]\sum_{n=0}^{\infty}\frac{\langle\vec{\mu},\vec{\beta};q\rangle_n x^n}{\langle 1,\vec{\gamma};q\rangle_n}(1-q)^m\frac{\langle\mu\,\vec{+}\,n,1;q\rangle_{\vec{\infty}}}{\langle\alpha\,\vec{+}\,n,\mu\,\vec{-}\,\alpha;q\rangle_{\vec{\infty}}}\\&\overset{\text{by}[9,1.45,1.46]}{=}\mathrm{LHS}.\end{aligned}\tag{27}$$

Theorem 6. *A q-analogue of Erdélyi [7, (5.2) p. 273]. Assume that $\vec{\gamma}$, $\vec{\delta}$ and $\vec{s}$ are vectors of length m and $\vec{\alpha}$ and $\vec{\beta}$ are vectors of length $p+1$ and $p-m$, respectively where $p>m$. Then we have the q-Euler integral representation*

$$\begin{aligned}&{}_{p+1}\phi_p\left[\begin{array}{c}\vec{\alpha}\\ \vec{\gamma},\vec{\beta}\end{array}\middle|q;x\right]\\&=\Gamma_q\left[\begin{array}{c}\vec{\gamma}\\ \vec{\delta},\gamma\,\vec{-}\,\delta\end{array}\right]\int_{\vec{s}=\vec{0}}^{\vec{1}}\vec{s}^{\,\delta\vec{-}1}(q\vec{s};q)_{\gamma-\vec{\delta}-1}\;{}_{p+1}\phi_p\left[\begin{array}{c}\vec{\alpha}\\ \vec{\delta},\vec{\beta}\end{array}\middle|q;x\vec{s}\right]d_q(\vec{s}).\end{aligned}\tag{28}$$

Proof. We compute the right hand side:

$$
\begin{aligned}
\text{RHS} &\overset{\text{by}[9,6.54]}{=} \Gamma_q\begin{bmatrix}\vec{\gamma}\\ \vec{\delta},\gamma\vec{-}\delta\end{bmatrix}\sum_{n=0}^{\infty}\sum_{\vec{k}=\vec{0}}^{\vec{\infty}}\frac{\langle\vec{\alpha};q\rangle_n x^n}{\langle 1,\vec{\delta},\vec{\beta};q\rangle_n}(1-q)^m q^{\overline{k(\delta+n)}}\langle 1\vec{+}k;q\rangle_{\gamma-\vec{\delta}-1}\\
&\overset{\text{by}[9,6.8,6.10]}{=} \Gamma_q\begin{bmatrix}\vec{\gamma}\\ \vec{\delta},\gamma\vec{-}\delta\end{bmatrix}\sum_{n=0}^{\infty}\sum_{\vec{k}=\vec{0}}^{\vec{\infty}}\frac{\langle\vec{\alpha};q\rangle_n x^n}{\langle 1,\vec{\delta},\vec{\beta};q\rangle_n}(1-q)^m q^{\overline{k(\delta+n)}}\frac{\langle\gamma\vec{-}\delta;q\rangle_{\vec{k}}\langle\vec{1};q\rangle_{\vec{\infty}}}{\langle\vec{1};q\rangle_{\vec{k}}\langle\gamma\vec{-}\delta;q\rangle_{\vec{\infty}}} \qquad (29)\\
&\overset{\text{by}[9,7.27]}{=} \Gamma_q\begin{bmatrix}\vec{\gamma}\\ \vec{\delta},\gamma\vec{-}\delta\end{bmatrix}\sum_{n=0}^{\infty}\frac{\langle\vec{\alpha};q\rangle_n x^n}{\langle 1,\vec{\delta},\vec{\beta};q\rangle_n}(1-q)^m\frac{\langle\gamma\vec{+}n,1;q\rangle_{\vec{\infty}}}{\langle\delta\vec{+}n,\gamma\vec{-}\delta;q\rangle_{\vec{\infty}}}\\
&\overset{\text{by}[9,1.45,1.46]}{=} \text{LHS}.
\end{aligned}
$$

We can now combine the two previous theorems.

Theorem 7. *A q-analogue of Erdélyi [7, (6.1) p. 274]. Assume that $\vec{\mu}$, $\vec{\alpha}$ and $\vec{s}$ are vectors of length m and $\vec{\gamma}$, $\vec{\epsilon}$ and $\vec{t}$ are vectors of length n, where $m+n=p+1$.*

Then we have the q-Euler integral representation

$$
\begin{aligned}
{}_{p+1}\phi_p\left[\begin{array}{c|}\vec{\alpha},\vec{\beta}\\ \vec{\gamma},\vec{\delta}\end{array}\,q;x\right] &= \Gamma_q\begin{bmatrix}\vec{\mu},\vec{\gamma}\\ \vec{\alpha},\mu\vec{-}\alpha,\vec{\epsilon},\gamma\vec{-}\epsilon\end{bmatrix}\int_{\vec{s}=\vec{0}}^{\vec{1}}\vec{s}^{\,\alpha\vec{-}1}(q\vec{s};q)_{\mu-\vec{\alpha}-1}\\
&\times\int_{\vec{t}=\vec{0}}^{\vec{1}}\vec{t}^{\,\epsilon\vec{-}1}(q\vec{t};q)_{\gamma-\vec{\epsilon}-1}\;{}_{p+1}\phi_p\left[\begin{array}{c|}\vec{\mu},\vec{\beta}\\ \vec{\epsilon},\vec{\delta}\end{array}\,q;x\vec{s}\vec{t}\right]d_q(\vec{s})\,d_q(\vec{t}).
\end{aligned} \qquad (30)
$$

Proof. Put

$$
D\equiv\Gamma_q\begin{bmatrix}\vec{\mu},\vec{\gamma}\\ \vec{\alpha},\mu\vec{-}\alpha,\vec{\epsilon},\gamma\vec{-}\epsilon\end{bmatrix}\sum_{i=0}^{\infty}\frac{\langle\vec{\mu},\vec{\beta};q\rangle_i x^i}{\langle 1,\vec{\epsilon},\vec{\delta};q\rangle_i}(1-q)^{m+n}. \qquad (31)
$$

Then we have

$$
\begin{aligned}
\text{RHS} &\overset{\text{by}[9,6.54]}{=} D\sum_{\vec{k}=\vec{0}}^{\vec{\infty}}q^{\overline{k(\alpha+i)}}\langle 1\vec{+}k;q\rangle_{\mu-\vec{\alpha}-1}\sum_{\vec{l}=\vec{0}}^{\vec{\infty}}q^{\overline{l(\epsilon+i)}}\langle 1\vec{+}l;q\rangle_{\gamma-\vec{\epsilon}-1}\\
&\overset{\text{by}[9,6.8,6.10]}{=} D\sum_{\vec{k}=\vec{0}}^{\vec{\infty}}q^{\overline{k(\alpha+i)}}\frac{\langle\mu\vec{-}\alpha;q\rangle_{\vec{k}}\langle\vec{1};q\rangle_{\vec{\infty}}}{\langle\vec{1};q\rangle_{\vec{k}}\langle\mu\vec{-}\alpha;q\rangle_{\vec{\infty}}}\sum_{\vec{l}=\vec{0}}^{\vec{\infty}}q^{\overline{l(\epsilon+i)}}\frac{\langle\gamma\vec{-}\epsilon;q\rangle_{\vec{l}}\langle\vec{1};q\rangle_{\vec{\infty}}}{\langle\vec{1};q\rangle_{\vec{l}}\langle\gamma\vec{-}\epsilon;q\rangle_{\vec{\infty}}} \qquad (32)\\
&\overset{\text{by}[9,7.27]}{=} D\frac{\langle\gamma\vec{+}i,\mu\vec{+}i,1,1;q\rangle_{\vec{\infty}}}{\langle\gamma\vec{-}\epsilon,\epsilon\vec{+}i,\alpha\vec{+}i,\mu\vec{-}\alpha;q\rangle_{\vec{\infty}}}\overset{\text{by}[9,1.45,1.46]}{=}\text{LHS}.
\end{aligned}
$$

We quote a few theorems from our book [9].

Lemma 3. *Linear substitution in a q-integral [9, 6.64].*

$$\int_0^x f(t,q)\ d_q(t) = a\int_0^{\frac{x}{a}} f(at,q)\ d_q(t). \tag{33}$$

Definition 2. *[9, 8.116]*

$$\mathrm{P}_{\alpha,q}(x,a) \equiv x^{\alpha}\frac{(\frac{a}{x};q)_\infty}{(\frac{a}{x}q^{\alpha};q)_\infty},\ \ \tfrac{a}{x} \neq q^{-m-\alpha}, m = 0,1,\ldots. \tag{34}$$

We remark that a totally different approach to the following formulas was made in [2, p.124-125].

Definition 3. *[9, 8.117] The fractional q-integral is defined in the following way, $\nu \in \mathbb{C}$:*

$$\mathrm{D}_q^{-\nu} f(x) \equiv \frac{1}{\Gamma_q(\nu)}\int_0^x \mathrm{P}_{\nu-1,q}(x,qt) f(t)\, d_q(t). \tag{35}$$

We infer that

Theorem 8. *[9, 8.118]*

$$\mathrm{D}_q^{-\nu} x^{\mu} = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu+\nu+1)} x^{\mu+\nu}. \tag{36}$$

Theorem 9. *A q-analogue of Mathai, Haubold [21, 3.2.1], Holmgren [16, p. 3 (4)], Kampé de Fériet [19, p. 200]. Assume that $n-1 < \mathrm{Re}(-\nu) < n$. Then the fractional q-integral (35) can also be expressed as*

$$\mathrm{D}_q^{-\nu} f(x) = \frac{1}{\Gamma_q(n+\nu)}\mathrm{D}_{q,x}^n \int_a^x \mathrm{P}_{n+\nu-1,q}(x,qt) f(t)\ d_q(t). \tag{37}$$

Proof. The proof is by induction.

$$\begin{aligned}
&\frac{1}{\Gamma_q(1+\nu)}\mathrm{D}_{q,x}\int_a^x \mathrm{P}_{\nu,q}(x,qt)f(t)\ d_q(t)\\
&=\frac{1-q}{\Gamma_q(1+\nu)x(q-1)}\sum_{n=0}^{\infty}q^n\left[(qx)^{\nu+1}\frac{(q^{n+1};q)_\infty}{(q^{n+1+\nu};q)_\infty}f(xq^{n+1})\right.\\
&-a(qx)^\nu\frac{(\frac{a}{x}q^n;q)_\infty}{(\frac{a}{x}q^{n+\nu};q)_\infty}f(aq^n)\\
&\left.-x^{\nu+1}\frac{(q^{n+1};q)_\infty}{(q^{n+1+\nu};q)_\infty}f(xq^n)+ax^\nu\frac{(\frac{a}{x}q^{n+1};q)_\infty}{(\frac{a}{x}q^{n+1+\nu};q)_\infty}f(aq^n)\right]\\
&=\frac{x^\nu(1-q)}{\Gamma_q(\nu)}\left[\sum_{n=0}^{\infty}q^n\frac{(q^{n+1};q)_\infty}{(q^{n+\nu};q)_\infty}f(xq^n)\frac{1-q^{n+\nu}}{1-q^\nu}\right.\\
&\left.-\sum_{n=1}^{\infty}q^{n+\nu}\frac{(q^n;q)_\infty}{(q^{n+\nu};q)_\infty}f(xq^n)\frac{1-q^n}{1-q^\nu}\right]\\
&+\frac{ax^{\nu-1}(1-q)}{\Gamma_q(\nu)(1-q^\nu)}\sum_{n=0}^{\infty}q^n\frac{(\frac{a}{x}q^{n+1};q)_\infty}{(\frac{a}{x}q^{n+\nu};q)_\infty}f(aq^n)\left[-(1-\frac{a}{x}q^{n+\nu})+q^\nu(1-\frac{a}{x}q^n)\right]\\
&=\frac{x^{\nu-1}(1-q)}{\Gamma_q(\nu)}\sum_{n=0}^{\infty}q^n\left[\frac{x(q^n;q)_\infty}{(q^{n+\nu};q)_\infty}f(xq^n)-a\frac{(\frac{a}{x}q^{n+1};q)_\infty}{(\frac{a}{x}q^{n+\nu};q)_\infty}f(aq^n)\right]\\
&=\frac{1}{\Gamma_q(\nu)}\int_a^x \mathrm{P}_{\nu-1,q}(x,qt)f(t)\ d_q(t).
\end{aligned}\tag{38}$$

We can continue this process to an arbitrary integer n like in the previous q-Taylor expansions.

The following lemma enables a series expansion for $\mathrm{D}_q^{-\nu}f(x)$.

Lemma 4.

$$D_{q,t}\left(-\frac{\mathrm{P}_{\nu,q}(x,t)}{\{\nu\}_q}\right)=\mathrm{P}_{\nu-1,q}(x,qt).\tag{39}$$

Theorem 10. *A q-analogue of [16, p.8 (19)].*

$$\begin{aligned}
&\mathrm{D}_q^{-\nu}f(x)\\
&=\sum_{j=0}^{k-1}\frac{(\mathrm{D}_q^j f)(a)\mathrm{P}_{j+\nu,q}(x,a)}{\Gamma_q(\nu+j+1)}+\frac{1}{\Gamma_q(k+\nu)}\int_a^x \mathrm{P}_{k+\nu-1,q}(x,qt)(\mathrm{D}_q^k f)(t)\ d_q(t).
\end{aligned}\tag{40}$$

Proof. At each step we only use q-integration by parts [9, 6.58] and formula (39). Put

$$\mathrm{D}_q v(t)=\mathrm{P}_{\nu-1,q}(x,qt),\ \ u(t)=f(t).\tag{41}$$

Then

$$\mathrm{LHS}=\frac{1}{\Gamma_q(\nu)}\left[-\left[\frac{f(t)}{\{\nu\}_q}x^\nu(\frac{t}{x};q)_\nu\right]_a^x+\int_a^x \mathrm{P}_{\nu,q}(x,qt)\frac{(\mathrm{D}_q f)(t)}{\{\nu\}_q}\ d_q(t)\right].\tag{42}$$

We can continue this process k times.

Theorem 11. *Assume that the convergence region for $\Phi_2(\alpha;\beta_1,\beta_2;,\gamma_1,\gamma_2|q;x,y)$ is [10].*

$$|x| \oplus_q |y| < 1. \tag{43}$$

A q-analogue of Koschmieder [20, p. 252]. Put

$$F(x,y) \equiv \Phi^{1:3}_{1:2}\left[\begin{array}{c} \alpha:\beta_1,\mu_1,\infty;\beta_2,\mu_2,\infty \\ \infty:\nu_1,\tau_1;\nu_2,\tau_2 \end{array}\middle| q;x,y\right]. \tag{44}$$

Then we have the q-Euler integral representation

$$\begin{aligned} &\Phi_2(\alpha;\beta_1,\beta_2;,\gamma_1,\gamma_2|q;x,y) = \Gamma_q\left[\begin{array}{c} \gamma_1,\gamma_2,\mu_1,\mu_2 \\ \nu_1,\gamma_1-\nu_1,\nu_2,\gamma_2-\nu_2,\tau_1,\tau_2 \end{array}\right] \\ &\times \int_{s=0}^{1}\int_{t=0}^{1} s^{\nu_1-\mu_1}t^{\nu_2-\mu_2}(qs;q)_{\gamma_1-\nu_1-1}(qt;q)_{\gamma_2-\nu_2-1} \\ &\mathrm{D}^{\tau_1-\mu_1}_{q,s}\mathrm{D}^{\tau_2-\mu_2}_{q,t}\left[s^{\tau_1-1}t^{\tau_2-1}F(sx,ty)\right]\,d_q(s)\,d_q(t). \end{aligned} \tag{45}$$

Proof. Put

$$\begin{aligned} D \equiv \Gamma_q&\left[\begin{array}{c} \gamma_1,\gamma_2,\mu_1,\mu_2 \\ \nu_1,\gamma_1-\nu_1,\nu_2,\gamma_2-\nu_2,\tau_1,\tau_2 \end{array}\right] \\ &\sum_{m,n=0}^{\infty}\frac{\langle\alpha;q\rangle_{m+n}\langle\beta_1,\mu_1;q\rangle_m\langle\beta_2,\mu_2;q\rangle_n}{\langle 1,\nu_1,\tau_1;q\rangle_m\langle 1,\nu_2,\tau_2;q\rangle_n}x^m y^n. \end{aligned} \tag{46}$$

We compute the right hand side:

$$\begin{aligned} &\text{RHS} \overset{\text{by}[9,8.118]}{=} D\int_{s=0}^{1}\int_{t=0}^{1}(qs;q)_{\gamma_1-\nu_1-1}(qt;q)_{\gamma_2-\nu_2-1} \\ &\Gamma_q\left[\begin{array}{c} m+\tau_1,n+\tau_2, \\ m+\mu_1,n+\mu_2 \end{array}\right] s^{m+\nu_1-1}t^{n+\nu_2-1}\,d_q(s)\,d_q(t) \\ &\overset{\text{by}[9,6.54]}{=} D(1-q)^2\Gamma_q\left[\begin{array}{c} m+\tau_1,n+\tau_2, \\ m+\mu_1,n+\mu_2 \end{array}\right] \\ &\sum_{k,l=0}^{\infty} q^{k(\nu_1+m)+l(\nu_2+n)}\langle 1+k;q\rangle_{\gamma_1-\nu_1-1}\langle 1+l;q\rangle_{\gamma_2-\nu_2-1} \\ &\overset{\text{by}[9,6.8,6.10]}{=} D(1-q)^2\Gamma_q\left[\begin{array}{c} m+\tau_1,n+\tau_2, \\ m+\mu_1,n+\mu_2 \end{array}\right] \\ &\sum_{k,l=0}^{\infty} q^{k(\nu_1+m)+l(\nu_2+n)}\frac{\langle\gamma_1-\nu_1;q\rangle_k\langle\gamma_2-\nu_2;q\rangle_l\langle 1,1;q\rangle_\infty}{\langle 1;q\rangle_k\langle 1;q\rangle_l\langle\gamma_1-\nu_1,\gamma_2-\nu_2;q\rangle_\infty} \\ &\overset{\text{by}[9,7.27]}{=} D(1-q)^2\Gamma_q\left[\begin{array}{c} m+\tau_1,n+\tau_2, \\ m+\mu_1,n+\mu_2 \end{array}\right] \\ &\frac{\langle\gamma_1+m,\gamma_2+n,1,1;q\rangle_\infty}{\langle\gamma_1-\nu_1,\gamma_2-\nu_2,\nu_1+m,\nu_2+n;q\rangle_\infty} \overset{\text{by}[9,1.45,1.46]}{=} \text{LHS}. \end{aligned} \tag{47}$$

We make a new proof of the following theorem by fractional q-integration.

Theorem 12. *[9, (7.50) p. 251], a q-analogue of [8, (1) p. 176].*

$$
{}_2\phi_1(\alpha,\beta;\gamma|q;z)\cong\frac{\Gamma_q(\gamma)}{\Gamma_q(\beta)\Gamma_q(\gamma-\beta)}\int_0^1 t^{\beta-1}\frac{(qt;q)_{\gamma-\beta-1}}{(zt;q)_a}\,d_q(t). \tag{48}
$$

Proof.

$$
\begin{aligned}
\text{LHS}&\overset{\text{by}(36)}{=}\frac{\Gamma_q(\gamma)}{\Gamma_q(\beta)}z^{1-\gamma}\mathrm{D}_q^{\beta-\gamma}\sum_{k=0}^{\infty}\frac{\langle\alpha;q\rangle_k}{\langle 1;q\rangle_k}z^{\beta+k-1}\\
&\overset{\text{by}[9,(7.27)p.247]}{=}\frac{\Gamma_q(\gamma)}{\Gamma_q(\beta)}z^{1-\gamma}\mathrm{D}_q^{\beta-\gamma}\frac{z^{\beta-1}}{(z;q)_\alpha}\\
&\overset{\text{by}(35)}{=}\frac{\Gamma_q(\gamma)z^{1-\gamma}}{\Gamma_q(\beta)\Gamma_q(\gamma-\beta)}\int_0^z\frac{t^{\beta-1}}{(t;q)_\alpha}\frac{z^{\gamma-\beta-1}(\frac{qt}{z};q)_\infty}{(\frac{t}{z}q^{\gamma-\beta};q)_\infty}\,d_q(t)\overset{\text{by}(33)}{=}\text{RHS}.
\end{aligned} \tag{49}
$$

Theorem 13. *A q-analogue of Erdélyi [8, (3) p. 176].*

$$
\begin{aligned}
&{}_2\phi_1(\alpha,\beta;\gamma|q;z)\\
&\cong\frac{\Gamma_q(\gamma)}{\Gamma_q(\lambda)\Gamma_q(\gamma-\lambda)}\int_0^1 t^{\lambda-1}\frac{(qt;q)_\infty}{(tq^{\gamma-\lambda}q)_\infty}\ {}_2\phi_1(\alpha,\beta;\lambda|q;tz)\ \ d_q(t).
\end{aligned} \tag{50}
$$

Proof.

$$
\begin{aligned}
\text{LHS}&\overset{\text{by}(36)}{=}\frac{\Gamma_q(\gamma)}{\Gamma_q(\lambda)}z^{1-\gamma}\mathrm{D}_q^{\lambda-\gamma}\sum_{k=0}^{\infty}\frac{\langle\alpha,\beta;q\rangle_k}{\langle\lambda,1;q\rangle_k}z^{\lambda+k-1}\\
&=\frac{\Gamma_q(\gamma)}{\Gamma_q(\lambda)}z^{1-\gamma}\mathrm{D}_q^{\lambda-\gamma}\left(z^{\lambda-1}\ {}_2\phi_1(\alpha,\beta;\lambda|q;z)\right)\\
&\overset{\text{by}(35)}{=}\frac{\Gamma_q(\gamma)z^{1-\gamma}}{\Gamma_q(\lambda)\Gamma_q(\gamma-\lambda)}\int_0^z z^{\gamma-\lambda-1}t^{\lambda-1}\frac{(\frac{qt}{z};q)_\infty}{(\frac{t}{z}q^{\gamma-\lambda}q)_\infty}\ {}_2\phi_1(\alpha,\beta;\lambda|q;t)\ \ d_q(t)\\
&\overset{\text{by}(33)}{=}\text{RHS}.
\end{aligned} \tag{51}
$$

Theorem 14. *A q-analogue of Erdélyi [8, p. 182].*

$$
\begin{aligned}
&{}_2\phi_1(\alpha,\beta;\gamma|q;z)\cong\Gamma_q\left[\begin{matrix}\gamma\\ \mu,\gamma-\lambda\end{matrix}\right]\\
&\int_0^1\frac{(qt;q)_{\gamma-\lambda-1}}{(tz;q)_{\alpha+\beta-\lambda}}\mathrm{D}_{q,t}^{\mu-\lambda}\left(t^{\mu-1}\ {}_2\phi_1(\lambda-\alpha,\lambda-\beta;\mu|q;tz)\right)\ \ d_q(t).
\end{aligned} \tag{52}
$$

Proof. We find that

$$\begin{aligned}&\text{LHS} \overset{\text{by}[9,(7.52)],(50)}{=} \\ &\Gamma_q \left[\begin{array}{c} \gamma \\ \lambda, \gamma-\lambda \end{array} \right] \int_0^1 t^{\lambda-1} \frac{(qt;q)_{\gamma-\lambda-1}}{(tz;q)_{\alpha+\beta-\lambda}} \,{}_2\phi_1(\lambda-\alpha,\lambda-\beta;\lambda|q;tz)\, d_q(t).\end{aligned} \tag{53}$$

On the other hand,

$$\begin{aligned}&\frac{t^{\lambda-1}}{\Gamma_q(\lambda)} \,{}_2\phi_1(\lambda-\alpha,\lambda-\beta;\lambda|q;tz) = \mathrm{D}_{q,t}^{\mu-\lambda} \sum_{k=0}^{\infty} \frac{\langle \lambda-\alpha,\lambda-\beta;q\rangle_k}{\langle 1;q\rangle_k \Gamma_q(\lambda+k)} t^{\mu+k-1} z^k \\ &= \mathrm{D}_{q,t}^{\mu-\lambda} \left(\frac{t^{\mu-1}}{\Gamma_q(\mu)} \,{}_2\phi_1(\lambda-\alpha,\lambda-\beta;\mu|q;tz) \right).\end{aligned} \tag{54}$$

Theorem 15. *A q-analogue of Erdélyi [8, p. 184].*

$$\begin{aligned}&{}_2\phi_1(\alpha,\beta;\gamma|q;z) \cong \Gamma_q \left[\begin{array}{c} \gamma, \mu \\ \lambda, \gamma-\lambda, \nu \end{array} \right] \\ &\int_0^1 t^{\lambda-\mu} (qt;q)_{\gamma-\lambda-1} \mathrm{D}_{q,t}^{\nu-\mu} \left(t^{\nu-1} \,{}_3\phi_2(\alpha,\beta,\mu;\lambda,\nu|q;tz) \right) \ d_q(t).\end{aligned} \tag{55}$$

Proof. We find that

$$\begin{aligned}&t^{\mu-1} \,{}_2\phi_1(\alpha,\beta;\lambda|q;tz) = \mathrm{D}_{q,t}^{\nu-\mu} \left(\sum_{k=0}^{\infty} \frac{\langle \alpha,\beta;q\rangle_k \Gamma_q(\mu+k)}{\langle \lambda,1;q\rangle_k \Gamma_q(\nu+k)} t^{\nu+k-1} z^k \right) \\ &= \mathrm{D}_{q,t}^{\nu-\mu} \left(\frac{\Gamma_q(\mu)}{\Gamma_q(\nu)} t^{\nu-1} \,{}_3\phi_2(\alpha,\beta,\mu;\lambda,\nu|q;tz) \right).\end{aligned} \tag{56}$$

Finally, put this into formula (50).

The following formula, whose proof can be found in (80) for $x = 0$ and permutation of the parameters, corresponds to Rodriguez formula.

Theorem 16. *A q-analogue of Koschmieder [20, (10.5)p.252] and Erdélyi [8, (9) p. 178]. Almost a q-analogue of Kampé de Fériet [19, p. 26].*

$$ {}_2\phi_1(\alpha,\beta;\gamma|q;z) = z^{1-\gamma} \Gamma_q \left[\begin{array}{c} \gamma \\ \beta \end{array} \right] \mathrm{D}_{q,z}^{\beta-\gamma} \left[\frac{z^{\beta-1}}{(z;q)_\alpha} \right]. \tag{57}$$

\section\label{Feld}

Definition 4. *The first q-Lauricella function is*

$$\Phi_{\mathrm{A}}^{(n)}(a,\vec{b};\vec{c}|q;\vec{x}) \equiv \sum_{\vec{m}} \frac{\langle a;q\rangle_m \langle \vec{b};q\rangle_{\vec{m}} \vec{x}^{\vec{m}}}{\langle \vec{c},\vec{1};q\rangle_{\vec{m}}}, \tag{58}$$

where [10]

$$|x_1| \oplus_q \cdots \oplus_q |x_n| < 1. \tag{59}$$

The following formula, similar to [12], was not included there.

Theorem 17. *(A q-analogue of Feldheim [14, (9) p. 244]).*

$$\begin{aligned}&\Gamma_q\left[\begin{array}{c}\alpha,\delta-\alpha\\ \delta\end{array}\right]\Phi_{\mathrm{A}}^{(n)}(\alpha,\vec{\beta};\vec{\gamma}|q;\vec{x})\\ &\cong\int_{s=0}^{1}s^{\alpha-1}(qs;q)_{\delta-\alpha-1}\Phi_{\mathrm{A}}^{(n)}(\delta,\vec{\beta};\vec{\gamma}|q;s\vec{x})\ d_q(s).\end{aligned}\tag{60}$$

Proof. Compute the RHS:

$$\begin{aligned}\text{RHS}&\overset{\text{by}[9,6.54]}{=}\sum_{\vec{m}=\vec{0}}^{\vec{\infty}}\frac{\langle\delta;q\rangle_m\langle\vec{\beta};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\vec{1},\vec{\gamma};q\rangle_{\vec{m}}}(1-q)\sum_{k=0}^{\infty}q^{k(\alpha+m)}\langle1+k;q\rangle_{\delta-\alpha-1}\\ &\overset{\text{by}[9,6.8,6.10]}{=}\sum_{\vec{m}=\vec{0}}^{\vec{\infty}}\frac{\langle\delta;q\rangle_m\langle\vec{\beta};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\vec{1},\vec{\gamma};q\rangle_{\vec{m}}}(1-q)\sum_{k=0}^{\infty}q^{k(\alpha+m)}\frac{\langle\delta-\alpha;q\rangle_k\langle1;q\rangle_\infty}{\langle1;q\rangle_k\langle\delta-\alpha;q\rangle_\infty}\\ &\overset{\text{by}[9,7.27]}{=}\sum_{\vec{m}=\vec{0}}^{\vec{\infty}}\frac{\langle\delta;q\rangle_m\langle\vec{\beta};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\vec{1},\vec{\gamma};q\rangle_{\vec{m}}}(1-q)\frac{\langle m+\delta,1;q\rangle_\infty}{\langle\delta-\alpha,\alpha+m;q\rangle_\infty}\overset{\text{by}[9,1.45,1.46]}{=}\text{LHS}.\end{aligned}\tag{61}$$

Definition 5. *Convergence regions for the following triple functions were given in [11]. The q-analogues of the Srivastava triple hypergeometric functions are*

$$\begin{aligned}&\mathrm{H}_{\mathrm{A}}(a,b_1,b_2;c_1,c_2|q;x_1,x_2,x_3)\equiv\\ &\sum_{m,n,p=0}^{\infty}\frac{\langle a;q\rangle_{m+p}\langle b_1;q\rangle_{m+n}\langle b_2;q\rangle_{n+p}}{\langle1,c_1;q\rangle_m\langle1;q\rangle_n\langle1;q\rangle_p\langle c_2;q\rangle_{n+p}}x_1^m x_2^n x_3^p.\end{aligned}\tag{62}$$

$$\begin{aligned}&\mathrm{H}_{\mathrm{B}}(a,b_1,b_2;c_1,c_2,c_3|q;x_1,x_2,x_3)\equiv\\ &\sum_{m,n,p=0}^{\infty}\frac{\langle a;q\rangle_{m+p}\langle b_1;q\rangle_{m+n}\langle b_2;q\rangle_{n+p}}{\langle1,c_1;q\rangle_m\langle1,c_2;q\rangle_n\langle1,c_3;q\rangle_p}x_1^m x_2^n x_3^p.\end{aligned}\tag{63}$$

$$\begin{aligned}&\mathrm{H}_{\mathrm{C}}(a,b_1,b_2;c|q;x_1,x_2,x_3)\equiv\\ &\sum_{m,n,p=0}^{\infty}\frac{\langle a;q\rangle_{m+p}\langle b_1;q\rangle_{m+n}\langle b_2;q\rangle_{n+p}}{\langle1;q\rangle_m\langle1;q\rangle_n\langle1;q\rangle_p\langle c;q\rangle_{m+n+p}}x_1^m x_2^n x_3^p.\end{aligned}\tag{64}$$

Theorem 18. *A q-integral representation of* H_{A}*. A q-analogue of [5, (2.1) p. 115].*

$$\begin{aligned}&\mathrm{H}_{\mathrm{A}}(a_1,a_2,a_3;c_1,c_2|q;x,y,z)\\ &=\Gamma_q\left[\begin{array}{c}b\\ a_1,b-a_1\end{array}\right]\int_{s=0}^{1}s^{a_1-1}(qs;q)_{b-a_1-1}\mathrm{H}_{\mathrm{A}}(b,a_2,a_3;c_1,c_2|q;xs,y,zs)\,d_q(s).\end{aligned}\tag{65}$$

Proof. Put

$$D \equiv \Gamma_q \left[\begin{array}{c} b \\ a_1, b-a_1 \end{array} \right] \sum_{m,n,p=0}^{\infty} \frac{\langle b; q\rangle_{m+p} \langle a_2; q\rangle_{m+n} \langle a_3; q\rangle_{n+p}}{\langle 1, c_1; q\rangle_m \langle 1; q\rangle_n \langle 1; q\rangle_p \langle c_2; q\rangle_{n+p}} x^m y^n z^p. \tag{66}$$

Then we have

$$\begin{aligned} \text{RHS} &\overset{\text{by}[9,6.54]}{=} D(1-q) \sum_{k=0}^{\infty} q^{k(a_1+m+p)} \langle 1+k; q\rangle_{b-a_1-1} \\ &\overset{\text{by}[9,6.8,6.10]}{=} D(1-q) \sum_{k=0}^{\infty} q^{k(a_1+m+p)} \frac{\langle b-a_1; q\rangle_k \langle 1; q\rangle_\infty}{\langle 1; q\rangle_k \langle b-a_1; q\rangle_\infty} \\ &\overset{\text{by}[9,7.27]}{=} D(1-q) \frac{\langle b+n+p, 1; q\rangle_\infty}{\langle a_1+m+p, b-a_1; q\rangle_\infty} \overset{\text{by}[9,1.45,1.46]}{=} \text{LHS}. \end{aligned} \tag{67}$$

Theorem 19. *A q-integral representation of* $\mathrm{H_A}$. *A q-analogue of [5, (2.2) p. 115].*

$$\begin{aligned} &\mathrm{H_A}(a_1, a_2, a_3; c_1, c_2|q; x, y, z) \\ &= \Gamma_q \left[\begin{array}{c} b \\ a_2, b-a_2 \end{array} \right] \int_{s=0}^{1} s^{a_2-1} (qs; q)_{b-a_2-1} \mathrm{H_A}(a_1, b, a_3; c_1, c_2|q; xs, ys, z)\, d_q(s). \end{aligned} \tag{68}$$

Theorem 20. *A q-integral representation of* $\mathrm{H_A}$. *A q-analogue of [5, (2.3) p. 115].*

$$\begin{aligned} &\mathrm{H_A}(a_1, a_2, a_3; c_1, c_2|q; x, y, z) \\ &= \Gamma_q \left[\begin{array}{c} b \\ a_3, b-a_3 \end{array} \right] \int_{s=0}^{1} s^{a_3-1} (qs; q)_{b-a_3-1} \mathrm{H_A}(a_1, a_2, b; c_1, c_2|q; x, ys, zs)\, d_q(s). \end{aligned} \tag{69}$$

Theorem 21. *A q-integral representation of* $\mathrm{H_A}$. *A q-analogue of [5, (2.4) p. 115].*

$$\begin{aligned} &\mathrm{H_A}(a_1, a_2, a_3; c_1, c_2|q; x, y, z) \\ &= \Gamma_q \left[\begin{array}{c} c_1 \\ c_1-b, b \end{array} \right] \int_{s=0}^{1} s^{b-1} (qs; q)_{c_1-b-1} \mathrm{H_A}(a_1, a_2, a_3; b, c_2|q; xs, y, z)\, d_q(s). \end{aligned} \tag{70}$$

Proof. Put

$$D \equiv \frac{\Gamma_q(c_1)}{\Gamma_q(c_1-b)\Gamma_q(b)} \sum_{m,n,p=0}^{\infty} \frac{\langle a_1; q\rangle_{m+p} \langle a_2; q\rangle_{m+n} \langle a_3; q\rangle_{n+p}}{\langle 1, b; q\rangle_m \langle 1; q\rangle_n \langle 1; q\rangle_p \langle c_2; q\rangle_{n+p}} x^m y^n z^p.$$

Then we have

$$
\begin{aligned}
&\mathrm{RHS} \overset{\text{by}[9,6.54]}{=} D(1-q)\sum_{k=0}^{\infty} q^{k(b+m)}\langle 1+k;q\rangle_{c_1-b-1} \\
&\overset{\text{by}[9,6.8,6.10]}{=} D(1-q)\sum_{k=0}^{\infty} q^{k(b+m)}\frac{\langle c_1-b;q\rangle_k\langle 1;q\rangle_\infty}{\langle 1;q\rangle_k\langle c_1-b;q\rangle_\infty} \\
&\overset{\text{by}[9,7.27]}{=} D(1-q)\frac{\langle c_1+m,1;q\rangle_\infty}{\langle b+m,c_1-b;q\rangle_\infty} \overset{\text{by}[9,1.45,1.46]}{=} \mathrm{LHS}.
\end{aligned}
\tag{71}
$$

Theorem 22. *A q-integral representation of* $\mathrm{H_A}$*. A q-analogue of [5, (2.5) p. 116].*

$$
\begin{aligned}
&\mathrm{H_A}(a_1,a_2,a_3;c_1,c_2|q;x,y,z) \\
&= \Gamma_q\left[\begin{array}{c} c_2 \\ c_2-b,b \end{array}\right]\int_{s=0}^{1} s^{b-1}(qs;q)_{c_2-b-1}\mathrm{H_A}(a_1,a_2,a_3;c_1,b|q;x,ys,zs)\,d_q(s).
\end{aligned}
\tag{72}
$$

Theorem 23. *A q-integral representation of* $\mathrm{H_C}$*. A q-analogue of [6, (2.1) p. 115].*

$$
\begin{aligned}
&\mathrm{H_C}(a_1,a_2,a_3;c|q;x,y,z) \\
&= \Gamma_q\left[\begin{array}{c} b \\ a_1,b-a_1 \end{array}\right]\int_{s=0}^{1} s^{a_1-1}(qs;q)_{b-a_1-1}\mathrm{H_C}(b,a_2,a_3;c|q;xs,y,zs)\,d_q(s).
\end{aligned}
\tag{73}
$$

Theorem 24. *A q-integral representation of* $\mathrm{H_B}$*. A q-analogue of [4, (3.1) p. 2757].*

$$
\begin{aligned}
&\mathrm{H_B}(a_1,a_2,a_3;c_1,c_2,c_3|q;x,y,z) \\
&= \Gamma_q\left[\begin{array}{c} a_1+a_2 \\ a_1,a_2 \end{array}\right]\sum_{m,n,p=0}^{\infty}\frac{\langle a_1+a_2;q\rangle_{2m+n+p}\langle a_3;q\rangle_{n+p}}{\langle 1,c_1;q\rangle_m\langle 1,c_2;q\rangle_n\langle 1,c_3;q\rangle_p}x_1^m x_2^n x_3^p \\
&\int_{s=0}^{1} s^{a_1+m+p-1}(qs;q)_{a_2+m+n-1}\,d_q(s).
\end{aligned}
\tag{74}
$$

Proof. We compute the right hand side:

$$
\begin{aligned}
&\Gamma_q\left[\begin{array}{c} a_1+a_2 \\ a_1,a_2 \end{array}\right]\sum_{m,n,p=0}^{\infty}\frac{\langle a_1+a_2;q\rangle_{2m+n+p}\langle a_3;q\rangle_{n+p}}{\langle 1,c_1;q\rangle_m\langle 1,c_2;q\rangle_n\langle 1,c_3;q\rangle_p}x_1^m x_2^n x_3^p \\
&\qquad\times\Gamma_q\left[\begin{array}{c} a_1+m+p,a_2+m+n \\ a_1+a_2+2m+n+p \end{array}\right] \\
&= \sum_{m,n,p=0}^{\infty}\frac{\langle a_1+a_2;q\rangle_{2m+n+p}\langle a_3;q\rangle_{n+p}}{\langle 1,c_1;q\rangle_m\langle 1,c_2;q\rangle_n\langle 1,c_3;q\rangle_p}\frac{\langle a_1;q\rangle_{m+p}\langle a_2;q\rangle_{m+n}}{\langle a_1+a_2;q\rangle_{2m+n+p}}x_1^m x_2^n x_3^p = \mathrm{LHS}.
\end{aligned}
\tag{75}
$$

Definition 6. *The fourth q-Lauricella function is defined by*

$$\Phi_{\mathrm{D}}^{(n)}(a,\vec{b};c|q;\vec{x}) \equiv \sum_{\vec{m}} \frac{\langle a;q\rangle_m \langle \vec{b};q\rangle_{\vec{m}} \vec{x}^{\vec{m}}}{\langle c;q\rangle_m \langle \vec{1};q\rangle_{\vec{m}}},\ \max(|x_1|,\ldots,|x_n|) < 1. \tag{76}$$

Theorem 25. *A q-analogue of Srivastava and Manocha [23, p.289 (17)].*

$$\begin{aligned} &\mathrm{D}_{q,z}^{\lambda-\mu}\left[\frac{z^{\lambda-1}}{(az,q)_\alpha (bz,q)_\beta (cz,q)_\gamma}\right] \\ &= \frac{\Gamma_q(\lambda)}{\Gamma_q(\mu)} z^{\mu-1} \Phi_{\mathrm{D}}^{(3)}(\lambda,\alpha,\beta,\gamma;\mu|q;az,bz,cz). \end{aligned} \tag{77}$$

Proof.

$$\begin{aligned} &\text{LHS} = \mathrm{D}_{q,z}^{\lambda-\mu}\left[z^{\lambda-1}\sum_{m,n,p=0}^{\infty} \frac{\langle\alpha;q\rangle_m \langle\beta;q\rangle_n \langle\gamma;q\rangle_p}{\langle 1;q\rangle_m \langle 1;q\rangle_n \langle 1;q\rangle_p} a^m b^n c^p z^{m+n+p}\right] \overset{\text{by}[9,8.118]}{=} \\ &z^{\mu-1}\sum \frac{\langle\alpha;q\rangle_m \langle\beta;q\rangle_n \langle\gamma;q\rangle_p (az)^m (bz)^n (cz)^p}{\langle 1;q\rangle_m \langle 1;q\rangle_n \langle 1;q\rangle_p} \Gamma_q\left[\begin{array}{c} \lambda+m+n+p \\ \mu+m+n+p \end{array}\right] \\ &\overset{\text{by}[9,1.45,1.46]}{=} \text{RHS}. \end{aligned} \tag{78}$$

The following operator formula is a generalization of (57).

Theorem 26. *A q-analogue of Srivastava and Manocha [23, p.289 (18)].*

$$\begin{aligned} &\mathrm{D}_{q,y}^{\lambda-\mu}\left[y^{\lambda-1}\frac{1}{(y;q)_\alpha}\ {}_2\phi_1(\alpha,\beta;\gamma|q;x||-;yq^\alpha)\right] \\ &= \frac{\Gamma_q(\lambda)}{\Gamma_q(\mu)} y^{\mu-1} \Phi_2(\alpha,\beta;\lambda;\gamma,\mu|q;x,y). \end{aligned} \tag{79}$$

Proof.

$$\text{LHS} = \mathrm{D}_{q,y}^{\lambda-\mu}\left[\sum_{m,n=0}^{\infty} \frac{\langle\alpha;q\rangle_{m+n} \langle\beta;q\rangle_m}{\langle\gamma,1;q\rangle_m \langle 1;q\rangle_n} x^m y^{\lambda+n-1}\right] \tag{80}$$

$$\overset{\text{by}[9,8.118]}{=} \sum_{m,n=0}^{\infty} \frac{\langle\alpha;q\rangle_{m+n} \langle\beta;q\rangle_m x^m}{\langle\gamma,1;q\rangle_m \langle 1;q\rangle_n} y^{\mu+n-1} \Gamma_q\left[\begin{array}{c} \lambda+n \\ \mu+n \end{array}\right] \overset{\text{by}[9,1.45,1.46]}{=} \text{RHS}.$$

Definition 7. *Assume that $\vec{m} \equiv (m_1,\ldots,m_n)$, $m \equiv m_1+\ldots+m_n$ and $a \in \mathbb{R}^\star$. The vector q-multinomial-coefficient $\binom{a}{\vec{m}}_q^\star$ [13] is defined by the symmetric expression*

$$\binom{a}{\vec{m}}_q^\star \equiv \frac{\langle -a;q\rangle_m (-1)^m q^{-\binom{\vec{m}}{2}+am}}{\langle 1;q\rangle_{m_1} \langle 1;q\rangle_{m_2} \ldots \langle 1;q\rangle_{m_n}}. \tag{81}$$

The following formula applies for a q–deformed hypercube of length 1 in $\mathbb{R}^n$. Note that formulas (82) and (83) are symmetric in the x_i.

Definition 8. *[13] Assuming that the right hand side converges, and $a \in \mathbb{R}^\star$:*

$$(1 \boxminus_q q^a x_1 \boxminus_q \ldots \boxminus_q q^a x_n)^{-a} \equiv \sum_{m_1,\ldots,m_n=0}^{\infty} \prod_{j=1}^{n} (-x_j)^{m_j} \binom{-a}{\vec{m}}_q^{\star} q^{\binom{\vec{m}}{2}+am}. \tag{82}$$

Corollary 1. *A generalization of the q-binomial theorem [13]:*

$$(1 \boxminus_q q^a x_1 \boxminus_q \ldots \boxminus_q q^a x_n)^{-a} = \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle a; q\rangle_m \vec{x}^{\vec{m}}}{\langle \vec{1}; q\rangle_{\vec{m}}}, \ a \in \mathbb{R}^\star. \tag{83}$$

Proof. Use formulas (81) and (82), the terms with factors $q^{-\binom{\vec{m}}{2}+am}$ cancel each other.

Theorem 27. *A q-analogue of Srivastava and Manocha [23, p.306].*

$$\mathrm{D}_{q,z_1}^{\beta_1-\gamma_1} \ldots \mathrm{D}_{q,z_n}^{\beta_n-\gamma_n} \left[z_1^{\beta_1-1} \ldots z_n^{\beta_n-1} (1 \boxminus_q q^\alpha z_1 \boxminus_q \ldots \boxminus_q q^\alpha z_n)^{-\alpha} \right] \tag{84}$$

$$= \prod_{j=1}^{n} \left[\frac{\Gamma_q(\beta_j)}{\Gamma_q(\gamma_j)} z_j^{\gamma_j - 1} \right] \Phi_{\mathrm{A}}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n | q; z_1, \ldots, z_n), \ \alpha \in \mathbb{R}^\star.$$

Proof.

$$\mathrm{LHS} = \mathrm{D}_{q,z_1}^{\beta_1-\gamma_1} \ldots \mathrm{D}_{q,z_n}^{\beta_n-\gamma_n} \left[z_1^{\beta_1-1} \ldots z_n^{\beta_n-1} \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle \alpha; q\rangle_m \vec{z}^{\vec{m}}}{\langle \vec{1}; q\rangle_{\vec{m}}} \right] \overset{\text{by}[9,8.118]}{=} \tag{85}$$

$$\sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle \alpha; q\rangle_m}{\langle \vec{1}; q\rangle_{\vec{m}}} \Gamma_q \left[\begin{matrix} m \vec{+} \beta \\ m \vec{+} \gamma \end{matrix} \right] \vec{z}^{m \vec{+} \gamma - 1} \overset{\text{by}[9,1.45,1.46]}{=} \vec{z}^{\vec{\gamma}-1} \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle \alpha; q\rangle_m \langle \vec{\beta}; q\rangle_{\vec{m}}}{\langle \vec{1}, \vec{\gamma}; q\rangle_{\vec{m}}} = \mathrm{RHS}.$$

Conclusion: The Euler q-integral proofs are made in a similar style. First the (multiple) q-integral is replaced by its definition [9, 6.54]. Then function(s) in the infinite sums are replaced by equivalent q-shifted factorials with the use of [9, 6.8,6.10]. The number of sums is reduced by using the q-binomial theorem [9, 7.27], observe that we always use the result quotient of q-shifted factorials. In the final step, we have a (multiple) sum equivalent to the expected result. The factors are rewritten by using formulas [9, 1.45,1.46] to give the left hand side. Observe that we always have a factor $(1-q)^m$, where m is the dimension of the (multiple) q-integral. This factor is deleted by using the definition of the q-gamma function [9, 1.45]. Formula [9, 1.46] never gives this factor. The definition of fractional q-integral [9, 8.118] can also be used in the first step.

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