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Finite Rank Solution for Conformable Degenerate First-Order Abstract Cauchy Problem In Hilbert Spaces

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Abstract. In this paper, we find a solution of finite rank form of fractional Abstract Cauchy Problem. The fractional derivative used is the Conformable derivative. The main idea of the proofs are based on theory of tensor product of Banach spaces.

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1. Introduction

Let X be a Banach space and $I = [0, 1]$. Let $C(I)$ be the Banach space of all real valued continuous functions defined on I under the sup-norm. Let $C(I, X)$ be the Banach space of all continuous function defined on I with values in X.
A classical and important differential equation is the so called abstract Cauchy problem. One form such equation is

$$\begin{aligned} Bu^{'} &= Au(t) + f(t)z \\ u(0) &= x_0 \end{aligned}$$

Here $u \in C^1(I, X)$ and A,B are densely defined linear operators on the codomain of u.
If $f = 0$ or $z = 0$, then the equation is homogeneous otherwise it is called non-homogeneous. Now in the non-homogeneous problem we have two cases. The first type if u is unknown and f is given and this is called the direct problem, the second type u and f are unknown and it is called the inverse problem.
If B is not invertible, then the equation is called degenerate otherwise it is called non-degenerate.
In this paper we will look for certain solutions called finite rank solutions for the fractional

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abstract Cauchy problem, using the tensor product technique.
First let us present some basic facts on conformable fractional derivative.
For $f : [0;\infty) \to \mathbb{R}$ and $0 < \alpha \leq 1$, the conformable fractional derivative of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, if f is α differentiable on $(0, b)$ where $b > 0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$.
We denote $f^{(\alpha)}(t)$ for $T_\alpha(f)(t)$ and we say f is α differentiable if the conformable fractional derivative of f of order α exists.
For $0 < \alpha \leq 1$ and f, g be α differentiable at a point $t > 0$, we have the following properties:
(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.
(2) $T_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$.
(3) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
(4) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
(5) $T_\alpha(\lambda) = 0$, for all λ is constant function.
(6) if f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.
The α fractional integral of a function f starting from $a \geq 0$ is:

$$I^a_\alpha(f(t)) = I^a_1(t^{\alpha-1}f(t)) = \int_a^t \frac{f(s)}{s^{1-\alpha}} ds$$

For more on conformable fractional derivative we refer to [1], [6]-[18], [20] and [21].

2. Basic Facts of the Tensor Product of Banach Space

Let X and Y be Banach spaces, X^* denotes the dual of X. For $x \in X$ and $y \in Y$ define the map $x \otimes y : X^* \to Y$ as: $x \otimes y(x^*) = \langle x, x^*\rangle y$, for all $x^* \in X^*$.
Cleaarly, $x \otimes y$ is a bounded linear operator and $\| x \otimes y \| = \| x \| \| y \|$[2]. Such an operator $x \otimes y$ is called an atom. The set $X \otimes Y = span\{x \otimes y : x \in X \text{ and } y \in Y\}$ is a subspace of $L(X^*, Y)$. The following lemma, [5], is needed in our paper.

Lemma 1. *Let $x_1 \otimes y_1$ and $x_2 \otimes y_2$ be two nonzero atoms in $X \otimes Y$ such that*

$$x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3.$$

Then either x_1, x_2 or y_1, y_2 are linearly dependent.

We can define many norms on $X \otimes Y$. The most important one is: the injective norm For $T = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ define

$$\| T \|_\vee = \sup\{| \sum_{i=1}^n \langle x_i, x^*\rangle\langle y_i, y^*\rangle | : x^* \in B_1(X^*) \ \ and \ \ y^* \in B_1(Y^*)\}.$$

So, $\| \, . \, \|_\vee$ is just the operator norm on $L(X^*, Y)$. This called the injective norm of T. The space $(X \otimes Y, \| \, . \, \|_\vee)$ need not be complete. Let $X \overset{\vee}{\otimes} Y$ denote the completion of $(X \otimes Y, \| \, . \, \|_\vee)$ and it is called the completed injective tensor product of X with Y. A nice result that is used in theory of differential equations is:

Theorem 1. *For any compact Housdorff space K, and any Banach space X , $C(K, X)$ is isometrically isomorphic to $C(K) \overset{\vee}{\otimes} X$.*
In particular, for any two compact metric spaces I and J, one has $C(I \times J) = C(I) \overset{\vee}{\otimes} C(J)$.

3. Main results

3.1. Direct Problem

Let u be an α-differentiable on $I = [0, 1]$ with values in the Hilbert space $X = \ell^2$, where $\ell^2 = \{(x_n) : \sum_{n=1}^{\infty} \mid x_n \mid^2 < \infty\}$. The natural basis of ℓ^2 is denotes by $\{\delta_1, \delta_2, ...\}$. In ℓ^2 we write $[x_1, x_2, ..., x_n]$ to denote the span of $\{x_1, x_2, ..., x_n\}$.
Let $A : Dom(A) \subseteq \ell^2 \to \ell^2, B : Dom(B) \subseteq \ell^2 \to \ell^2$ be two densely defined linear operators on ℓ^2, where domains of A and B contain the elements of the natural basis of ℓ^2.

The homogeneous degenerate fractional abstract Cauchy problem is

$$\begin{cases} Bu^{(\alpha)}(t) = Au(t) \\ u(0) = x_0 \end{cases} \tag{1}$$

The nonhomogeneous degenerate fractional abstract Cauchy problem is

$$\begin{cases} Bu^{(\alpha)}(t) = Au(t) + f(t)z \\ u(0) = x_0 \end{cases} \tag{2}$$

Where $u(t) \in Dom(A) \cap Dom(B)$, $u^{(\alpha)}(t) \in Dom(B)$, $f \in C(I)$ and $z \in \ell^2$.
In this section we look for a solution to problems (1) and (2) among finite rank function of the form $u(t) = \sum_{i=1}^{n} u_i(t)\delta_i$, where $u_i^{(\alpha)} \in C(I), i = 1, 2, ...n$.

Theorem 2. *In problem (P1), let $u(t) = \sum_{i=1}^{n} u_i(t)\delta_i$, where $u_i^{(\alpha)} \in C(I), i = 1, 2, ...n$ and assume $B = I$, then the problem (1) has a unique solution.*

Proof. We have, $u(t) = \sum_{i=1}^{n} u_i(t)\delta_i$, then $u^{(\alpha)}(t) = \sum_{i=1}^{n} u_i^{(\alpha)}(t)\delta_i$, thus

$$\sum_{i=1}^{n} u_i^{(\alpha)}(t)\delta_i = \sum_{i=1}^{n} u_i(t)A\delta_i. \tag{3}$$

So, $Au(t) \in [\delta_1, \delta_2, ..., \delta_n]$, since $u^{(\alpha)}(t)$ is linear combination of $\delta_1, \delta_2, ..., \delta_n$. Hence $[\delta_1, \delta_2, ..., \delta_n]$ is invariant subspace of A.

Let $\hat{A} = A\mid_{[\delta_1,\delta_2,...,\delta_n]}$ be the restriction of A on $[\delta_1, \delta_2, ..., \delta_n]$ and so $\hat{A}$ has a matrix representation which is $\hat{A} = [a_{ij}]$, such that $a_{ij} = \langle A\delta_j, \delta_i\rangle$.
Taking the inner product of δ_j with both sides of equation (3), we get

$$\sum_{i=1}^{n} u_i^{(\alpha)}(t)\langle\delta_i, \delta_j\rangle = \sum_{i=1}^{n} u_i(t)\langle A\delta_i, \delta_j\rangle.$$

Since $\{\delta_i\}_{i=1}^n$ is orthonormal, we obtain

$$u_j^{(\alpha)}(t) = \sum_{i=1}^{n} u_i(t)\langle A\delta_i, \delta_j\rangle. \tag{4}$$

Which is a homogeneous linear system of differential equations

$$U^{(\alpha)}(t) = \hat{A}U(t),\ where\ U(t) = (u_1(t), u_2(t), ..., u_n(t))^T.$$

This is system has a unique solution of the form

$$U(t) = \phi(t)c.$$

Here $\phi(t)$ is the fundamental matrix, which is invertible. By the initial condition, we have

$$c_i = \langle\phi^{-1}(0)x_0, \delta_i\rangle, \quad i = 1, ..., n.$$

Consequently, the problem (1) has a unique solution.

Theorem 3. *In problem (2), let $u(t) = \sum_{i=1}^n u_i(t)\delta_i$, where $u_i^{(\alpha)} \in C(I), i = 1, 2, ...n$ and assume $B = I$ and $z \in [\delta_1, ..., \delta_n]$, then the problem (2) has a unique solution.*

Proof. We have, $u(t) = \sum_{i=1}^n u_i(t)\delta_i$, then $u^{(\alpha)}(t) = \sum_{i=1}^n u_i^{(\alpha)}(t)\delta_i$, thus

$$\sum_{i=1}^{n} u_i^{(\alpha)}(t)\delta_i = \sum_{i=1}^{n} u_i(t)A\delta_i + f(t)z. \tag{5}$$

Let $\hat{A} = A\mid_{[\delta_1,\delta_2,...,\delta_n]}$ the restriction of A on $[\delta_1, \delta_2, ..., \delta_n]$ and so $\hat{A}$ has a matrix representation which is $\hat{A} = [a_{ij}]$, such that $a_{ij} = \langle A\delta_j, \delta_i\rangle$.
Taking the inner product of δ_j with both sides of equation (5), we get

$$\sum_{i=1}^{n} u_i^{(\alpha)}(t)\langle\delta_i, \delta_j\rangle = \sum_{i=1}^{n} u_i(t)\langle A\delta_i, \delta_j\rangle + f(t)\langle z, \delta_j\rangle.$$

Since $\{\delta_i\}_{i=1}^n$ is orthonormal, we obtain

$$u_j^{(\alpha)}(t) = \sum_{i=1}^{n} u_i(t)\langle A\delta_i, \delta_j\rangle + f(t)\langle z, \delta_j\rangle. \tag{6}$$

We set, $U(t) = (u_1(t), ..., u_n(t))^T$ and $F(t) = f(t)(\langle z, \delta_1\rangle, ..., \langle z, \delta_n\rangle)^T$, so equation (6) can be written in the form

$$U^{(\alpha)}(t) = \hat{A}U(t) + F(t).$$

This system has a unique solution of the form

$$U(t) = \phi(t)c + \phi(t)\int_0^t \frac{\phi^{-1}(s)F(s)}{s^{1-\alpha}}ds.$$

Where $\phi(t)$ is the fundamental matrix. This is an invertible matrix. Now we use the initial condition to find the constant c. Consequently, the problem (2) has a unique solution.

Now, let $B \neq I$ and $u(t)$ is finite rank function. In addition assume that $[\delta_1, \delta_2, ..., \delta_n]$ is invariant under both A and B and let A_n, B_n be the restriction of A and B to $[\delta_1, \delta_2, ..., \delta_n]$.

Theorem 4. *In problem (1), let B_n be orthogonally diagonalizable linear operator such that $A_n\mid_{Ker(B_n)}$ is invertible. Then problem (1) has a unique solution.*

Proof. Let $\{\theta_1, \theta_2, ..., \theta_n\}$ be an orthonormal basis such that the matrix representation of B_n with respect this basis is $\tilde{D} = diag(\lambda_1, ..., \lambda_n)$, when $\lambda_1, ..., \lambda_n$ the corresponding eigenvalues of B_n. Now, if $\lambda_i \neq 0$ for all $i = 1, 2, ...n$, then problem (1) becomes $u^{(\alpha)}(t) = B_n^{-1}A_nu(t)$ and hence has a unique solution by theorem 3.1.
Suppose $\lambda_i \neq 0$ for $i = 1, 2, ...r$, and $\lambda_i = 0$ for $i = r+1, r+2, ...n$. Let $u(t) = \sum_{i=1}^n v_i(t)\theta_i$: Then

$$\sum_{i=1}^n v_i^{(\alpha)}(t)B_n\theta_i = \sum_{i=1}^n v_i(t)A_n\theta_i. \tag{7}$$

Taking the inner product of θ_j with both sides of (7), we obtain

$$\sum_{i=1}^n v_i^{(\alpha)}(t)\langle B_n\theta_i, \theta_j\rangle = \sum_{i=1}^n v_i(t)\langle A_n\theta_i, \theta_j\rangle.$$

So, we get the following system

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1^{(\alpha)}(t) \\ . \\ . \\ . \\ v_n^{(\alpha)}(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & \tilde{A} \end{bmatrix}\begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix}. \tag{8}$$

where $D = diag(\lambda_1, ., ., ., \lambda_r)$ and $\tilde{A} = A_n\mid_{Ker(B_n)} = [\langle A_n\theta_j, \theta_i\rangle]_{i,j=r+1,...,n}$.
Multiplying (8) by $\begin{bmatrix} I & 0 \\ 0 & \tilde{A}^{-1} \end{bmatrix}$, we obtain

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1^{(\alpha)}(t) \\ . \\ . \\ . \\ v_n^{(\alpha)}(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ \tilde{A}^{-1}A_3 & I_{n-r} \end{bmatrix}\begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix}.$$

Thus, we get

$$D\begin{bmatrix} v_1^{(\alpha)}(t) \\ . \\ . \\ . \\ v_r^{(\alpha)}(t) \end{bmatrix} = A_1\begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_r(t) \end{bmatrix} + A_2\begin{bmatrix} v_{r+1}(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix}, \tag{9}$$

and

$$\tilde{A}^{-1}A_3\begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_r(t) \end{bmatrix} + I_{n-r}\begin{bmatrix} v_{r+1}(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix} = 0. \tag{10}$$

From equation (10), we have

$$\begin{bmatrix} v_{r+1}(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix} = -\tilde{A}^{-1}A_3\begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_r(t) \end{bmatrix}. \tag{11}$$

Substitute (11) in equation (9), to get

$$\begin{bmatrix} v_1^{(\alpha)}(t) \\ . \\ . \\ . \\ v_r^{(\alpha)}(t) \end{bmatrix} = D^{-1}(A_1 - A_2\tilde{A}^{-1}A_3)\begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_r(t) \end{bmatrix}.$$

We put, $U_1(t) = \begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_r(t) \end{bmatrix}$, $U_2(t) = \begin{bmatrix} v_{r+1}(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix}$ and $M = D^{-1}(A_1 - A_2\tilde{A}^{-1}A_3)$.

We get the system, $U_1^{(\alpha)}(t) = MU_1(t)$, which has a unique solution $U_1(t) = \phi(t)c$, where $\phi(t)$ is the fundamental matrix. So we have $U_2(t) = -\tilde{A}^{-1}A_3U_1(t)$. Therefore $u(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix}^T \begin{bmatrix} \theta_1 \\ . \\ . \\ . \\ \theta_n \end{bmatrix}$.

We conclude the problem (1) has a unique solution.

Theorem 5. *In problem (2), let B_n be orthogonally diagonalizable linear operator such that $A_n\mid_{Ker(B_n)}$ is invertible. Then problem (2) has a unique solution.*

Proof. Let $\{\theta_1, \theta_2, ..., \theta_n\}$ be an orthonormal basis such that the matrix representation of B_n with respect this basis is $\tilde{D} = diag(\lambda_1, ..., \lambda_n)$, when $\lambda_1, ..., \lambda_n$ the corresponding eigenvalues of B_n. Now, if $\lambda_i \neq 0$, for all $i = 1, 2, ...n$, then the problem (2) becomes

$$u^{(\alpha)}(t) = B_n^{-1} A_n u(t) + f(t) B_n^{-1} z.$$

Hence has a unique solution by theorem 3.2.
Suppose $\lambda_i \neq 0$, for $i = 1, 2, ...r$, and $\lambda_i = 0$, for $i = r+1, r+2, ...n$. Let $u(t) = \sum_{i=1}^{n} v_i(t)\theta_i$: Then

$$\sum_{i=1}^{n} v_i^{(\alpha)}(t) B_n \theta_i = \sum_{i=1}^{n} v_i(t) A_n \theta_i + f(t) z. \tag{12}$$

Taking the inner product of θ_j with both sides of equation (12), we obtain

$$\sum_{i=1}^{n} v_i^{(\alpha)}(t) \langle B_n \theta_i, \theta_j \rangle = \sum_{i=1}^{n} v_i(t) \langle A_n \theta_i, \theta_j \rangle + f(t) \langle z, \theta_j \rangle.$$

So, we get the following system

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{(\alpha)}(t) \\ . \\ . \\ . \\ v_n^{(\alpha)}(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & \tilde{A} \end{bmatrix} \begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix} + f(t) \begin{bmatrix} \langle z, \theta_1 \rangle \\ . \\ . \\ . \\ \langle z, \theta_n \rangle \end{bmatrix}. \tag{13}$$

where $D = diag(\lambda_1, ., ., ., \lambda_r)$ and $\tilde{A} = A_n \mid_{Ker(B_n)} = [\langle A_n \theta_j, \theta_i \rangle]_{i,j=r+1,...,n}$.
Multiplying (13) by $\begin{bmatrix} I & 0 \\ 0 & \tilde{A}^{-1} \end{bmatrix}$, we obtain

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{(\alpha)}(t) \\ . \\ . \\ . \\ v_n^{(\alpha)}(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ \tilde{A}^{-1} A_3 & I_{n-r} \end{bmatrix} \begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix} + f(t) \begin{bmatrix} I & 0 \\ 0 & \tilde{A}^{-1} \end{bmatrix} \begin{bmatrix} \langle z, \theta_1 \rangle \\ . \\ . \\ . \\ \langle z, \theta_n \rangle \end{bmatrix}.$$

Thus, we get

$$D \begin{bmatrix} v_1^{(\alpha)}(t) \\ . \\ . \\ . \\ v_r^{(\alpha)}(t) \end{bmatrix} = A_1 \begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_r(t) \end{bmatrix} + A_2 \begin{bmatrix} v_{r+1}(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix} + f(t) \begin{bmatrix} \langle z, \theta_1 \rangle \\ . \\ . \\ . \\ \langle z, \theta_r \rangle \end{bmatrix}, \tag{14}$$

and

$$\tilde{A}^{-1}A_3\begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_r(t) \end{bmatrix} + I_{n-r}\begin{bmatrix} v_{r+1}(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix} + f(t)\tilde{A}^{-1}\begin{bmatrix} \langle z,\theta_{r+1}\rangle \\ . \\ . \\ . \\ \langle z,\theta_n\rangle \end{bmatrix} = 0. \qquad (15)$$

From equation (15), we have

$$\begin{bmatrix} v_{r+1}(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix} = -\tilde{A}^{-1}A_3\begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_r(t) \end{bmatrix} - f(t)\tilde{A}^{-1}\begin{bmatrix} \langle z,\theta_{r+1}\rangle \\ . \\ . \\ . \\ \langle z,\theta_n\rangle \end{bmatrix}. \qquad (16)$$

Substitute (16) in equation (14), we get

$$\begin{bmatrix} v_1^{(\alpha)}(t) \\ . \\ . \\ . \\ v_r^{(\alpha)}(t) \end{bmatrix} = D^{-1}(A_1 - A_2\tilde{A}^{-1}A_3)\begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_r(t) \end{bmatrix} + f(t)D^{-1}(\begin{bmatrix} \langle z,\theta_1\rangle \\ . \\ . \\ . \\ \langle z,\theta_r\rangle \end{bmatrix} - A_2\tilde{A}^{-1}\begin{bmatrix} \langle z,\theta_{r+1}\rangle \\ . \\ . \\ . \\ \langle z,\theta_n\rangle \end{bmatrix}).$$

We put, $U_1(t) = \begin{bmatrix} v_1(t) \\ . \\ . \\ . \\ v_r(t) \end{bmatrix}$, $U_2(t) = \begin{bmatrix} v_{r+1}(t) \\ . \\ . \\ . \\ v_n(t) \end{bmatrix}$, $M = D^{-1}(A_1 - A_2\tilde{A}^{-1}A_3)$ and $F(t) = f(t)D^{-1}(\begin{bmatrix} \langle z,\theta_1\rangle \\ . \\ . \\ . \\ \langle z,\theta_r\rangle \end{bmatrix} - A_2\tilde{A}^{-1}\begin{bmatrix} \langle z,\theta_{r+1}\rangle \\ . \\ . \\ . \\ \langle z,\theta_n\rangle \end{bmatrix})$.

Then we obtain the system

$$U_1^{(\alpha)}(t) = MU_1(t) + F(t).$$

Which is has a unique solution

$$U_1(t) = \phi(t)c + \phi(t)\int_0^t \frac{\phi^{-1}(s)F(s)}{s^{1-\alpha}}ds,$$

where $\phi(t)$ is the fundamental matrix and we have

$$U_2(t) = -\tilde{A}^{-1}A_3U_1(t) - f(t)\tilde{A}^{-1}\begin{bmatrix} \langle z,\theta_{r+1}\rangle \\ . \\ . \\ . \\ \langle z,\theta_n\rangle \end{bmatrix}.$$

Hence, $u(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix}^T \begin{bmatrix} \theta_1 \\ . \\ . \\ . \\ \theta_n \end{bmatrix}.$

Therefore, the problem (2) has a unique solution.

3.2. Inverse Problem Case

Let $X = \ell^2$ be the Hilbert space. Let $A : Dom(A) \subseteq \ell^2 \to \ell^2, B : Dom(B) \subseteq \ell^2 \to \ell^2$ be two densely defined linear operators on ℓ^2, where domains of A and B contain the elements of the natural basis of ℓ^2.

Consider the two inverse problems (P3) and (P4) respectively

$$\begin{cases} u^{(\alpha)}(t) = Au(t) + f(t) \\ u(0) = x_0 \end{cases}$$

$$\begin{cases} Bu^{(\alpha)}(t) = Au(t) + f(t) \\ u(0) = x_0 \end{cases}$$

Where $u^{(\alpha)} \in C^(I, X)$, $f \in C(I, X)$.

In this section we look for a solution to problems (P3) and (P4) among finite rank functions of the form $u(t) = \sum_{i=1}^{n} u_i(t)\delta_i$, and $f(t) = \sum_{i=1}^{n} f_i(t)\delta_i$, where, $u_i^{(\alpha)} \in C(I)$ and $f_i \in C(I)$, for $i = 1, 2, ...n$. Here we use a condition similar to that used in [22].

Theorem 6. *In problem (P3), let* $u(t) = \sum_{i=1}^{n} u_i(t)\delta_i$, *and* $f(t) = \sum_{i=1}^{n} f_i(t)\delta_i$ *where,* $u_i^{(\alpha)} \in C(I)$ *and* $f_i \in C(I)$, *for* $i = 1, 2, ...n$.
Assume the following two condition are satisfied:
1)There exist, $x \in \ell^2$ *such that* $\langle u_i(t)\delta_i, x\rangle = g_i(t)$ *where* $g_i^{(\alpha)} \in C(I)$ *and* $\langle \delta_i, x\rangle \neq 0$.
2) A is diagonal with respect to the basis $\{\delta_i\}_{i=1}^{n}$. *That is,* $A\delta_i = \lambda_i\delta_i$ *for all* $i = 1, ..., n$.
Then the problem (P3) has a unique solution.

Proof. Substitute $u(t) = \sum_{i=1}^{n} u_i(t)\delta_i$, and $f(t) = \sum_{i=1}^{n} f_i(t)\delta_i$, in (P3), we get

$$\sum_{i=1}^{n} u_i^{(\alpha)}(t)\delta_i = \sum_{i=1}^{n} u_i(t)A\delta_i + \sum_{i=1}^{n} f_i(t)\delta_i.$$

Since A is diagonal with respect to $\{\delta_i\}_{i=1}^{n}$, we have

$$\sum_{i=1}^{n} u_i^{(\alpha)}(t)\delta_i = \sum_{i=1}^{n} \lambda_i u_i(t)\delta_i + \sum_{i=1}^{n} f_i(t)\delta_i. \tag{17}$$

Taking the inner product of δ_j with both sides of equation (17), we obtain

$$u_j^{(\alpha)}(t) = \lambda_j u_j(t) + f_j(t). \tag{18}$$

Multiplying equation (18) by δ_j and use condition (1), we obtain

$$g_j^{(\alpha)}(t) = \lambda_j g_j(t) + \langle f_j(t)\delta_j, x\rangle.$$

Thus, we have

$$f_j(t) = \frac{g_j^{(\alpha)}(t) - \lambda_j g_j(t)}{\langle \delta_j, x\rangle}.$$

Hence, $f_j(t)$ is determined uniquely for $j = 1, ...n$ and thus $f(t)$ is determined uniquely. Now to find $u(t)$. Since $f(t)$ is determined, then we have

$$u_j(t) = u_j(0)e^{\lambda_j \frac{t^\alpha}{\alpha}} + e^{\lambda_j \frac{t^\alpha}{\alpha}} \int_0^t \frac{e^{-\lambda_j \frac{s^\alpha}{\alpha}} f_j(s)}{s^{1-\alpha}} ds.$$

Consequently, the problem (P3) has a unique solution.

Now, to solve problem (P4) we need to assume the following satisfy:
Assumption 1. $B_n = B \mid_{[\delta_1,...,\delta_n]}$ is orthogonally diagonalizable linear operator with respect to the orthonormal basis $\{\theta_1, ..., \theta_n\}$ and corresponding eigenvalues $\lambda_1, ..., \lambda_n$ such that $A_n \mid_{Ker(B_n)}$ is invertible, where $A_n = A \mid_{[\delta_1,...,\delta_n]}$.
Assumption 2. A_n is diagonal with respect to $\{\theta_1, ..., \theta_n\}$ ie $A_n\theta_j = \mu_j\theta_j$ for $j = 1, ..., n$. Now, let $u(t) = \sum_{i=1}^n u_i(t)\theta_i$
Assumption 3. There exist, $x \in \ell^2$ such that $\langle u_i(t)\theta_i, x\rangle = g_i(t)$ where $g_i^{(\alpha)} \in C(I)$.
Assumption 4. $M = [\langle \delta_i, \theta_j\rangle\langle \theta_j, x\rangle]_{i,j=1,...,n}$is invertible.

Theorem 7. *Under assumptions 1,2, 3 and 4, problem (P4) has a unique solution.*

Proof. Since $u(t) = \sum_{i=1}^n u_i(t)\theta_i$, then we substitute in problem (P4), we have

$$\sum_{i=1}^n u_i^{(\alpha)}(t)B_n\theta_i = \sum_{i=1}^n u_i(t)A_n\theta_i + \sum_{i=1}^n f_i(t)\delta_i.$$

This implies

$$\sum_{i=1}^n u_i^{(\alpha)}(t)\lambda_i\theta_i = \sum_{i=1}^n u_i(t)\mu_i\theta_i + \sum_{i=1}^n f_i(t)\delta_i. \tag{19}$$

Taking the inner product of θ_j with both sides of equation (19), we obtain

$$\lambda_j u_j^{(\alpha)}(t) = \mu_j u_j(t) + \sum_{i=1}^n f_i(t)\langle \delta_i, \theta_j\rangle. \tag{20}$$

Multiplying equation (20) by θ_j and using assumption 3, we obtain

$$\lambda_j g_j^{(\alpha)}(t) = \mu_j g_j(t) + \sum_{i=1}^n f_i(t)\langle\langle \delta_i, \theta_j\rangle\langle \theta_j, x\rangle.$$

Hence, we get the following system:

$$\begin{bmatrix} \lambda_1 g_1^{(\alpha)}(t) - \mu_1 g_1(t) \\ . \\ . \\ . \\ \lambda_n g_n^{(\alpha)}(t) - \mu_n g_n(t) \end{bmatrix} = M^T \begin{bmatrix} f_1(t) \\ . \\ . \\ . \\ f_n(t) \end{bmatrix}.$$

Where, $M = [\langle \delta_i, \theta_j \rangle \langle \theta_j, x \rangle]_{i,j=1,\ldots,n}$. By assumption 4 M is invertible, then $M^T$is also invertible and $(M^T)^{-1} = (M^{-1})^T$, thus

$$\begin{bmatrix} f_1(t) \\ . \\ . \\ . \\ f_n(t) \end{bmatrix} = (M^{-1})^T \begin{bmatrix} \lambda_1 g_1^{(\alpha)}(t) - \mu_1 g_1(t) \\ . \\ . \\ . \\ \lambda_n g_n^{(\alpha)}(t) - \mu_n g_n(t) \end{bmatrix}.$$

Therefore f is determined uniquely.
Now to find $u(t)$, we have
• If $\lambda_j = 0$, then $u_j(t) = \sum_{i=1}^{n} \frac{f_i(t)\langle \delta_i, \theta_j \rangle}{-\mu_j}$.
• If $\lambda_j \neq 0$, then

$$u_j(t) = u_j(0) e^{\frac{\mu_j t^\alpha}{\lambda_j \alpha}} + \sum_{i=1}^{n} e^{\frac{\mu_j t^\alpha}{\lambda_j \alpha}} \langle \delta_i, \theta_j \rangle \int_0^t \frac{f_i(s) e^{\frac{-\mu_j s^\alpha}{\lambda_j \alpha}}}{s^{1-\alpha}} ds.$$

Consequently, the problem (P4) has a unique solution.

References

[1] R. Khalil, M. Al Horani, A. Yousef. and M. Sababheh, A new Definition of Fractional Derivative, J. Comput. Appl. Math., 264:65-70, (2014).

[2] W. A. Light, E. W. Cheney, Approximation theory in tensor product spaces. Lecture notes in math. 1169. Springer-Verlag, New York, 1985.

[3] A. M. Ziqan, M. H. Al Horani, and R. Khalil, Tensor Product Technique and the Degenerate Homogeneous Abstract Cauchy Problem, J. Appl. Funct. Anal., 5 (1):121-138, (2010).

[4] A. M. Ziqan, M. H. Al Horani, and R. Khalil, Tensor Product Technique and the Degenerate Nonhomogeneous Abstract Cauchy Problem, J. Appl. Funct. Anal., 23 (1):137-158, (2010).

[5] R. Khalil, Isometries on $L_p \otimes L_p$, Tamkang J. Math. 16(2):77-85, (1985).

[6] T. Abdeljawad, Conformable Fractional Calculus, J. Comput. Appl. Math. 279:57-66, (2015).

[7] M. Abu Hammad, R. Khalil, Systems of Linear Fractional Differential Equations, Asian J. Math.Comput. Res., 12(2):120-126, (2016).

[8] B. Thaller, S.Thaller, Factorization of Degenerate Cauchy Problem, the linear case, J.Oper. Theory 36:121-146, (1996).

[9] A. Atangana, D. Baleanu, A. Alsaedi, New properties of conformable derivative, Open Mathematics 13(2015).

[10] M. AlHorani, R. Khalil, Total Fractional Differentials With Applications to Exact Fractional Differential Equations, International Journal of Computer Mathematics, 95:1444-1452, (2018).

[11] R. Khalil, M. Al Horani, D. Anderson, Undetermined coeficients for local fractional differential equations J. Math. Comput. Sci 16:140-146, (2016).

[12] D. R. Anderson, E. Camud, and D. J. Ulness, On the nature of the conformable derivative and its applications to physics, Journal of fractional Calculus and applications, 14:92-135, (2019).

[13] M. Mhailan, M. AbuHammad, M. AlHorani, R. Khalil, Fractional vector analysis, Journal of Mathematical and Computational Science,10:2320-2326, (2020).

[14] A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, in : Math. Studies., North-Holland, New York, 2006.

[15] M. Abu Hammad, R. Khalil, Conformable fractional Heat differential equation, International Journal of pure and applied mathematics 94(2):215- 221, (2014).

[16] M. Abu Hammad, R. Khalil., Fractional Fourier Series with Applications, American Journal of Computational and Applied Mathematics 4(6):187-191, (2014).

[17] W. S Chung, Fractional Newton mechanics with conformable fractional derivative, Journal of computational and applied mathematics, 290:150-158, (2015).

[18] M. AlHorani, M. AbuHammad, R. Khalil, Variation of parameters for local fractional nonhomogenous linear-differential equations J. Math. Computer Sci 16:140-146, (2016).

[19] W. Deeb, R. Khalil, Best approximation in $L(X;Y)$, Mathematical Proceedings of the Cambridge Philosophical Society 104:527-531, (1988).

[20] M. Al-Horani, R. Khalil and Aldarawi, Fractional Cauchy Euler Differential Equation, J. COMPUTATIONAL ANALYSIS AND APPLICATIONS 28(2):226-233, (2019).

[21] R. Khalil, M. Al Horani, M. A. Hammad, Geometric meaning of conformable derivative via fractional cords, J. Math. Computer Sci., 19:241-245, (2019).

[22] M. Al Horani, M. Fabrizio, A. Favini, and H. Tanabe, Fractional Cauchy problems for infnite interval case, Discrete Continuous Dynamical Systems-S, 3(12):3285, (2020).

[23] O. A. Arqub, M. Al-Smadi, Fuzzy conformable fractional differential equations: novel extended approach and new numerical solutions, Soft Computing., 1-22, (2020).

[24] M. Al-Smadi, O. A. Arqub, S. Hadid, An attractive analytical technique for coupled system of fractional partial differential equations in shallow water waves with conformable derivative. Communications in Theoretical Physics, 72(8):085001, (2020).

[25] O. Abu Arqub, Solutions of timefractional Tricomi and Keldysh equations of Dirichlet functions types in Hilbert space. Numerical Methods for Partial Differential Equations, 34(5):1759-1780, (2018).