



Truncated Tangent Polynomials

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Abstract. In this paper, we introduce a class of truncated tangent polynomials which generalizes tangent numbers and polynomials, and establish various properties and identities. Moreover, we obtain some interesting correlations of truncated tangent polynomials with the Stirling numbers of the second kind and with the hypergeometric Bernoulli polynomials.

2020 Mathematics Subject Classifications: 11B68, 11B73, 11B83, 33C15

Key Words and Phrases: Tangent numbers and polynomials, Truncated tangent polynomials, truncated Bernoulli polynomials, Hypergeometric Bernoulli polynomials, Stirling numbers of the second kind, truncated Stirling numbers

1. Introduction

The hypergeometric Bernoulli numbers $B_{m,n}$ (see [8–11, 13]) are defined by

$$\frac{1}{{}_1F_1(1; m+1; t)} = \frac{\frac{t^m}{m!}}{e^t - \sum_{j=0}^{m-1} \frac{t^j}{j!}} = \sum_{n=0}^{\infty} B_{m,n} \frac{t^n}{n!}, \quad (1)$$

where

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)} z^n}{(b)^{(n)} n!} \quad (2)$$

is the confluent hypergeometric function with

$$(x)^{(n)} = x(x+1) \cdots (x+n-1) \text{ for } n \geq 1, \text{ and } (x)^{(0)} = 1.$$

When $m = 1$, $B_n := B_{1,n}$ are the classical Bernoulli numbers given by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

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DOI: <https://doi.org/10.29020/nybg.ejpam.v13i4.3839>

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The hypergeometric Bernoulli polynomials $B_{m,n}(x)$ were also introduced in [8, 9] and defined by

$$\frac{e^{xt}}{{}_1F_1(1; m + 1; t)} = \frac{\frac{t^m}{m!}e^{xt}}{e^t - \sum_{j=0}^{m-1} \frac{t^j}{j!}} = \sum_{n=0}^{\infty} B_{m,n}(x) \frac{t^n}{n!}. \tag{3}$$

When $m = 1$, $B_n(x) := B_{1,n}(x)$ are the classical Bernoulli polynomials given by the exponential generating function

$$\frac{te^x}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Hypergeometric Bernoulli polynomials (numbers) are also called truncated Bernoulli polynomials (numbers).

In [15], Komatsu and Pita introduced truncated Euler polynomials via the generating function

$$\frac{\frac{2t^m}{m!}e^{xt}}{e^t + 1 - \sum_{j=0}^{m-1} \frac{t^j}{j!}} = \sum_{n=0}^{\infty} E_{m,n}(x) \frac{t^n}{n!}. \tag{4}$$

These polynomials satisfy recurrence relation

$$E_{n,m}(x) = 0, \quad n = 0, 1, 2, \dots, m - 1, \quad \text{and}$$

$$E_{n,n+m}(x) = 2 \binom{n+m}{n} x^n - \sum_{k=0}^n \binom{n+m}{k} E_{n,k}(x), \quad n \geq 0.$$

When $m = 0$ in (4), $E_n(x) := E_{0,n}(x)$ are the classical Euler polynomials given by

$$\frac{2e^x}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

In recent years, extensive researches on various families of truncated exponential polynomials have become popular. Truncation of exponential polynomials have played crucial importance to evaluate integrals including products of special functions [4]. Some of the recent works on truncated numbers polynomials include truncated Fubini polynomials [5], truncated-exponential based Apostol-type polynomials [26], truncated-exponential-based Frobenius-Euler polynomials [17], hypergeometric Cauchy numbers [13], truncated Bernoulli-Carlitz and truncated Cauchy-Carlitz numbers [14, 16], truncated exponential-based Appell polynomials [12] and many others. In [6], Duran and Acikgoz introduced degenerate truncated exponential polynomials and obtain truncated degenerate versions of some special polynomials such as Stirling polynomials of the second kind, Bernoulli polynomials, Euler polynomials, and Bell polynomials.

In the next section, we introduce truncated tangent numbers and polynomials and explore some of their interesting properties and formula.

2. Truncated tangent polynomials

In this section, we give a generalization of tangent polynomials in terms of the truncated exponential function.

For nonnegative integer m , we define the truncated tangent polynomials $T_{m,n}(x)$ through the generating function

$$\frac{2^{\frac{t^m}{m!}}}{e^{2t} + 1 - \sum_{j=0}^{m-1} 2^j \frac{t^j}{j!}} e^{xt} = \sum_{n=0}^{\infty} T_{m,n}(x) \frac{t^n}{n!}. \tag{5}$$

When $m = 0$, $T_n(x) := T_{0,n}(x)$ are the tangent polynomials (see [20, 21]) defined by

$$\frac{2}{e^{2t} + 1} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!},$$

and $T_n := T_n(0)$ are called tangent numbers.

When $x = 0$ in (5), $T_{m,n} := T_{m,n}(0)$ are called the truncated tangent numbers given by

$$\frac{2^{\frac{t^m}{m!}}}{e^{2t} + 1 - \sum_{j=0}^{m-1} 2^j \frac{t^j}{j!}} = \sum_{n=0}^{\infty} T_{m,n} \frac{t^n}{n!}. \tag{6}$$

Several extensions and generalizations of tangent numbers and polynomials can be seen in [1, 22–25].

The following identities follow directly from the generating function (5).

Theorem 1. For $m, n \geq 0$,

$$T_{m,n}(x) = \sum_{r=0}^n \binom{n}{r} T_{m,r} \cdot x^{n-r} \tag{7}$$

$$T_{m,n}(x) = \sum_{r=0}^n \binom{n}{r} T_{m,r}(y) (x - y)^{n-r} \tag{8}$$

$$T_{m,n}(x + y) = \sum_{r=0}^n \binom{n}{r} T_{m,r}(x) y^{n-r} \tag{9}$$

$$T_{m,n}(px) = \sum_{r=0}^n \binom{n}{r} T_{m,r}(x) (p - 1)^{n-r} x^{n-r}, \quad p \neq 1. \tag{10}$$

The truncated tangent polynomials satisfy the following derivative and integral properties.

Theorem 2. For $m \geq 0$ and $n \geq 1$,

$$\frac{d}{dx} T_{m,n}(x) = n T_{m,n-1}(x). \tag{11}$$

$$\int T_{m,n}(x)dx = \frac{1}{n+1}T_{m,n+1} \tag{12}$$

$$T_{m,n}(x) = T_{m,n} + n \int_0^x T_{m,n-1}(t)dt. \tag{13}$$

Proof. Taking the derivative of both sides of (7) with respect to x , we obtain

$$\begin{aligned} \frac{d}{dx}T_{m,n}(x) &= \sum_{r=0}^{n-1} \binom{n}{r} (n-r)T_{m,r} \cdot x^{(n-r-1)} \\ &= n \sum_{r=0}^{n-1} \binom{n-1}{r} T_{m,r} x^{(n-1)-r} \\ &= nT_{m,n-1}(x). \end{aligned}$$

Equation (12) follows from (11). From (12), we have

$$\int_0^x T_{m,n-1}(t)dt = \frac{1}{n} (T_{m,n}(x) - T_{m,n}(0)),$$

which gives (13). □

Theorem 3. For $n \geq 0$,

$$T_{1,n}(x) = 2n(x-2)^{n-1}. \tag{14}$$

Proof. When $m = 1$ in (5), we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_{1,n}(x) \frac{t^{n-1}}{n!} &= 2te^{(x-2)t} \\ &= 2 \sum_{n=0}^{\infty} (x-2)^n \frac{t^{n+1}}{n!} \\ &= 2 \sum_{n=1}^{\infty} n(x-2)^{n-1} \frac{t^n}{n!} \\ &= 2 \sum_{n=0}^{\infty} n(x-2)^{n-1} \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof. □

Theorem 4. The truncated tangent polynomials satisfy the following recurrence relation:

$$T_{m,n}(x) = 0, \quad n = 0, 1, 2, \dots, m-1$$

and

$$T_{m,n+m}(x) = 2 \binom{n+m}{n} x^n - \sum_{j=0}^n 2^{n+m-j} \binom{n+m}{j} T_{m,j}, \quad n \geq 0. \tag{15}$$

Proof. It follows from (5) that

$$\begin{aligned} \frac{2t^m}{m!} \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} &= \sum_{n=0}^{\infty} T_{m,n}(x) \frac{t^n}{n!} \left(1 + \sum_{j=m}^{\infty} 2^j \frac{t^j}{j!} \right) \\ &= \sum_{n=0}^{\infty} T_{m,n}(x) \frac{t^n}{n!} + \left(\sum_{n=0}^{\infty} T_{m,n}(x) \frac{t^n}{n!} \right) \left(\sum_{j=m}^{\infty} 2^j \frac{t^j}{j!} \right). \end{aligned}$$

Thus,

$$\left(\sum_{n=0}^{\infty} T_{m,n}(x) \frac{t^n}{n!} \right) \left(\sum_{j=m}^{\infty} 2^j \frac{t^j}{j!} \right) \tag{16}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{2x^n t^{n+m}}{n!m!} - \sum_{n=0}^{\infty} T_{m,n+m}(x) \frac{t^{n+m}}{(n+m)!} - \sum_{n=0}^{m-1} T_{m,n}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(2 \binom{n+m}{n} x^n - T_{m,n+m}(x) \right) \frac{t^{n+m}}{(n+m)!} - \sum_{n=0}^{m-1} T_{m,n}(x) \frac{t^n}{n!}. \end{aligned} \tag{17}$$

Note that expression (16) can be written as

$$\begin{aligned} \left(\sum_{n=0}^{\infty} T_{m,n}(x) \frac{t^n}{n!} \right) \left(\sum_{j=m}^{\infty} 2^j \frac{t^j}{j!} \right) &= \sum_{n=0}^{\infty} \sum_{j=0}^n T_{m,j}(x) \frac{t^j}{j!} 2^{n-j+m} \frac{t^{n-j+m}}{(n-j+m)!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^n 2^{n+m-j} \binom{n+m}{j} T_{m,j}(x) \frac{t^{n+m}}{(n+m)!}. \end{aligned} \tag{18}$$

Comparing (17) and (18) gives the desired result. □

Example 1. For $m = 1$, we have $T_{1,0}(x) = 0$. Using recurrence relation (15), we obtain

$$T_{1,n+1}(x) = 2 \binom{n+1}{n} x^n - \sum_{j=0}^n 2^{n+1-j} \binom{n+1}{j} T_{1,j}. \tag{19}$$

Computing for $n = 0, 1, 2, 3, 4$, we get the following polynomials:

$$\begin{aligned} T_{1,1} &= 2 \\ T_{1,2} &= 2 \binom{2}{1} x - 2 \binom{2}{1} T_{1,1} \\ &= 4x - 8 = 4(x - 2) \\ T_{1,3} &= 2 \binom{3}{2} x^2 - 2^2 \binom{3}{1} T_{1,1}(x) - 2 \binom{3}{2} T_{1,2}(x) \end{aligned}$$

$$\begin{aligned}
 &= 6x^2 - 12(2) - 2(3)4(x - 2) = 6(x - 2)^2 \\
 T_{1,4} &= 2 \binom{4}{3} x^3 - 2^3 \binom{4}{1} T_{1,1}(x) - 2^2 \binom{4}{2} T_{1,2}(x) - 2 \binom{4}{3} T_{1,3}(x) \\
 &= 8x^3 - 8(4)(2) - 24(4)(x - 2) - 8(6)(x - 2) = 8(x - 2)^3.
 \end{aligned}$$

Note that the above computations can be easily done using (14). Furthermore, taking $m = 2$, we obtain the recurrence relation:

$$T_{2,n+2}(x) = 2 \binom{n+2}{n} x^n - \sum_{j=0}^n 2^{n+2-j} \binom{n+2}{j} T_{2,j}, \tag{20}$$

which yields the following polynomials:

$$\begin{aligned}
 T_{2,0}(x) &= T_{2,1}(x) = 0 \\
 T_{2,2}(x) &= 2 \\
 T_{2,3}(x) &= 6x \\
 T_{2,4}(x) &= 12(x^2 - 4).
 \end{aligned}$$

Theorem 5. For $m \geq 0$ and $n > 0$,

$$2^{m-1} \sum_{k=0}^n \binom{n}{k} T_{m,n-k}(y) T_{m+1,k}(x) = \sum_{k=0}^n \binom{n}{k} T_{m+1,n-k}(x) y^k - \frac{n}{m+1} \sum_{k=0}^{n-1} \binom{n-1}{k} T_{m,n-1-k}(y) x^k.$$

Proof. It follows from (5) that

$$\begin{aligned}
 2e^{xt} \frac{t^{m+1}}{(m+1)!} &= \left(e^{2t} + 1 - \sum_{j=0}^m \frac{(2t)^j}{j!} \right) \sum_{n=0}^{\infty} T_{m+1,n}(x) \frac{t^n}{n!} \\
 &= \left(e^{2t} + 1 - \sum_{j=0}^{m-1} \frac{(2t)^j}{j!} \right) \sum_{n=0}^{\infty} T_{m+1,n}(x) \frac{t^n}{n!} - \frac{2^m t^m}{m!} \sum_{n=0}^{\infty} T_{m+1,n}(x) \frac{t^n}{n!}.
 \end{aligned}$$

Hence,

$$2e^{xt} \frac{t^{m+1}}{(m+1)!} \sum_{n=0}^{\infty} T_{m,n}(y) \frac{t^n}{n!} = \frac{2^m t^m}{m!} e^{yt} \sum_{n=0}^{\infty} T_{m+1,n}(x) \frac{t^n}{n!} - \frac{2^m t^m}{m!} \sum_{n=0}^{\infty} T_{m,n}(y) \frac{t^n}{n!} \sum_{n=0}^{\infty} T_{m+1,n}(x) \frac{t^n}{n!}.$$

Consequently,

$$\begin{aligned}
 \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} T_{m,n-k}(y) x^k \frac{t^{n+1}}{(m+1)n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} T_{m+1,n-k}(x) y^k \frac{t^n}{n!} \\
 &\quad - 2^{m-1} \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} T_{m,n-k}(y) T_{m+1,k}(x) \frac{t^n}{n!},
 \end{aligned}$$

which provides the desired result. □

Theorem 6. For $m, n \geq 0$,

$$T_{n,m}(x) = \frac{m!n!}{(n+m)!} \sum_{l=0}^n \binom{n+m}{l} \sum_{k=0}^l \binom{l}{k} T_{m,k} B_{m,l-k}(x) \frac{t^n}{n!}. \tag{21}$$

Proof. Applying (5), we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_{m,n}(x, y) \frac{t^n}{n!} &= \frac{2^{\frac{t^m}{m!}}}{\left(e^{2t} + 1 - \sum_{j=0}^{m-1} 2^j \frac{t^j}{j!}\right)} e^{xt} \frac{\frac{t^m}{m!}}{\left(e^t - \sum_{j=0}^{m-1} \frac{t^j}{j!}\right)} \cdot \frac{\left(e^t - \sum_{j=0}^{m-1} \frac{t^j}{j!}\right)}{\frac{t^m}{m!}} \\ &= \frac{m!}{t^m} \left(\sum_{n=0}^{\infty} T_{m,n} \frac{t^n}{n!} \sum_{n=0}^{\infty} B_{m,n}(x) \frac{t^n}{n!} \right) \sum_{j=m}^{\infty} \frac{t^j}{j!} \\ &= m! \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} T_{m,k} B_{m,n-k}(x) \frac{t^n}{n!} \sum_{j=0}^{\infty} \frac{t^j}{(j+m)!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^n \frac{m!n! \binom{n+m}{l}}{(n+m)!} \sum_{k=0}^l \binom{l}{k} T_{m,k} B_{m,l-k}(x) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof. □

3. Relations with Stirling numbers of the second kind and its associated truncated Stirling numbers

The Stirling numbers of the second kind are given by the generating function

$$\frac{(e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n, k) \frac{t^n}{n!}, \tag{22}$$

or by the recurrence relation for a fixed nonnegative integer n ,

$$x^n = \sum_{k=0}^n S_2(n, k)(x)_k, \tag{23}$$

where $(x)_{(k)}$ is the falling factorial defined as

$$(x)_k = x(x-1) \cdots (x-k+1) \text{ for } k \geq 0, \text{ and } (x)_0 = 1.$$

For the detailed discussion of Stirling numbers of the second kind, see [3, 7].

In [5], Duran and Acikgoz introduced the truncated Stirling numbers of the second kind:

$$\frac{\left(e^t - 1 - \sum_{j=0}^{m-1} \frac{t^j}{j!}\right)^k}{k!} = \sum_{n=0}^{\infty} S_{2,m}(n, k) \frac{t^n}{n!}, \tag{24}$$

which reduce to the Stirling numbers of the second kind when $m = 0$. In this section, we derive some relationships between the truncated tangent polynomials and Stirling numbers of the second kind and its associated truncated version.

Theorem 7. For $m, n \geq 0$,

$$T_{m,n}(x) = \sum_{k=0}^n \sum_{r=k}^n \binom{n}{r} S_2(r, k) T_{m,n-r} \cdot (x)_k, \tag{25}$$

$$T_{m,n}(x) = \sum_{k=0}^n \sum_{r=k}^n \binom{n}{r} S_2(r, k) T_{m,n-r}(-k) x^{(k)}. \tag{26}$$

Proof. Applying relation (23) to (7), we get

$$T_{m,n}(x) = \sum_{r=0}^n \binom{n}{r} T_{m,n-r} \sum_{k=0}^r S_2(r, k) (x)_k \tag{27}$$

$$= \sum_{k=0}^n \sum_{r=k}^n \binom{n}{r} S_2(r, k) T_{m,n-r} \cdot (x)_k. \tag{28}$$

To do (26), we note that $e^{xt} = (1 - (1 - e^{-t}))^{-x}$. Hence, equation (5) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} T_{m,n}(x) \frac{t^n}{n!} &= \frac{2^{\frac{t^m}{m!}}}{e^{2t} + 1 - \sum_{j=0}^{m-1} 2^j \frac{t^j}{j!}} (1 - (1 - e^{-t}))^{-x} \\ &= \frac{2^{\frac{t^m}{m!}}}{e^{2t} + 1 - \sum_{j=0}^{m-1} 2^j \frac{t^j}{j!}} \sum_{k=0}^{\infty} \binom{x+k-1}{k} (1 - e^{-t})^k \\ &= \frac{2^{\frac{t^m}{m!}}}{e^{2t} + 1 - \sum_{j=0}^{m-1} 2^j \frac{t^j}{j!}} \sum_{k=0}^{\infty} x^{(k)} \frac{(e^t - 1)^k}{k!} e^{-kt} \\ &= \sum_{k=0}^{\infty} x^{(k)} \left(\sum_{n=0}^{\infty} S_2(n, k) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} T_{m,n}(-k) \frac{t^n}{n!} \right) \\ &= \sum_{k=0}^{\infty} x^{(k)} \sum_{n=0}^{\infty} \sum_{r=0}^n \binom{n}{r} S_2(r, k) T_{m,n-r}(-k) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{r=k}^n \binom{n}{r} S_2(r, k) T_{m,n-r}(-k) x^{(k)} \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (26). □

Theorem 8. For $m, n \geq 0$,

$$T_{m,n+m}(x) = \sum_{k=0}^n \binom{n+m}{m} (-1)^k k! 2^{n-k} S_{2,m}(n, k). \tag{29}$$

Proof. Note that (5) can be expressed as

$$\sum_{n=0}^{\infty} T_{m,n}(x) \frac{t^n}{n!} = \frac{\frac{t^m}{m!}}{\left(1 + \frac{1}{2} \left(e^{2t} - 1 - \sum_{j=0}^{m-1} \frac{(2t)^j}{j!}\right)\right)} \tag{30}$$

$$= \frac{t^m}{m!} \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \left(e^{2t} - 1 - \sum_{j=0}^{m-1} \frac{(2t)^j}{j!}\right)^k \tag{31}$$

$$= \frac{t^m}{m!} \sum_{k=0}^{\infty} (-1)^k k! 2^{-k} \sum_{n=0}^{\infty} S_{2,m}(n, k) \frac{(2t)^n}{n!} \tag{32}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k k! 2^{n-k} S_{2,m}(n, k) \frac{t^{n+m}}{m!n!} \tag{33}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n+m}{m} (-1)^k k! 2^{n-k} S_{2,m}(n, k) \frac{t^{n+m}}{(n+m)!}, \tag{34}$$

which gives the desired result. □

The case when $m = 1$, we obtain an interesting identity involving Apostol-type Stirling numbers of the second kind. These numbers are defined (see [18] by means of the generating function

$$\frac{(\lambda e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n, k; \lambda) \frac{t^n}{n!}. \tag{35}$$

Theorem 9. For $n \geq 0$,

$$T_{1,n+1}(x) = (n+1)2^n \sum_{k=0}^{\infty} (-1)^k k! S_2\left(n, k; \frac{1}{2}\right). \tag{36}$$

Proof. When $m = 1$ in (5), we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_{1,n}(x) \frac{t^n}{n!} &= \frac{t}{\left(\frac{1}{2}e^{2t} - 1\right) + 1} \\ &= t \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}e^{2t} - 1\right)^k \\ &= t \sum_{k=0}^{\infty} (-1)^k k! \frac{\left(\frac{1}{2}e^{2t} - 1\right)^k}{k!} \\ &= t \sum_{k=0}^{\infty} (-1)^k k! \sum_{n=0}^{\infty} S_2\left(n, k; \frac{1}{2}\right) \frac{(2t)^n}{n!} \\ &= \sum_{n=0}^{\infty} (n+1)2^n \sum_{k=0}^{\infty} (-1)^k k! S_2\left(n, k; \frac{1}{2}\right) \frac{t^{n+1}}{(n+1)!}, \end{aligned}$$

which gives the desired result. □

4. Relations with hypergeometric Bernoulli polynomials

In this section, we show several relations of truncated tangent polynomials with hypergeometric Bernoulli polynomials and hypergeometric Bernoulli numbers.

Theorem 10. For $m, n \geq 0$,

$$\binom{n+m}{n} \sum_{k=0}^n \binom{n}{k} \left(2^{k-(n+m)} B_{m,k}(x) y^{n-k} - 2^{-(k+1)} T_{m,k}(y) x^{n-k} \right) \tag{37}$$

$$= \sum_{k=0}^{n+m} \binom{n+m}{k} 2^{-(k+1)} T_{m,k}(y) B_{m,n+m-k}(x). \tag{38}$$

Proof. Applying (5), we obtain

$$\begin{aligned} \frac{2t^m}{m!} e^{yt} &= \left(\sum_{n=0}^{\infty} T_{m,n}(y) \frac{t^n}{n!} \right) \left(e^{2t} + 1 - \sum_{j=0}^{m-1} \frac{(2t)^j}{j!} \right) \\ &= \left(\sum_{n=0}^{\infty} T_{m,n}(y) \frac{t^n}{n!} \right) \left(e^{2t} - \sum_{j=0}^{m-1} \frac{(2t)^j}{j!} \right) + \left(\sum_{n=0}^{\infty} T_{m,n}(y) \frac{t^n}{n!} \right). \end{aligned}$$

Thus,

$$\frac{2t^m}{m!} e^{yt} \sum_{n=0}^{\infty} B_{m,n}(x) \frac{(2t)^n}{n!} = \frac{(2t)^m}{m!} e^{2xt} \left(\sum_{n=0}^{\infty} T_{m,n}(y) \frac{t^n}{n!} \right) + \left(\sum_{n=0}^{\infty} T_{m,n}(y) \frac{t^n}{n!} \right) \sum_{n=0}^{\infty} B_{m,n}(x) \frac{(2t)^n}{n!}.$$

Expanding the exponential functions into series and applying Cauchy product, we have

$$\begin{aligned} \frac{t^m}{m!} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} \left(2^{k+1} B_{m,k}(x) y^{n-k} - 2^{m+n-k} T_{m,k}(y) x^{n-k} \right) \right) \frac{t^n}{n!} \\ = \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} 2^{n-k} T_{m,k}(y) B_{m,n-k}(x) \frac{t^n}{n!}. \end{aligned}$$

Avoiding the zero-terms on the right-hand side of the above equation leads to

$$\begin{aligned} \frac{1}{m!} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} \left[2^{k+1} B_{m,k}(x) y^{n-k} - 2^{m+n-k} T_{m,k}(y) x^{n-k} \right] \right) \frac{t^{n+m}}{n!} \\ = \sum_{n=0}^{\infty} \sum_{k=0}^{n+m} \binom{n+m}{k} 2^{n+m-k} T_{m,k}(y) B_{m,n+m-k}(x) \frac{t^{n+m}}{(n+m)!}. \end{aligned}$$

Comparing the coefficients of both sides gives

$$\frac{2^{n+m+1}}{n!m!} \sum_{k=0}^n \binom{n}{k} \left[2^{k-(n+m)} B_{m,k}(x) y^{n-k} - 2^{-(k+1)} T_{m,j}(y) x^{n-k} \right]$$

$$= \frac{2^{n+m+1}}{(n+m)!} \sum_{k=0}^{n+m} \binom{n+m}{k} 2^{-(k+1)} T_{m,k}(y) B_{m,n+m-k}(x),$$

from where the desired result follows. □

Lemma 1. (see Theorem 1[19]) *The polynomial identity*

$$\sum_{k=0}^n a_{n,k}(x + \alpha)^k = \sum_{k=0}^n b_{n,k}(x + \beta)^k \tag{39}$$

implies the Bernoulli polynomials identity

$$\sum_{k=0}^n a_{n,k} B_k(x + \alpha) = \sum_{k=0}^n b_{n,k} B_k(x + \beta). \tag{40}$$

When $m = 0$ in Theorem 10, we obtain the following corollary.

Corollary 1. *For $n \geq 0$,*

$$\sum_{k=0}^n \binom{n}{k} 2^{-(k+1)} T_k(y) \left((x-1)^{n-k} + x^{n-k} \right) = \left(x-1 + \frac{y}{2} \right)^n, \tag{41}$$

$$\sum_{k=0}^n \binom{n}{k} 2^{-(k+1)} T_k(y) (B_{n-k}(x-1) + B_{n-k}(x)) = B_n \left(x-1 + \frac{y}{2} \right). \tag{42}$$

Proof. Note that $B_{0,n}(x) = (x-1)^n$ and $T_{0,n}(x) = T_n(x)$. Setting $m = 0$ in Theorem 10, we get

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} 2^{-(k+1)} T_k(y) \left((x-1)^{n-k} + x^{n-k} \right) &= \sum_{k=0}^n \binom{n}{k} 2^{k-n} (x-1)^k y^{n-k} \\ &= \left(x-1 + \frac{y}{2} \right)^n. \end{aligned} \tag{43}$$

Applying Lemma 1 in (43), we obtain

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} 2^{-(k+1)} T_k(y) (B_{n-k}(x-1) + B_{n-k}(x)) &= \sum_{k=0}^n \binom{n}{k} 2^{k-n} B_k(x-1) y^{n-k} \\ &= B_n \left(x-1 + \frac{y}{2} \right). \end{aligned}$$

□

Corollary 2. *For $n \geq 0$,*

$$\sum_{k=0}^n \binom{n}{k} \left(2^{k-n-1} B_k(x) y^{n-k} - 2^{-k} k (y-2)^{k-1} x^{n-k} \right)$$

$$= \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{n-k} 2^{-k-1} (k+1) (y-2)^k B_{n-k}(x). \tag{44}$$

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \left(2^{k-n-1} B_k(x) B_{n-k}(y) - 2^{-k} k B_{k-1}(y-2) x^{n-k} \right) \\ &= \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{n-k} 2^{-k-1} (k+1) B_k(y-2) B_{n-k}(x). \end{aligned} \tag{45}$$

Proof. Using $B_{1,n}(x) = B_n(x)$ and $T_{1,n}(x) = 2n(x-2)^{n-1}$, and applying Theorem 10 when $m = 1$, we obtain

$$(n+1) \sum_{k=0}^n \binom{n}{k} \left(2^{k-n-1} B_k(x) y^{n-k} - 2^{-k} k (y-2)^{k-1} x^{n-k} \right) \tag{46}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} 2^{-k} k (y-2)^{k-1} B_{n+1-k}(x) \tag{47}$$

$$= \sum_{k=0}^n \binom{n+1}{n-k} 2^{-(k+1)} (k+1) (y-2)^k B_{n-k}(x), \tag{48}$$

which gives (44). Applying Lemma 1 in (44), we obtain (45). □

Theorem 11.

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} T_{m+1,n-k}(x) y^k - \frac{n}{m+1} \sum_{k=0}^{n-1} \binom{n-1}{k} T_{m,n-k-1}(y) x^k \\ &= 2^{m-1} \sum_{k=0}^n \binom{n}{k} T_{m+1,n-k}(x) T_{m,k}(y). \end{aligned}$$

Proof. From (5),

$$\begin{aligned} \frac{2t^{m+1}}{(m+1)!} e^{xt} &= \left(e^{2t} + 1 - \sum_{j=0}^{m-1} \frac{(2t)^j}{j!} - \frac{2^m t^m}{m!} \right) \sum_{n=0}^{\infty} T_{m+1,n}(x) \frac{t^n}{n!} \\ &= \left(e^{2t} + 1 - \sum_{j=0}^{m-1} \frac{(2t)^j}{j!} \right) \sum_{n=0}^{\infty} T_{m+1,n}(x) \frac{t^n}{n!} - \frac{2^m t^m}{m!} \sum_{n=0}^{\infty} T_{m+1,n}(x) \frac{t^n}{n!}. \end{aligned}$$

Consequently,

$$\frac{2t^{m+1}}{(m+1)!} e^{xt} \sum_{n=0}^{\infty} T_{m,n}(y) \frac{t^n}{n!} = \frac{2t^m}{m!} e^{yt} \sum_{n=0}^{\infty} T_{m+1,n}(x) \frac{t^n}{n!}$$

$$- \frac{2^m t^m}{m!} \sum_{n=0}^{\infty} T_{m+1,n}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} T_{m,n}(y) \frac{t^n}{n!}.$$

Expanding the exponential functions into series and applying Cauchy product, we get

$$\begin{aligned} \frac{n}{m+1} \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n-1}{k} T_{m,n-1-k}(y) x^k \right) \frac{t^n}{n!} &= \sum_{n=1}^{\infty} \sum_{k=0}^n \binom{n}{k} T_{m+1,n-k}(x) y^k \frac{t^n}{n!} \\ &\quad - 2^{m-1} \sum_{n=1}^{\infty} \sum_{k=0}^n \binom{n}{k} T_{m+1,n-k}(x) T_{m,k}(y) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof. □

Corollary 3. For $n \geq 0$,

$$\sum_{k=0}^n \binom{n}{k} \left(2(x-2)^{n-k} y^k - T_{n-k}(y) x^k \right) = T_n(x-2+y) \tag{49}$$

$$\sum_{k=0}^n \binom{n}{k} \left(2B_{n-k}(x-2) y^k - T_{n-k}(y) B_k(x) \right) = \sum_{k=0}^n \binom{n}{k} B_{n-k}(x-2) T_k(y). \tag{50}$$

Proof. Setting $m = 0$ in Theorem 11, we have

$$\begin{aligned} \sum_{k=0}^{n-1} \binom{n}{k} 2(n-k)(x-2)^{n-k-1} y^k - n \sum_{k=0}^{n-1} \binom{n-1}{k} T_{n-k-1}(y) x^k \\ = \sum_{k=0}^{n-1} \binom{n}{k} (n-k)(x-2)^{n-k-1} T_k(y). \end{aligned}$$

Using the identity $\binom{n}{k}(n-k) = \binom{n-1}{k}n$, the above equation simplifies to

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \left(2(x-2)^{n-k-1} y^k - T_{n-k-1}(y) x^k \right) = \sum_{k=0}^{n-1} \binom{n-1}{k} (x-2)^{n-k-1} T_k(y),$$

which is equivalent to

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \left(2(x-2)^{n-k} y^k - T_{n-k}(y) x^k \right) &= \sum_{k=0}^n \binom{n}{k} (x-2)^{n-k} T_k(y) \\ &= T_n(x-2+y). \end{aligned} \tag{51}$$

Applying Lemma 1 to (51), we obtain (50). □

Lastly, we obtain a relation of truncated tangent polynomials with the Frobenius-Euler polynomials.

Theorem 12. For $m, n \geq 0$,

$$T_{m,n}(x) = \sum_{k=0}^n \frac{\binom{n}{k}}{(1-\lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} T_{m,n-k}(j) H_k^{(r)}(x|\lambda), \quad (52)$$

where

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x; \lambda) \frac{t^n}{n!}$$

are the Frobenius-Euler polynomials (see [2]).

Proof. We express (5) as

$$\begin{aligned} \sum_{n=0}^{\infty} T_{m,n}(x) \frac{t^n}{n!} &= \left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} \frac{2^{\frac{t^m}{m!}}}{e^{2t} + 1 - \sum_{j=0}^{m-1} 2^j \frac{t^j}{j!}} \\ &= \frac{1}{(1-\lambda)^r} \left(\sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!}\right) \left(\sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j}\right) \frac{2^{\frac{t^m}{m!}} e^{jt}}{e^{2t} + 1 - \sum_{j=0}^{m-1} 2^j \frac{t^j}{j!}} \\ &= \frac{1}{(1-\lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} \left(\sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!}\right) \left(\sum_{n=0}^{\infty} T_{m,n}(j) \frac{t^n}{n!}\right) \\ &= \frac{1}{(1-\lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} H_k^{(r)}(x|\lambda) T_{m,n-k}(j)\right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{\binom{n}{k}}{(1-\lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} T_{m,n-k}(j) H_k^{(r)}(x|\lambda)\right) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof. \square

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