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Solving a class of integral equations via contractive mapping with rational type

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Abstract. In this paper, using rational type contractive conditions, the existence and uniqueness of common coupled fixed point theorem in the set up of G_b-metric spaces is studied. The derived result cover and generalize some well-known comparable results in the existing literature. Then we use the derived results to prove the existence and uniqueness solution for some classes of integral equations. Further more, an example of such type of integral equation is presented.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Integral equations, Complete G_b-metric space, Cauchy sequence, common coupled fixed point.

1. Introduction and Preliminaries

The notion of metric space has been generalized in different directions. In particular, Mustafa and Sims [18] introduced a new generalization of metric space known as G-metric space. In fact, they assigned a non-negative real number to every triplet of elements of a metric space M and studied fixed point results. Further, in the setting of G-metric space, Mustafa et al. [20] investigated some fixed point results for mappings via rational type contractions. Abbas and Rhoades [2] opened the study of finding common fixed points in G-metric spaces. Shatanawi [15] studied applications to integral equations via fixed point results for two weakly increasing mappings f and g with respect to partial ordering relation $\preceq$ in G- metric spaces. After that several fixed point results were proved in these spaces. Some of these works are noted in $\big[$[8],[3],[19],[22],[23]$\big]$.

Recently Aghajani et al. [1] introduced the concept of G_b-metric spaces by combining the definition of G-metric and b-metric spaces and studied a common fixed result for six mappings. Jamal Rezaei et al. [13] obtained common fixed point results for three maps in discontinuous G_b-metric spaces. Sedghi et al. [14] derived coupled fixed point theorems

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in G_b-metric spaces. Khomdram et al. [4] obtained a coupled fixed point theorem in G_b-metric space using rational type contractive conditions. In addition Mustafa et al. [21] worked on applications to a system of integral equations with the help of tripled coincidence point in ordered G_b-metric spaces.

Guo and Lakshmikantham [5] gave the concept of coupled fixed point of non-linear operator with applications. Lakshmikantham et al. [16] introduced Ćirić type mappings in the framework of G_b-metric spaces. Sintunavarat et al. [17] used the nonlinear contraction to established a coupled fixed point results in complete metric spaces without the mixed monotone property. Using generalized contractive condition, Nashie et al. [6] proved the existence and uniqueness of a common coupled fixed point theorem for a pair of mappings in complete cone metric space. Radenović [10] presented some remarks on some recent coupled coincidence point results in symmetric G-metric spaces and reduce some coupled coincidence point results to more general forms. For more details on coupled fixed point results we refer the reader to [[12], [9],[10], [11], [7]].

In the current work we will obtain a common triple fixed point in G_b-metric space using contraction with rational type.

Definition 1. *[18] Let $S \neq \emptyset$ and $G : S^3 \to \mathbb{R}^+$ satisfies :*
$G1)$ $G(p,q,r) = 0$ *if* $p = q = r$;
$G2)$ $G(p,p,q) > 0$ *for all* $p,q \in S$ *with* $p \neq q$;
$G3)$ $G(p,q,q) \leq G(p,q,r)$ *for all* $p,q,r \in S$ *with* $q \neq r$;
$G4)$ $G(p,q,r) = G(q,r,p) = G(r,p,q) = \cdots$ *(symmetry in all three variables);*
$G5)$ $G(p,q,r) \leq G(p,t,t) + G(t,q,r)$ *for all* $p,q,r,t \in S$.

Then G is called generalized metric on S, and the pair (S,G) is called G-metric space.

Definition 2. *[1] Let $Y \neq \emptyset$ and $s \geq 1$ be a given real number. Suppose that a mapping $G_b : Y \times Y \times Y \to \mathbb{R}^+$ satisfies:*
(G_b1) $G_b(a,b,c) = 0$ *if* $a = b = c$,
(G_b2) $G_b(a,a,b) > 0$ *for all* $a,b \in Y$ *with* $a \neq b$,
(G_b3) $G_b(a,b,b) \leq G_b(a,b,c)$ *for all* $a,b,c \in Y$ *with* $b \neq c$,
(G_b4) $G_b(a,b,c) = G_b(p\{a,b,c\})$ *where* p *is a permutation of* a,b,c *(symmetry)*
(G_b5) $G_b(a,b,c) \leq s(G_b(a,u,u) + G_b(u,b,c))$ *for all* $a,b,c,u \in Y$.

Then G_b is called G_b-metric on Y, and the pair (Y,G_b) is called G_b-metric space. If $s = 1$, then G_b reduce to G-metric space. G_b is the generalization of G-metric, evidently every G-metric is a G_b-metric.

Remark 1. *Every G-metric space (X,G) defines a G_b-metric space with $s = 2^{p-1}$ by $G_b(x,y,z) = (G(x,y,z))^p$, where $p > 1$ is a real number. Further every G_b-metric is a G-metric when $s = 1$, but, in general, not every G_b-metric is a G-metric, for example, let $X = \mathbb{R}$ and the metric G_b is defined by $G_b(x,y,z) = \max\left\{|x-y|^2, |y-z|^2, |z-x|^2\right\}$, for all $x,y,z \in \mathbb{R}$. Then G_b is a G_b-metric on $\mathbb{R}$ with $s = 2^{2-1} = 2$, but it is not a G-metric on $\mathbb{R}$.*

Proposition 1. *[1] Let X be a G_b-metric space, then for each $x, y, z, a \in X$ the following holds:*

(1) if $G_b(x, y, z) = 0$ then $x = y = z$.
(2) $G(x, y, z) \leq s(G_b(x, x, y) + G_b(x, x, z))$.
(3) $G_b(x, y, y) \leq 2sG_b(y, x, x)$.
(4) $G_b(x, y, z) \leq s(G(x, a, z) + G(a, y, z))$.

Definition 3. *[1]*

Let X' be a G_b-metric space. A sequence $\{x'_n\}$ in X is said to be:

(1) G_b-Cauchy sequence if, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{Z}_+$ such that for all $m, n, l \geq n_0, G_b(x_n, x_m, x_l) < \epsilon$.

(2) G_b-convergent to a point $x \in X$ if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}_+$ such that, for all $m, n \geq n_0, G_b(x_n, x_m, x) < \epsilon$.

Proposition 2. *[1] Let X be a G_b-metric space, Then the following are equivalent:*

(1) the sequence x_n is G_b-Cauchy.
(2) for any $\epsilon > 0$, there exists $n_0 \in N$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $m, n \geq n_0$.

Proposition 3. *[1] The following are equivalent in (X, G_b) metric spaces:*

(A1) $\{x_n\}$ is G_b-convergent to x.
(A2) $G_b(x_n, x_n, x) \to 0$ as $n \to +\infty$.
(A3) $G_b(x_n x, x) \to 0$ as $n \to +\infty$.

Definition 4. *[1] A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.*

Definition 5. *[5] Let M' be a metric space and let $F : M \times M \to M$ be a function. An element $(m, n) \in M \times M$ is said to be a coupled fixed point of the mapping F if $F(m, n) = m, F(n, m) = n$.*

2. Main Result

Throughout the paper we will use the following generalized contraction. Assume that (X, G_b) is generalized G_b-metric space. The mappings $F, H, T : X \times X \to X$ are said to satisfy the generalized contraction if for $a, b, u, v, x, y \in X$

$$G_b\Big(F(a, b), H(u, v), T(x, y)\Big) \leq \mathcal{N}(a, b, u, v, x, y),$$

where

$$\mathcal{N}(a, b, u, v, x, y) = \alpha_1(a, u) \cdot \frac{G_b(a, u, x) + G_b(b, v, y)}{2}$$
$$+\alpha_2(a, u)\frac{G_b\Big(F(a, b), H(u, v), T(x, y)\Big)G_b(a, u, x)}{\Omega_s(a, u, x, b, v, y)}$$
$$+\alpha_3(a, u)\frac{G_b\Big(F(a, b), H(u, v), T(x, y)\Big)G_b(b, v, y)}{\Omega_s(a, u, x, b, v, y)}$$

$$+\alpha_4(a,u)\frac{G_b\big(a,a,F(a,b)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}+\alpha_5(u,x)\frac{G_b\big(a,a,F(a,b)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_6(u,x)\frac{G_b\big(u,u,H(u,v)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}+\alpha_7(u,x)\frac{G_b\big(u,u,H(u,v)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_8(u,x)\frac{G_b\big(x,x,T(x,y)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}+\alpha_9(x,a)\frac{G_b\big(x,x,T(x,y)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_{10}(x,a)\frac{G_b\big(a,u,T(x,y)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}+\alpha_{11}(x,a)\frac{G_b\big(u,x,F(a,b)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_{12}(x,a)\frac{G_b\big(x,a,H(u,v)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}, \tag{1}$$

$\Omega_1(a,u,x,b,v,y) = [1+G_b(a,u,x)+G_b(b,v,y)]$, $\Omega_s(a,u,x,b,v,y) = s\Omega_1(a,u,x,b,v,y)$ and $\alpha_i : X\times X\to[0,1)$, $i=1,2,3,\cdots,12$. such that

$$0\le s[\alpha_1(x,y)+\alpha_4(x,y)+\alpha_5(x,y)+\alpha_{10}(x,y)+\alpha_{12}(x,y)]+[\alpha_2(x,y)+\alpha_3(x,y)+\sum_{i=6}^{11}\alpha_i(x,y)]<1. \tag{2}$$

Theorem 1. *Let (X,G_b) be a complete G_b-metric space and let $F,H,T : X\times X\to X$ be mappings satisfying the above generalized contraction(1) and the following conditions satisfied $\forall i, i=1,\cdots,12$:*

(i) $\alpha_i(F(x_1,y_1),y)\le\alpha_i(x_1,y)$ *and* $\alpha_i(x,F(x_1,y_1))\le\alpha_i(x,x_1))$;

(ii) $\alpha_i(H(x_1,y_1),y)\le\alpha_i(x_1,y)$ *and* $\alpha_i(x,H(x_1,y_1))\le\alpha_i(x,x_1))$;

(iii) $\alpha_i(T(x_1,y_1),y)\le\alpha_i(x_1,y)$ *and* $\alpha_i(x,T(x_1,y_1))\le\alpha_i(x,x_1))$.

Then F,H,T have a unique common coupled fixed point in X.

Proof. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X by the rule:

$$\begin{aligned} x_{3k+1} &= F(x_{3k},y_{3k}), \\ y_{3k+1} &= F(y_{3k},x_{3k}),. \end{aligned} \tag{3}$$

$$\begin{aligned} x_{3k+2} &= H(x_{3k+1},y_{3k+1}), \\ y_{3k+2} &= H(y_{k+1},x_{3k+1}), \end{aligned} \tag{4}$$

$$\begin{aligned} x_{3k+3} &= T(x_{3k+2},y_{3k+2}) \\ y_{3k+3} &= T(y_{k+2},x_{3k+2}), \end{aligned} \tag{5}$$

where $k=0,1,2,3,\cdots$ and x_0,y_0 to be arbitrary in X. Now, consider

$$G_b(x_{3k+1},x_{3k+2},x_{3k+3}) = G_b(F(x_{3k},y_{3k}),H(x_{3k+1},y_{3k+1}),T(x_{3k+2},y_{3k+2}))$$

$$
\begin{aligned}
&\leq \alpha_1(x_{3k}, x_{3k+1}).\frac{G_b(x_{3k}, x_{3k+1}, x_{3k+2}) + G_b(y_{3k}, y_{3k+1}, y_{3k+2})}{2} \\
&+\alpha_2(x_{3k}, x_{3k+1}) \cdot \frac{G_b\Big(F(x_{3k}, y_{3k}), H(x_{3k+1}, y_{3k+1}), T(x_{3k+2}, y_{3k+2})\Big)G_b(x_{3k}, x_{3k+1}, x_{3k+2})}{\Omega_s(x_{3k}, x_{3k+1}, x_{3k+2}, y_{3k}, y_{3k+1}, y_{3k+2})} \\
&+\alpha_3(x_{3k}, x_{3k+1}) \cdot \frac{G_b\Big(F(x_{3k}, y_{3k}), H(x_{3k+1}, y_{3k+1}), T(x_{3k+2}, y_{3k+2})\Big)G_b(y_{3k}, y_{3k+1}, y_{3k+2})}{\Omega_s(x_{3k}, x_{3k+1}, x_{3k+2}, y_{3k}, y_{3k+1}, y_{3k+2})} \\
&+\alpha_4(x_{3k}, x_{3k+1}) \cdot \frac{G_b(x_{3k}, x_{3k}, F(x_{3k}, y_{3k}))G_b(x_{3k}, x_{3k+1}, x_{3k+2})}{\Omega_s(x_{3k}, x_{3k+1}, x_{3k+2}, y_{3k}, y_{3k+1}, y_{3k+2})} \\
&+\alpha_5(x_{3k+1}, x_{3k+2}) \cdot \frac{G_b(x_{3k}, x_{3k}, F(x_{3k}, y_{3k}))G_b(y_{3k}, y_{3k+1}, y_{3k+2})}{\Omega_s(x_{3k}, x_{3k+1}, x_{3k+2}, y_{3k}, y_{3k+1}, y_{3k+2})} \\
&+\alpha_6(x_{3k+1}, x_{3k+2}) \cdot \frac{G_b(x_{3k+1}, x_{3k+1}, H(x_{3k+1}, y_{3k+1}))G_b(x_{3k}, x_{3k+1}, x_{3k+2})}{\Omega_s(x_{3k}, x_{3k+1}, x_{3k+2}, y_{3k}, y_{3k+1}, y_{3k+2})} \\
&+\alpha_7(x_{3k+1}, x_{3k+2}) \cdot \frac{G_b(x_{3k+1}, x_{3k+1}, H(x_{3k+1}, y_{3k+1}))G_b(y_{3k}, y_{3k+1}, y_{3k+2})}{\Omega_s(x_{3k}, x_{3k+1}, x_{3k+2}, y_{3k}, y_{3k+1}, y_{3k+2})} \\
&+\alpha_8(x_{3k+1}, x_{3k+2}) \cdot \frac{G_b(x_{3k+2}, x_{3k+2}, T(x_{3k+2}, y_{3k+2}))G_b(x_{3k}, x_{3k+1}, x_{3k+2})}{\Omega_s(x_{3k}, x_{3k+1}, x_{3k+2}, y_{3k}, y_{3k+1}, y_{3k+2})} \\
&+\alpha_9(x_{3k+2}, x_{3k}) \cdot \frac{G_b(x_{3k+2}, x_{3k+2}, T(x_{3k+2}, y_{3k+2}))G_b(y_{3k}, y_{3k+1}, y_{3k+2})}{\Omega_s(x_{3k}, x_{3k+1}, x_{3k+2}, y_{3k}, y_{3k+1}, y_{3k+2})} \\
&+\alpha_{10}(x_{3k+2}, x_{3k}) \cdot \frac{G_b(x_{3k}, x_{3k+1}, T(x_{3k+2}, y_{3k+2}))G_b(x_{3k}, x_{3k+1}, x_{3k+2})}{\Omega_s(x_{3k}, x_{3k+1}, x_{3k+2}, y_{3k}, y_{3k+1}, y_{3k+2})} \\
&+\alpha_{11}(x_{3k+2}, x_{3k}) \cdot \frac{G_b(x_{3k+1}, x_{3k+2}, F(x_{3k}, y_{3k}))G_b(y_{3k}, y_{3k+1}, y_{3k+2})}{\Omega_s(x_{3k}, x_{3k+1}, x_{3k+2}, y_{3k}, y_{3k+1}, y_{3k+2})} \\
&+\alpha_{12}(x_{3k+2}, x_{3k}) \cdot \frac{G_b(x_{3k+2}, x_{3k}, H(x_{3k+1}, y_{3k+1}))G_b(x_{3k}, x_{3k+1}, x_{3k+2})}{\Omega_s(x_{3k}, x_{3k+1}, x_{3k+2}, y_{3k}, y_{3k+1}, y_{3k+2})}
\end{aligned}
$$

Now, with the help of condition (i), we have

$$
\begin{aligned}
\alpha_i(x_{3k}, x_{3k+1}) \quad &= \alpha_i(x_{3k}, F(x_{3k}, y_{3k})) \leq \alpha_i(x_{3k}, x_{3k}) \\
&= \alpha_i(x_{3k}, F(x_{3k-1}, y_{3k-1})) \leq \alpha_i(x_{3k}, x_{3k-1}) \\
&= \alpha_i(F(x_{3k-1}, y_{3k-1}), x_{3k-1}) \\
&\leq \alpha_i(x_{3k-1}, x_{3k-1}) \leq \cdots \leq \alpha_i(x_0, x_0) \text{ for all } i = 1, 2, 3, 4.
\end{aligned}
$$

Similarly by using (ii) and (iii) one can show that

$$
\begin{aligned}
&\alpha_i(x_{3k+1}, x_{3k+2}) \leq \alpha_i(x_0, x_0) \text{ for all } i = 5, 6, 7, 8. \\
&\alpha_i(x_{3k+2}, x_{3k}) \leq \alpha_i(x_0, x_0) \text{ for all } i = 9, 10, 11, 12.
\end{aligned}
$$

So by using the above and (3) and (4) one has

$$
G_b(x_{3k+1}, x_{3k+2}, x_{3k+3}) \leq \alpha_1(x_0, x_0).\frac{G_b(x_{3k}, x_{3k+1}, x_{3k+2}) + G_b(y_{3k}, y_{3k+1}, y_{3k+2})}{2}
$$

$$+\alpha_2(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+2},x_{3k+3})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_3(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+2},x_{3k+3})G_b(y_{3k},y_{3k+1},y_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_4(x_0,x_0)\cdot\frac{G_b(x_{3k},x_{3k},x_{3k+1})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_5(x_0,x_0)\cdot\frac{G_b(x_{3k},x_{3k},x_{3k+1})G_b(y_{3k+1},y_{3k+1},y_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_6(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+1},x_{3k+2})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_7(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+1},x_{3k+2})G_b(y_{3k},y_{3k+1},y_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_8(x_0,x_0)\cdot\frac{G_b(x_{3k+2},x_{3k+2},x_{3k+3})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_9(x_0,x_0)\cdot\frac{G_b(x_{3k+2},x_{3k+2},x_{3k+3})G_b(y_{3k},y_{3k+1},y_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_{10}(x_0,x_0)\cdot\frac{G_b(x_{3k},x_{3k+1},x_{3k+3})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_{11}(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+2},x_{3k+1})G_b(y_{3k},y_{3k+1},y_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_{12}(x_0,x_0)\cdot\frac{G_b(x_{3k+2},x_{3k},x_{3k+2})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$

Now using (G_b3), (G_b4) and (G_b5) of Definition 2 we have

$$G_b(x_{3k+1},x_{3k+2},x_{3k+3}\leq\alpha_1(x_0,x_0).\frac{G_b(x_{3k},x_{3k+1},x_{3k+2})+G_b(y_{3k},y_{3k+1},y_{3k+2})}{2}$$
$$+\alpha_2(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+2},x_{3k+3})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_3(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+2},x_{3k+3})G_b(y_{3k},y_{3k+1},y_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_4(x_0,x_0)\cdot\frac{G_b(x_{3k},x_{3k+1},x_{3k+2})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_5(x_0,x_0)\cdot\frac{G_b(x_{3k},x_{3k+1},x_{3k+2})G_b(y_{3k+1},y_{3k+1},y_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_6(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+2},x_{3k+3})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_7(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+2},x_{3k+3})G_b(y_{3k},y_{3k+1},y_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$

$$+\alpha_8(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+2},x_{3k+3})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$

$$+\alpha_9(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+2},x_{3k+3})G_b(y_{3k},y_{3k+1},y_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_{10}(x_0,x_0)\cdot\frac{G_b(x_{3k},x_{3k+1},x_{3k+2})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_1(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_{10}(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+2},x_{3k+3})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_1(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_{11}(x_0,x_0)\cdot\frac{G_b(x_{3k+1},x_{3k+2},x_{3k+3})G_b(y_{3k},y_{3k+1},y_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$
$$+\alpha_{12}(x_0,x_0)\cdot\frac{G_b(x_{3k},x_{3k+1},x_{3k+2})G_b(x_{3k},x_{3k+1},x_{3k+2})}{\Omega_s(x_{3k},x_{3k+1},x_{3k+2},y_{3k},y_{3k+1},y_{3k+2})}$$

Which implies that

$$\begin{aligned}
&G_b(x_{3k+1},x_{3k+2},x_{3k+3})\leq\alpha_1(x_0,x_0).\frac{G_b(x_{3k},x_{3k+1},x_{3k+2})+G_b(y_{3k},y_{3k+1},y_{3k+2})}{2}\\
&+\alpha_2(x_0,x_0)\cdot G_b(x_{3k+1},x_{3k+2},x_{3k+3})+\alpha_3(x_0,x_0)\cdot G_b(x_{3k+1},x_{3k+2},x_{3k+3})\\
&+\alpha_4(x_0,x_0)\cdot G_b(x_{3k},x_{3k+1},x_{3k+2})+\alpha_5(x_0,x_0)\cdot G_b(x_{3k},x_{3k+1},x_{3k+2})\\
&+\alpha_6(x_0,x_0)\cdot G_b(x_{3k+1},x_{3k+2},x_{3k+3})+\alpha_7(x_0,x_0)\cdot G_b(x_{3k+1},x_{3k+2},x_{3k+3})\\
&+\alpha_8(x_0,x_0)\cdot G_b(x_{3k+1},x_{3k+2},x_{3k+3})+\alpha_9(x_0,x_0)\cdot G_b(x_{3k+1},x_{3k+2},x_{3k+3})\\
&+\alpha_{10}(x_0,x_0)\cdot G_b(x_{3k},x_{3k+1},x_{3k+2})+\alpha_{10}(x_0,x_0)\cdot G_b(x_{3k+1},x_{3k+2},x_{3k+3})\\
&+\alpha_{11}(x_0,x_0)\cdot G_b(x_{3k+1},x_{3k+2},x_{3k+3})+\alpha_{12}(x_0,x_0)\cdot G_b(x_{3k},x_{3k+1},x_{3k+2}).
\end{aligned}$$

Which further implies that

$$\begin{aligned}
&\Big(1-\alpha_2(x_0,x_0)-\alpha_3(x_0,x_0)-\sum_{i=6}^{11}\alpha_i(x_0,x_0)\Big)\cdot G_b(x_{3k+1},x_{3k+2},x_{3k+3})\\
&\leq\Big(\frac{\alpha_1(x_0,x_0)}{2}+\alpha_4(x_0,x_0)+\alpha_5(x_0,x_0)+\alpha_{10}(x_0,x_0)+\alpha_{12}(x_0,x_0)\Big)\cdot G_b(x_{3k},x_{3k+1},x_{3k+2})\\
&+\frac{\alpha_1(x_0,x_0)}{2}\cdot G_b(y_{3k},y_{3k+1},y_{3k+2}).
\end{aligned}$$

Which gives

$$\begin{aligned}
&G_b(x_{3k+1},x_{3k+2},x_{3k+3})\\
&\leq\frac{\Big(\frac{\alpha_1(x_0,x_0)}{2}+\alpha_4(x_0,x_0)+\alpha_5(x_0,x_0)+\alpha_{10}(x_0,x_0)+\alpha_{12}(x_0,x_0)\Big)}{\Big(1-\alpha_2(x_0,x_0)-\alpha_3(x_0,x_0)-\sum_{i=6}^{11}\alpha_i(x_0,x_0)\Big)}.G_b(x_{3k},x_{3k+1},x_{3k+2})
\end{aligned}\tag{6}$$

$$+\frac{\alpha_1(x_0,x_0)}{2\Big(1-\alpha_2(x_0,x_0)-\alpha_3(x_0,x_0)-\sum\limits_{i=6}^{11}\alpha_i(x_0,x_0)\Big)}\cdot G_b(y_{3k},y_{3k+1},y_{3k+2}).$$

Similarly for the sequence $\{y_n\}$, we have

$$\begin{aligned}&G_b(y_{3k+1},y_{3k+2},y_{3k+3}) \\ &\le \frac{\Big(\frac{\alpha_1(x_0,x_0)}{2}+\alpha_4(x_0,x_0)+\alpha_5(x_0,x_0)+\alpha_{10}(x_0,x_0)+\alpha_{12}(x_0,x_0)\Big)}{\Big(1-\alpha_2(x_0,x_0)-\alpha_3(x_0,x_0)-\sum\limits_{i=6}^{11}\alpha_i(x_0,x_0)\Big)}.G_b(y_{3k},y_{3k+1},y_{3k+2}) \\ &+\frac{\alpha_1(x_0,x_0)}{2\Big(1-\alpha_2(x_0,x_0)-\alpha_3(x_0,x_0)-\sum\limits_{i=6}^{11}\alpha_i(x_0,x_0)\Big)}\cdot G_b(x_{3k},x_{3k+1},x_{3k+2}).\end{aligned}\tag{7}$$

Adding inequalities (6), and (7) we get

$$G_b(x_{3k+1},x_{3k+2},x_{3k+3})+G_b(y_{3k+1},y_{3k+2},y_{3k+3})\le h.\Big(G_b(x_{3k},x_{3k+1},x_{3k+2})+G_b(y_{3k},y_{3k+1},y_{3k+2})\Big),$$

where

$$0\le h=\frac{\Big(\alpha_1(x_0,x_0)+\alpha_4(x_0,x_0)+\alpha_5(x_0,x_0)+\alpha_{10}(x_0,x_0)+\alpha_{12}(x_0,x_0)\Big)}{\Big(1-\alpha_2(x_0,x_0)-\alpha_3(x_0,x_0)-\sum\limits_{i=6}^{11}\alpha_i(x_0,x_0)\Big)}.$$

By (2) we have

$$0<h<\frac{1}{s}.\tag{8}$$

Similarly,

$$\begin{aligned}&G_b(x_{3k+2},x_{3k+3},x_{3k+4})+G_b(y_{3k+2},y_{3k+3},y_{3k+4}) \\ &\le h\cdot\Big(G_b(x_{3k+1},x_{3k+2},x_{3k+3})+G_b(y_{3k+1},y_{3k+2},y_{3k+3})\Big).\end{aligned}$$

Continuing this way, we have

$$\begin{aligned}&G_b(x_n,x_{n+1},x_{n+2})+G_b(y_n,y_{n+1},y_{n+2}) \\ &\le h\cdot(G_b(x_{n-1},x_n,x_{n+1})+G_b(y_{n-1},y_n,y_{n+1})) \\ &\le h^2\cdot(G_b(x_{n-2},x_{n-1},x_n)+G_b(y_{n-2},y_{n-1},y_n)) \\ &\le h^3\cdot(G_b(x_{n-3},x_{n-2},x_{n-1})+G_b(y_{n-3},y_{n-2},y_{n-1})) \\ &\le\cdots\le h^{n+1}\cdot(G_b(x_0,x_1,x_2)+G_b(y_0,y_1,y_2)).\end{aligned}\tag{9}$$

Let

$$G_b(x_n,x_{n+1},x_{n+2})+G_b(y_n,y_{n+1},y_{n+2})=\Psi_n.$$

Then the pattern (9) can be written as

$$\Psi_n \le h.\Psi_{n-1} \le h^2.\Psi_{n-2} \le \cdots \le h^n \cdot \Psi_0. \tag{10}$$

Now, we prove that sequences $\{x_n\}$ and $\{y_n\}$ are G_b-cauchy. By (8) we have $0 \le sh < 1$. Let $j > k$. Then

$$\begin{aligned}
&G_b(x_k, x_j, x_j) + G_b(y_k, y_j, y_j)\\
&\le s[G_b(x_k, x_{k+1}, x_{k+1}) + G_b(x_{k+1}, x_j, x_j) + G_b(y_k, y_{k+1}, y_{k+1}) + G_b(y_{k+1}, y_j, y_j)]\\
&\le s[G_b(x_k, x_{k+1}, x_{k+1}) + G_b(y_k, y_{k+1}, y_{k+1})] + s^2[G_b(x_{k+1}, x_{k+2}, x_{k+2}) + G_b(y_{k+1}, y_{k+2}, y_{k+2})]\\
&+s^3[G_b(x_{k+2}, x_{k+3}, x_{k+3}) + G_b(y_{k+2}, y_{k+3}, y_{k+3}] + \cdots\\
&+s^{j-k}[G_b(x_{j-1}, x_j, x_j) + G_b(y_{j-1}, y_j, y_j]\\
&\le sh^k\Psi_0 + s^2h^{k+1}\Psi_0 + s^3h^{k+2}\Psi_0 + \cdots + s^{j-k}h^{j-1}\Psi_0\\
&\le sh^k[1 + sh + (sh)^2 + (sh)^3 + \cdots]\Psi_0\\
&= \frac{sh^k}{1-sh}\Psi_0. \to 0 \quad as \quad k \to +\infty.
\end{aligned}$$

Hence $G_b(x_k, x_j, x_j) + G_b(y_k, y_j, y_j) \to 0$ as $k \to +\infty$, which shows that $\{x_n\}$ and $\{y_n\}$ are G_b-cauchy sequences in X by Proposition 2. But due to the completeness of G_b-metric spaces, we have $x_n \to x$ and $y_n \to y$ as $n \to +\infty$, for $x, y \in X$.

Now, we show that (x, y) is common coupled fixed points of F, H and T. Suppose that $x \ne F(x, y)$ and $G_b(x, F(x, y), F(x, y)) > 0$. Thus from definition of G_b-metric we have

$$\begin{aligned}
&G_b(x, F(x, y), F(x, y)) \le G_b(x, x_{3k+2}, F(x, y))\\
&\le s \cdot \Big[G_b(x, x_{3k+3}, x_{3k+3}) + G_b(x_{3k+3}, x_{3k+2}, F(x, y))\Big]\\
&= s \cdot G_b(x, x_{3k+3}, x_{3k+3}) + s \cdot G_b\big(T(x_{3k+2}, y_{3k+2}), H(x_{3k+1}, y_{3k+1}), F(x, y)\big)\\
&\le s \cdot G_b(x, x_{3k+3}, x_{3k+3}) + s\alpha_1(x_{3k+2}, x_{3k+1}). \frac{G_b(x_{3k+2}, x_{3k+1}, x) + G_b(y_{3k+2}, y_{3k+1}, y)}{2}\\
&+s.\alpha_2(x_{3k+2}, x_{3k+1}) \cdot \frac{G_b\big(T(x_{3k+2}, y_{3k+2}), H(x_{3k+1}, y_{3k+1}), F(x, y)\big)\big)G_b(x_{3k+2}, x_{3k+1}, x)}{\Omega_s(x_{3k+2}, x_{3k+1}, x, y_{3k+2}, y_{3k+1}, y)}\\
&+s.\alpha_3(x_{3k+2}, x_{3k+1}) \cdot \frac{G_b\big(T(x_{3k+2}, y_{3k+2}), H(x_{3k+1}, y_{3k+1}), F(x, y)\big)\big)G_b(y_{3k+2}, y_{3k+1}, y)}{\Omega_s(x_{3k+2}, x_{3k+1}, x, y_{3k+2}, y_{3k+1}, y)}\\
&+s.\alpha_4(x_{3k+2}, x_{3k+1}) \cdot \frac{G_b(x_{3k+2}, x_{3k+2}, T(x_{3k+2}, y_{3k+2}))G_b(x_{3k+2}, x_{3k+1}, x)}{\Omega_s(x_{3k+2}, x_{3k+1}, x, y_{3k+2}, y_{3k+1}, y)}\\
&+s.\alpha_5(x_{3k+1}, x) \cdot \frac{G_b(x_{3k+2}, x_{3k+2}, T(x_{3k+2}, y_{3k+2}))G_b(y_{3k+2}, y_{3k+1}, y)}{\Omega_s(x_{3k+2}, x_{3k+1}, x, y_{3k+2}, y_{3k+1}, y)}\\
&+s.\alpha_6(x_{3k+1}, x) \cdot \frac{G_b(x_{3k+1}, x_{3k+1}, H(x_{3k+1}, y_{3k+1}))G_b(x_{3k+2}, x_{3k+1}, x)}{\Omega_s(x_{3k+2}, x_{3k+1}, x, y_{3k+2}, y_{3k+1}, y)}\\
&+s.\alpha_7(x_{3k+1}, x) \cdot \frac{G_b(x_{3k+1}, x_{3k+1}, H(x_{3k+1}, y_{3k+1}))G_b(y_{3k+2}, y_{3k+1}, y)}{\Omega_s(x_{3k+2}, x_{3k+1}, x, y_{3k+2}, y_{3k+1}, y)}
\end{aligned}$$

$$+s.\alpha_8(x_{3k+1},x)\cdot\frac{G_b(x,x,F(x,y))G_b(x_{3k+2},x_{3k+1},x)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_9(x,x_{3k+2})\cdot\frac{G_b(x,x,F(x,y))G_b(y_{3k+2},y_{3k+1},y)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_{10}(x,x_{3k+2})\cdot\frac{G_b(x_{3k+2},x_{3k+1},F(x,y))G_b(x_{3k+2},x_{3k+1},x)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_{11}(x,x_{3k+2})\cdot\frac{G_b(x_{3k+1},x,T(x_{3k+2},y_{3k+2}))G_b(y_{3k+2},y_{3k+1},y)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_{12}(x,x_{3k+2})\cdot\frac{G_b(x,x_{3k+2},H(x_{3k+1},y_{3k+1}))G_b(x_{3k+2},x_{3k+1},x)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}.$$

Using (3) and (4), we have

$$G_b(x,F(x,y),F(x,y))\leq G_b(x,x_{3k+2},F(x,y))$$
$$\leq s\cdot G_b(x,x_{3k+3},x_{3k+3})+s\alpha_1(x_{3k+2},x_{3k+1}).\frac{G_b(x_{3k+2},x_{3k+1},x)+G_b(y_{3k+2},y_{3k+1},y)}{2}$$
$$+s.\alpha_2(x_{3k+2},x_{3k+1})\cdot\frac{G_b\big(x_{3k+3},x_{3k+2},F(x,y)\big)G_b(x_{3k+2},x_{3k+1},x)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_3(x_{3k+2},x_{3k+1})\cdot\frac{G_b\big(x_{3k+3},x_{3k+2},F(x,y)\big)G_b(y_{3k+2},y_{3k+1},y)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_4(x_{3k+2},x_{3k+1})\cdot\frac{G_b(x_{3k+2},x_{3k+2},x_{3k+3})G_b(x_{3k+2},x_{3k+1},x)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_5(x_{3k+1},x)\cdot\frac{G_b(x_{3k+2},x_{3k+2},x_{3k+3})G_b(y_{3k+2},y_{3k+1},y)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_6(x_{3k+1},x)\cdot\frac{G_b(x_{3k+1},x_{3k+1},x_{3k+2})G_b(x_{3k+2},x_{3k+1},x)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_7(x_{3k+1},x)\cdot\frac{G_b(x_{3k+1},x_{3k+1},x_{3k+2})G_b(y_{3k+2},y_{3k+1},y)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$

$$+s.\alpha_8(x_{3k+1},x)\cdot\frac{G_b(x,x,F(x,y))G_b(x_{3k+2},x_{3k+1},x)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_9(x,x_{3k+2})\cdot\frac{G_b(x,x,F(x,y))G_b(y_{3k+2},y_{3k+1},y)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_{10}(x,x_{3k+2})\cdot\frac{G_b(x_{3k+2},x_{3k+1},F(x,y))G_b(x_{3k+2},x_{3k+1},x)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_{11}(x,x_{3k+2})\cdot\frac{G_b(x_{3k+1},x,x_{3k+3})G_b(y_{3k+2},y_{3k+1},y)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}$$
$$+s.\alpha_{12}(x,x_{3k+2})\cdot\frac{G_b(x,x_{3k+2},x_{3k+2})G_b(x_{3k+2},x_{3k+1},x)}{\Omega_s(x_{3k+2},x_{3k+1},x,y_{3k+2},y_{3k+1},y)}.$$

Since $\{x_n\}$ and $\{y_n\}$ are G_b-convergent sequences converges to $x \quad and \quad y$ respectively.

Therefore, by taking lim sup as $k \to +\infty$ of the above and using condition (i), we have

$$\begin{aligned}
G_b(x, F(x,y), F(x,y)) &\leq s \cdot G_b(x,x,x) + s\alpha_1(x_0,x_0). \frac{G_b(x,x,x) + G_b(y,y,y)}{2} \\
&+ s.\alpha_2(x_0,x_0) \cdot \frac{\limsup_{k\to\infty} G_b\big(x_{3k+3}, x_{3k+2}, F(x,y)\big) G_b(x,x,x)}{\Omega_s(x,x,x,y,y,y)} \\
&+ s.\alpha_3(x_0,x_0) \cdot \frac{\limsup_{k\to\infty} G_b\big(x_{3k+3}, x_{3k+2}, F(x,y) G_b(y,y,y)}{\Omega_s(x,x,x,y,y,y)} \\
&+ s.\alpha_4(x_0,x_0) \cdot \frac{G_b(x,x,x) G_b(x,x,x)}{\Omega_s(x,x,x,y,y,y)} \\
&+ s.\alpha_5(x_0,x) \cdot \frac{G_b(x,x,x) G_b(y,y,y)}{\Omega_s(x,x,x,y,y,y)} \\
&+ s.\alpha_6(x_0,x) \cdot \frac{G_b(x,x,x) G_b(x,x,x)}{\Omega_s(x,x,x,y,y,y)} \\
&+ s.\alpha_7(x_0,x) \cdot \frac{G_b(x,x,x) G_b(y,y,y)}{\Omega_s(x,x,x,y,y,y)} \\
&+ s.\alpha_8(x_0,x) \cdot \frac{G_b(x,x,F(x,y)) G_b(x,x,x)}{\Omega_s(x,x,x,y,y,y)} \\
&+ s.\alpha_9(x,x_0) \cdot \frac{G_b(x,x,F(x,y)) G_b(y,y,y)}{\Omega_s(x,x,x,y,y,y)} \\
&+ s.\alpha_{10}(x,x_0) \cdot \frac{\limsup_{k\to\infty} G_b\big(x_{3k+2}, x_{3k+1}, F(x,y) G_b(x,x,x)}{\Omega_s(x,x,x,y,y,y)} \\
&+ s.\alpha_{11}(x,x_0) \cdot \frac{G_b(x,x,x) G_b(y,y,y)}{\Omega_s(x,x,x,y,y,y)} \\
&+ s.\alpha_{12}(x,x_0) \cdot \frac{G_b(x,x,x) G_b(x,x,x)}{\Omega_s(x,x,x,y,y,y)}.
\end{aligned}$$

Since $G_b(x,x,x) = G_b(y,y,y) = G_b(z,z,z) = 0$ for all $x, y, z \in X$. Therefore, $G_b(x, F(x,y), F(x,y)) \leq 0$. Which is contradiction to our assumption that $G_b(x, F(x,y), F(x,y)) > 0$.

Hence $G_b(x, F(x,y), F(x,y)) = 0$. Which implies that $x = F(x,y)$ and similarly $y = F(y,x)$. Thus (x,y) is a coupled fixed point of F. Similarly we can show that (x,y) is a coupled fixed point of H and T. Hence (x,y) is a common coupled fixed point of F, H and T.

To show the uniqueness, suppose that F, H and T have two common coupled fixed points (x,y) and (u,v). Then from given condition one has

$$\begin{aligned}
G_b(x,u,u) &= G_b(F(x,y), H(u,v), T(u,v)) \\
&\leq \alpha_1(x,u) \cdot \frac{G_b(x,u,u) + G_b(y,v,v)}{2} \\
&+ s.\alpha_2(x,u) \cdot \frac{G_b\big(F(x,y), H(u,v), T(u,v)\big) G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad s.\alpha_3(x,u)\cdot\frac{G_b\big(F(x,y),H(u,v),T(u,v)\big)G_b(y,v,v)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad s.\alpha_4(x,u)\cdot\frac{G_b(x,x,F(x,y))G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad s.\alpha_5(u,u)\cdot\frac{G_b(x,x,F(x,y))G_b(y,v,v)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad s.\alpha_6(u,u)\cdot\frac{G_b(u,u,H(u,v))G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad s.\alpha_7(u,u)\cdot\frac{G_b(u,u,H(u,v))G_b(y,v,v)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad s.\alpha_8(u,u)\cdot\frac{G_b(u,u,T(u,v))G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad s.\alpha_9(u,x)\cdot\frac{G_b(u,u,T(u,v))G_b(y,v,v)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad s.\alpha_{10}(u,x)\cdot\frac{G_b(x,u,T(u,v))G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad s.\alpha_{11}(u,x)\cdot\frac{G_b(u,u,F(x,y))G_b(y,v,v)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad s.\alpha_{12}(u,x)\cdot\frac{G_b(u,x,H(u,v))G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}.
\end{aligned}
$$

Which implies that

$$
\begin{aligned}
G_b(x,u,u) \quad \leq \quad & \alpha_1(x,u)\cdot\frac{G_b(x,u,u)+G_b(y,v,v)}{2}\\
&+ \quad \alpha_2(x,u)\cdot\frac{G_b(x,u,u)G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad \alpha_3(x,u)\cdot\frac{G_b(x,u,u)G_b(y,v,v)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad \alpha_4(x,u)\cdot\frac{G_b(x,x,x)G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad \alpha_5(u,u)\cdot\frac{G_b(x,x,x)G_b(y,v,v)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad \alpha_6(u,u)\cdot\frac{G_b(u,u,u)G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad \alpha_7(u,u)\cdot\frac{G_b(u,u,u)G_b(y,v,v)}{\Omega_s(x,u,u,y,v,v)}
\end{aligned}
$$

$$
+ \quad \alpha_8(u,u)\cdot\frac{G_b(u,u,u)G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}
$$

$$
\begin{aligned}
&+ \quad \alpha_9(u,x)\cdot \frac{G_b(u,u,u)G_b(y,v,v)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad \alpha_{10}(u,x)\cdot \frac{G_b(x,u,u)G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad \alpha_{11}(u,x)\cdot \frac{G_b(u,u,x)G_b(y,v,v)}{\Omega_s(x,u,u,y,v,v)}\\
&+ \quad \alpha_{12}(u,x)\cdot \frac{G_b(u,x,u)G_b(x,u,u)}{\Omega_s(x,u,u,y,v,v)}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
G_b(x,u,u) \quad &\leq \quad \alpha_1(x,u)\cdot \frac{G_b(x,u,u)+G_b(y,v,v)}{2} + \alpha_2(x,u)\cdot G_b(x,u,u)\\
&+ \quad \alpha_3(x,u)\cdot G_b(x,u,u) + \alpha_{10}(u,x)\cdot G_b(x,u,u)\\
&+ \quad \alpha_{11}(u,x)\cdot G_b(x,u,u) + \alpha_{12}(u,x)\cdot G_b(x,u,u)
\end{aligned}
$$

Further simplification gives

$$
\Big(1 - \frac{\alpha_1(x,u)}{2} - \alpha_2(x,u) - \alpha_3(x,u) - \sum_{i=10}^{12}\alpha_i(u,x)\Big) G_b(x,u,u) \leq \frac{\alpha_1(x,u)}{2}\cdot G_b(y,v,v). \tag{11}
$$

Similarly

$$
\Big(1 - \frac{\alpha_1(x,u)}{2} - \alpha_2(x,u) - \alpha_3(x,u) - \sum_{i=10}^{12}\alpha_i(u,x)\Big)\cdot G_b(y,v,v) \leq \frac{\alpha_1(x,u)}{2}\cdot G_b(x,u,u). \tag{12}
$$

Adding inequalities (11) and (12), we get

$$
\begin{aligned}
&\Big(1 - \frac{\alpha_1(x,u)}{2} - \alpha_2(x,u) - \alpha_3(x,u) - \sum_{i=10}^{12}\alpha_i(u,x)\Big)\cdot \Big[G_b(x,u,u) + G_b(y,v,v)\Big]\\
&\leq \frac{\alpha_1(x,u)}{2}\cdot \Big[G_b(x,u,u) + G_b(y,v,v)\Big].
\end{aligned}
$$

Which implies that

$$
\Big(1 - \sum_{i=1}^{3}\alpha_i(u,x) - \sum_{i=10}^{12}\alpha_i(u,x)\Big)\cdot \Big[G_b(x,u,u) + G_b(y,v,v)\Big] \leq 0,
$$

but

$$
1 - \sum_{i=1}^{3}\alpha_i(u,x) - \sum_{i=10}^{12}\alpha_i(u,x) > 0. \ \text{ Therefore, } \ G_b(x,u,u) + G_b(y,v,v) = 0.
$$

Thus we have $G_b(x,u,u) = G_b(y,v,v) = 0$ which implies $x = u$ and $y = v$. Hence (x,y) is the unique common coupled fixed point of F, H and T.

Now, we present some corollaries.

Corollary 1. *Let (X, G_b) be a complete G_b-metric space and let $H, T : X \times X \to X$ be mappings satisfying the following condition,*

$$(i) \quad \alpha_i(H(x_1,y_1),y) \leq \alpha_i(x_1,y), \alpha_i(x,H(x_1,y_1)) \leq \alpha_i(x,x_1)),$$
$$\alpha_i(T(x_1,y_1),y) \leq \alpha_i(x_1,y) \quad and \quad \alpha_i(x,T(x_1,y_1)) \leq \alpha_i(x,x_1).$$
$$(ii) \quad G_b\Big(H(a,b),H(u,v),T(x,y)\Big) \leq \alpha_1(a,u) \cdot \frac{G_b(a,u,x)+G_b(b,v,y)}{2}$$
$$+\alpha_2(a,u)\frac{G_b\Big(H(a,b),H(u,v),T(x,y)\Big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_3(a,u)\frac{G_b\Big(H(a,b),H(u,v),T(x,y)\Big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_4(a,u)\frac{G_b\big(a,a,H(a,b)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)} + \alpha_5(u,x)\frac{G_b\big(a,a,H(a,b)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_6(u,x)\frac{G_b\big(u,u,H(u,v)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)} + \alpha_7(u,x)\frac{G_b\big(u,u,H(u,v)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_8(u,x)\frac{G_b\big(x,x,T(x,y)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)} + \alpha_9(x,a)\frac{G_b\big(x,x,T(x,y)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_{10}(x,a)\frac{G_b\big(a,u,T(x,y)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)} + \alpha_{11}(x,a)\frac{G_b\big(u,x,H(a,b)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_{12}(x,a)\frac{G_b\big(x,a,H(u,v)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)},$$

where $0 \leq s[\alpha_1(x,y)+\alpha_4(x,y)+\alpha_5(x,y)+\alpha_{10}(x,y)+\alpha_{12}(x,y)] + [\alpha_2(x,y)+\alpha_3(x,y) + \sum_{i=6}^{11}\alpha_i(x,y)] < 1$. *Then H, T have a unique common coupled fixed point in X.*

Proof. Taking $F = H$ in Theorem 1 we get the required proof.

Corollary 2. *Let (X, G_b) be a complete G_b-metric space and let $T : X^2 \to X$ be a map satisfying the following condition,*

$$(i) \quad \alpha_i(T(x_1,y_1),y) \leq \alpha_i(x_1,y) \quad and \quad \alpha_i(x,T(x_1,y_1)) \leq \alpha_i(x,x_1))$$
$$(ii) \quad G_b\Big(T(a,b),T(u,v),T(x,y)\Big) \leq \alpha_1(a,u) \cdot \frac{G_b(a,u,x)+G_b(b,v,y)}{2}$$
$$+\alpha_2(a,u)\frac{G_b\Big(T(a,b),T(u,v),T(x,y)\Big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}$$

$$+\alpha_3(a,u)\frac{G_b\Big(T(a,b),T(u,v),T(x,y)\Big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_4(a,u)\frac{G_b\big(a,a,T(a,b)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}+\alpha_5(u,x)\frac{G_b\big(a,a,T(a,b)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_6(u,x)\frac{G_b\big(u,u,T(u,v)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}+\alpha_7(u,x)\frac{G_b\big(u,u,T(u,v)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_8(u,x)\frac{G_b\big(x,x,T(x,y)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}+\alpha_9(x,a)\frac{G_b\big(x,x,T(x,y)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_{10}(x,a)\frac{G_b\big(a,u,T(x,y)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}+\alpha_{11}(x,a)\frac{G_b\big(u,x,T(a,b)\big)G_b(b,v,y)}{\Omega_s(a,u,x,b,v,y)}$$
$$+\alpha_{12}(x,a)\frac{G_b\big(x,a,T(u,v)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)},$$

where $0 \leq s[\alpha_1(x,y)+\alpha_4(x,y)+\alpha_5(x,y)+\alpha_{10}(x,y)+\alpha_{12}(x,y)]+[\alpha_2(x,y)+\alpha_3(x,y)+\sum_{i=6}^{11}\alpha_i(x,y)]<1$. *Then* T *has a unique coupled fixed point in* X.

Proof. Taking $F=H=T$ in Theorem 1, we get the required proof.

The proof of the following corollary follows word by word by using the same argument as in the proof of Theorem 1.

Corollary 3. *Let* (X,G_b) *be a complete* G_b*-metric space and let* $F,H,T: X\times X\to X$ *be mappings satisfying the above generalized contraction*(1) *and the following conditions.*

(i) $\alpha_i(F(x_1,y_1),y)\leq\alpha_i(x_1,y)$ *and* $\alpha_i(x,F(x_1,y_1))\leq\alpha_i(x,x_1))$;

(ii) $\alpha_i(H(x_1,y_1),y)\leq\alpha_i(x_1,y)$ *and* $\alpha_i(x,H(x_1,y_1))\leq\alpha_i(x,x_1))$;

(iii) $\alpha_i(T(x_1,y_1),y)\leq\alpha_i(x_1,y)$ *and* $\alpha_i(x,T(x_1,y_1))\leq\alpha_i(x,x_1))$;

(iv) $G_b\Big(F(a,b),H(u,v),T(x,y)\Big)\leq\mathcal{N}(a,b,u,v,x,y)$.

where

$$\mathcal{N}(a,b,u,v,x,y)=\alpha_1(a,u)\cdot\frac{G_b(a,u,x)+G_b(b,v,y)}{2}$$
$$+\alpha_2(a,u)\frac{G_b\Big(F(a,b),H(u,v),T(x,y)\Big)G_b(a,u,x)}{\Omega_1(a,u,x,b,v,y)}$$
$$+\alpha_3(a,u)\frac{G_b\Big(F(a,b),H(u,v),T(x,y)\Big)G_b(b,v,y)}{\Omega_1(a,u,x,b,v,y)}$$
$$+\alpha_4(a,u)\frac{G_b\big(a,a,F(a,b)\big)G_b(a,u,x)}{\Omega_1(a,u,x,b,v,y)}+\alpha_5(u,x)\frac{G_b\big(a,a,F(a,b)\big)G_b(b,v,y)}{\Omega_1(a,u,x,b,v,y)}$$

$$+\alpha_6(u,x)\frac{G_b\big(u,u,H(u,v)\big)G_b(a,u,x)}{\Omega_1(a,u,x,b,v,y)}+\alpha_7(u,x)\frac{G_b\big(u,u,H(u,v)\big)G_b(b,v,y)}{\Omega_1(a,u,x,b,v,y)}$$
$$+\alpha_8(u,x)\frac{G_b\big(x,x,T(x,y)\big)G_b(a,u,x)}{\Omega_1(a,u,x,b,v,y)}+\alpha_9(x,a)\frac{G_b\big(x,x,T(x,y)\big)G_b(b,v,y)}{\Omega_1(a,u,x,b,v,y)}$$
$$+\alpha_{10}(x,a)\frac{G_b\big(a,u,T(x,y)\big)G_b(a,u,x)}{\Omega_s(a,u,x,b,v,y)}+\alpha_{11}(x,a)\frac{G_b\big(u,x,F(a,b)\big)G_b(b,v,y)}{\Omega_1(a,u,x,b,v,y)}$$
$$+\alpha_{12}(x,a)\frac{G_b\big(x,a,H(u,v)\big)G_b(a,u,x)}{\Omega_1(a,u,x,b,v,y)}, \tag{13}$$

and $\alpha_i : X\times X \to [0,1)$, $i=1,2,3,\cdots,12.$ *such that* $0\le s[\alpha_1(x,y)+\alpha_4(x,y)+\alpha_5(x,y)+\alpha_{10}(x,y)+\alpha_{12}(x,y)]+[\alpha_2(x,y)+\alpha_3(x,y)+\sum_{i=6}^{11}\alpha_i(x,y)]<1$. *Then* F,H,T *have a unique common coupled fixed point in* X.

Remark 2. *If we put* $\alpha_i(x,y)=\alpha_i$ *for* $i=1,2,3,\cdots,9$ *and* $\alpha_i(x,y)=0$ *for* $i=10,11,12$ *in condition* (13) *we get the Theorem 16 of Khomdram et.al [4]*

Remark 3. *If we put* $\alpha_i(x,y)=\alpha_i$ *for* $i=1,2,3,\cdots,9$ *and* $\alpha_i(x,y)=0$ *for* $i=10,11,12$ *in Corollary 1 we get the Corollary 17 of Khomdram et.al [4]*

Example 1. *Let* $X=[0,\infty)$ *with complete G-metric defined by*

$$G(x,y,z)=\begin{cases}0, & \text{if } x=y=z;\\ \max\{x,y,z\}, & \text{otherwise,}\end{cases}$$

and define the G_b *metric by*

$$G_b(x,y,z)=(G(x,y,z))^3.$$

Then, (X,G_b) *is a complete* G_b*-metric space with* $s=3$. *Define the mappings* F,H *and* T *by*

$$F(x,y)=\frac{2x+y}{160}, H(x,y)=\frac{x+3y}{170}, \text{ and } T(x,y)=\frac{4x+5y}{180}$$

for all $x,y,z\in X$. *We will use the following fact: For* $\delta,\beta,\ge 0$, *then*

$$(\delta+\beta)^p\le 2^{2p-2}(\delta^p+\beta^p) \tag{14}$$

Now for $a\ne u\ne x$ *and* $b\ne v\ne y$ *we have*

$$\begin{aligned}G_b(F(a,b),H(u,v),T(x,y)) &= (\max\{\frac{2a+b}{160},\frac{u+3v}{170},\frac{4x+5y}{180}\})^3\\ &= \max\{(\frac{2a+b}{160})^3,(\frac{u+3v}{170})^3,(\frac{4x+5y}{180})^3\}\\ &= \max\{\frac{(2a+b)^3}{(160)^3},\frac{(u+3v)^3}{(170)^3},\frac{(4x+5y)^3}{(180)^3}\}\end{aligned}$$

$$
\begin{aligned}
\text{(by using (14) for } p=3\text{)} \quad &\leq 16\max\{\frac{8(a)^3+(b)^3}{(160)^3}, \frac{(u)^3+27(v)^3}{(170)^3}, \frac{64(x)^3+125(y)^3}{(180)^3}\} \\
&\leq \frac{16}{(160)^3}\max\{8(a)^3+(b)^3, (u)^3+27(v)^3, 64(x)^3+125(y)^3\} \\
&\leq \frac{16}{(160)^3}\max\{125(a)^3+125(b)^3, 125(u)^3+125(v)^3, 125(x)^3+125(y)^3\} \\
&\leq \frac{2000}{(160)^3}\Big(\max\{a^3,u^3,x^3\}+\max\{b^3,v^3,y^3\}\Big) \\
&= \frac{4000}{(160)^3}\frac{\max\{a^3,u^3,x^3\}+\max\{b^3,v^3,y^3\}}{2} \\
&= \frac{4000}{(160)^3}\frac{(\max\{a,u,x\})^3+(\max\{b,v,y\})^3}{2} \\
&= \frac{1}{1024}\frac{G_b(a,u,x)+G_b(b,v,y)}{2} \\
&\leq \mathcal{N}(a,b,u,v,x,y),
\end{aligned}
$$

where $\alpha_1(x,y)=\frac{1}{1024}$ *and* $\alpha_i(x,y)=\frac{1}{(1024)^{2i-1}}$ *for* $i=2,3,\cdots,12$. *Note that for* $s=3$ *we have*

$$s[\alpha_1(x,y)+\alpha_4(x,y)+\alpha_5(x,y)+\alpha_{10}(x,y)+\alpha_{12}(x,y)]+[\alpha_2(x,y)+\alpha_3(x,y)+\sum_{i=6}^{11}\alpha_i(x,y)]<1.$$

Moreover, it is clear that the conditions (i), (ii) and (iii) of Theorem 1 are satisfied.

Hence, F,H,T *satisfy all conditions of Theorem 1, and* $(x,y)=(0,0)$ *is the unique common coupled fixed point of* F,H *and* T.

3. Application

Let $X=(C[a,b],\mathbf{R})$ denote the set of all continuous functions from $[a,b]$ to $\mathbf{R}$. In this section, we will use corollary 2 to show that there is a solution to the following integral equation:

$$u(t)=\int_a^b H(t,\kappa)f(\kappa,u(\kappa))d\kappa; \quad t\in[a,b], \tag{15}$$

where $u(\kappa)\in X$.

Theorem 2. *Consider equation* (15) *and suppose:*

(i) $H:[a,b]\times[a,b]\to[0,\infty)$ *is a continuous function,*

(ii) $f:[a,b]\times\boldsymbol{R}\to\boldsymbol{R}$*,is continuous,*

(iii) $\max_{t,\kappa\in[a,b]}H(t,\kappa)<\alpha=\frac{\alpha_1}{4(b-a)}$, *where* $\alpha_1=\frac{1}{30}$,

(iv) For all $u(\kappa) \in X$ we have $|f(\kappa, u(\kappa))| \leq (|u(\kappa)|)^3$.

Then equation (15) has a solution.

Proof. Let X be as defined above. Define a mapping $T : X \times X \to X$ by

$$T(u(t), v(t)) = \int_a^b H(t,\kappa) f(\kappa, \frac{u(\kappa)}{2} + \frac{v(\kappa)}{2}) d\kappa; \quad t \in [a,b]. \tag{16}$$

For all $u, v, w \in X$ define the G_b-metric on X by

$$G_b(u,v,w) = \left(\left\{ \begin{array}{ll} 0, & \text{if } u = v = w, \\ \max\{\sup_{\kappa\in[a,b]} |u|, \sup_{\kappa\in[a,b]} |v|, \sup_{\kappa\in[a,b]} |w|\}, & \text{otherwise.} \end{array} \right\} \right)^3 \tag{17}$$

Clearly that (X, G_b) is a complete G_b-metric space with constant $(s = 4)$. Now, for $l_i(\kappa) \in X$, where $i = 1,2,3,4,5,6$ we have
$G_b(T(l_1,l_2), T(l_3,l_4), T(l_5,l_6)) =$

$$\left(\left\{ \begin{array}{ll} 0, & \text{if } T(l_1,l_2) = T(l_3,l_4) = T(l_5,l_6), \\ \max \left\{ \begin{array}{l} \sup_{\kappa\in[a,b]} |T(l_1,l_2)|, \\ \sup_{\kappa\in[a,b]} |T(l_3,l_4)|, \\ \sup_{\kappa\in[a,b]} |T(l_5,l_6)| \end{array} \right\}, & \text{otherwise.} \end{array} \right\} \right)^3 \tag{18}$$

Now,

$$\begin{aligned}
\sup_{\kappa\in[a,b]} |T(l_1,l_2)| &= \sup_{\kappa\in[a,b]} |\int_a^b H(t,\kappa) f(\kappa, \frac{l_1(\kappa)}{2} + \frac{l_2(\kappa)}{2}) d\kappa| \qquad (19) \\
&\leq \sup_{\kappa\in[a,b]} \int_a^b |H(t,\kappa)| |f(\kappa, \frac{l_1(\kappa)}{2} + \frac{l_2(\kappa)}{2})| d\kappa \\
&\leq \sup_{\kappa\in[a,b]} \int_a^b \alpha |f(\kappa, \frac{l_1(\kappa)}{2} + \frac{l_2(\kappa)}{2})| d\kappa \\
&\leq \sup_{\kappa\in[a,b]} \int_a^b \alpha (|\frac{l_1(\kappa)}{2}| + |\frac{l_2(\kappa)}{2}|)^3 d\kappa \\
(\text{ by using (14) for } p = 3) \quad &\leq \sup_{\kappa\in[a,b]} \int_a^b 16\alpha \big[(|\frac{l_1(\kappa)}{2}|)^3 + (|\frac{l_2(\kappa)}{2}|)^3 \big] d\kappa \\
&\leq \sup_{\kappa\in[a,b]} \int_a^b 2\alpha \big[(|l_1(\kappa)|)^3 + (|l_2(\kappa)|)^3 \big] d\kappa \\
&\leq \int_a^b 2\alpha \big[G_b(l_1,l_3,l_3) + G_b(l_2,l_4,l_6) \big] d\kappa \\
&\leq 2\alpha \big[G_b(l_1,l_3,l_3) + G_b(l_2,l_4,l_6) \big] \int_a^b d\kappa
\end{aligned}$$

$$= 2\alpha\big[G_b(l_1,l_3,l_3)+G_b(l_2,l_4,l_6)\big]|b-a|$$
$$= \frac{\alpha_1}{2}\big[G_b(l_1,l_3,l_3)+G_b(l_2,l_4,l_6)\big].$$

Similarly, one can show that

$$\sup_{\kappa\in[a,b]}|T(l_3,l_4)| \leq \frac{\alpha_1}{2}\big[G_b(l_1,l_3,l_3)+G_b(l_2,l_4,l_6)\big]. \tag{20}$$

and

$$\sup_{\kappa\in[a,b]}|T(l_5,l_6)| \leq \frac{\alpha_1}{2}\big[G_b(l_1,l_3,l_3)+G_b(l_2,l_4,l_6)\big]. \tag{21}$$

Hence, by (19), (20) and (21) we have

$$\max\left\{\begin{array}{c}\sup_{\kappa\in[a,b]}|T(l_1,l_2)|,\\ \sup_{\kappa\in[a,b]}|T(l_3,l_4)|,\\ \sup_{\kappa\in[a,b]}|T(l_5,l_6)|\end{array}\right\} \leq \frac{\alpha_1}{2}\big[G_b(l_1,l_3,l_3)+G_b(l_2,l_4,l_6)\big] \tag{22}$$

Thus,

$$G_b(T(l_1,l_2),T(l_3,l_4),T(l_5,l_6)) \leq \frac{\alpha_1}{2}\big[G_b(l_1,l_3,l_3)+G_b(l_2,l_4,l_6)\big] \tag{23}$$

Therefore, all conditions of corollary 2 are satisfied for $\alpha_i = \frac{1}{30}$ for $i = 1, 2, \cdots, 12$.
As a result of corollary 2 the mapping T has a unique coupled fixed point in X which is a solution of (15).

The following example illustrate the validity of Theorem 2.

Example 2. *The following integral equation has a solution in* $X = (C[0,1], \boldsymbol{R})$.

$$u(t) = \int_0^1 \frac{t\kappa}{130}\,(e^{-\kappa})(u(\kappa))^3 d\kappa; \quad t\in[0,1]. \tag{24}$$

Proof. *Let* $T : X\times X \to X$ *be defined as*

$$T(u(t),v(t)) = \int_0^1 \frac{t\kappa}{130}\Big(\frac{u(\kappa)}{2}+\frac{v(\kappa)}{2}\Big)^3 e^{-\kappa}d\kappa; t\in[0,1].$$

By specifying $H(t,\kappa) = \frac{t\kappa}{130}$, $f(\kappa,t) = t^3e^{-\kappa}$ *in Theorem 2 we get that:*

(i) The function $H(t,\kappa)$ *is continuous on* $[0,1]\times[0,1]$,

(ii) $f(\kappa,t) = t^3e^{-\kappa}$ *is continuous on* $[0,1]\times\boldsymbol{R}$, .

(iii) $\max_{t,\kappa\in[0,1]} H(t,\kappa) = \max_{t,\kappa\in[0,1]}\frac{t\kappa}{130} = \frac{1}{130} < \frac{1}{120} = \frac{\alpha_1}{4(1-0)}$, *where* $\alpha_1 = \frac{1}{30}$.

(iv) also $|f(\kappa,u(\kappa))| = |e^{-\kappa}||(u(\kappa))^3| \leq (|u(\kappa)|)^3$.

Therefore, all conditions of Theorem 2 are satisfied, hence the mapping T has a fixed point in X, which is a solution to equation (24)*, that is*

$$T(u(t),u(t))=\int_0^1\frac{t\kappa}{130}(\frac{u(\kappa)}{2}+\frac{u(\kappa)}{2})^3e^{-\kappa}d\kappa=\int_0^1\frac{t\kappa}{130}(u(\kappa))^3e^{-\kappa}d\kappa;t\in[0,1]=u(t).$$

4. Conclusions

We have introduced rational type contractive conditions on three mappings to give the existence and uniqueness of common coupled fixed point theorem in the set up of G_b-metric spaces. The derived result cover and generalize some well-known comparable results in the existing literature. Also we presented examples to illustrate the main results as well as we gave an application to prove the existence and uniqueness of a solution for a class of integral equations.

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