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Total Perfect Hop Domination in Graphs Under Some Binary Operations

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Abstract. Let $G = (V(G), E(G))$ be a simple graph. A set $S \subseteq V(G)$ is a *perfect hop dominating set* of G if for every $v \in V(G) \setminus S$, there is exactly one vertex $u \in S$ such that $d_G(u, v) = 2$. The smallest cardinality of a perfect hop dominating set of G is called the *perfect hop domination number* of G, denoted by $\gamma_{ph}(G)$. A perfect hop dominating set $S \subseteq V(G)$ is called a *total perfect hop dominating set* of G if for every $v \in V(G)$, there is exactly one vertex $u \in S$ such that $d_G(u, v) = 2$. The *total perfect hop domination number* of G, denoted by $\gamma_{tph}(G)$, is the smallest cardinality of a total perfect hop dominating set of G. Any total perfect hop dominating set of G of cardinality $\gamma_{tph}(G)$ is referred to as a γ_{tph}*-set* of G. In this paper, we characterize the total perfect hop dominating sets in the join, corona and lexicographic product of graphs and determine their corresponding total perfect hop domination number.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph. The *open neighborhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. The maximum degree of a graph G, denoted by $\Delta(G)$, is the maximum $deg_G(u)$, for all $u \in V(G)$. Similarly, the minimum degree of a graph G, denoted by $\delta(G)$, is the minimum $deg_G(u)$, for all $u \in V(G)$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{u \in X} N_G(u)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

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A graph $H = (V(H), E(H))$ is a *subgraph* of a graph $G = (V(G), E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $C \subseteq V(G)$, then the *induced subgraph* $\langle C \rangle$ of G is the graph with vertex set C and such that $uv \in E(\langle C \rangle)$ whenever $u, v \in C$ and $uv \in E(G)$.

Domination in graphs is one of the fastest growing research areas in Graph Theory. Since then it has been an extensively investigated branch of graph theory. This is largely due to a variety of new parameters that can be developed from the basic definition of domination and its wide range of applications to other fields of study. Many authors contribute several interesting domination parameters to nurture the growth of this research area.

In 2015, Natarajan and Ayyaswamy [3] introduced a new domination parameter called the hop domination number of a graph. In 2016, some variations of hop domination was studied by Pabilona and Rara [4]. A subset S of $V(G)$ is a *hop dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The smallest cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$ is the *hop domination number* of G. A hop dominating set S of G with cardinality $\gamma_h(G)$ is called a γ_h-set of G. At the same time of this year, Saromines and Rara [5] introduced a new hop domination parameter called the perfect hop domination in graphs in which they characterized the perfect hop dominating set of the join and corona of graphs. A subset S of $V(G)$ is a *perfect hop dominating set* of G if for every $v \in V(G) \setminus S$, there is exactly one vertex $u \in S$ such that $d_G(u, v) = 2$. The smallest cardinality of a perfect hop dominating set of G, denoted by $\gamma_{ph}(G))$ is the *perfect hop domination number* of G. A perfect hop dominating set) S of G with cardinality $\gamma_{ph}(G))$ is called a γ_{ph}-set) of G.

In 2018, Rara and Rakim present a further study on perfect hop dominaton in graphs [5] and in the following year we also introduce connected perfect hop domination in graphs under some binary operations [6].

A subset S of $V(G)$ is a *total hop dominating set* [4] of G if for every $v \in V(G)$, there exists $u \in S$ such that $d_G(u, v) = 2$. The smallest cardinality of a total hop dominating set of G, denoted by $\gamma_{th}(G)$ is called the *total hop domination number* of G. Any total hop dominating set of G with cardinality $\gamma_{th}(G)$ is called a γ_{th}-set.

A set $S \subseteq V(G)$ is a *total point-wise non-dominating set* [4] of G if for every $v \in V(G)$, there is a vertex $u \in S$ such that $v \notin N_G(u)$. The smallest cardinality of a total point-wise non-dominating set of G, denoted by $tpnd(G)$ is called the *total point-wise non-domination number* of G. Any total point-wise non-dominating set S of G with $|S| = tppnd(G)$ is called a $tpnd$-set.

A set $S \subseteq V(G)$ is a $(1,2)^*$-*dominating set* [1] of G if for every $w \in V(G) \setminus S$, there exists vertex $x \in S$ such that $wx \in E(G)$ and for every $u \in V(G) \setminus S$, there is vertex $v \in S$ such that $d_G(u, v) = 2$. The smallest cardinality of a $(1,2)^*$-dominating set of G is called the $(1,2)^*$-*domination number* of G, denoted by $\gamma^*_{1,2}(G)$. A $(1,2)^*$-dominating set S of G with cardinality $\gamma^*_{1,2}(G)$ is called a $\gamma^*_{1,2}$-set of G.

For other terms not define here, refer to [2].

In the next section, we introduce total perfect hop dominating set and explore some of its properties.

2. Total Perfect Hop Dominating Set

Definition 2.1. *A perfect hop dominating set S of $V(G)$ is a total perfect hop dominating set of G if for every $v \in V(G)$, there is exactly one vertex $u \in S$ such that $d_G(u,v) = 2$. The smallest cardinality of a total perfect hop dominating set of G, denoted by $\gamma_{tph}(G)$ is called the total perfect hop domination number of G. Any total perfect hop dominating set of G with cardinality $\gamma_{tph}(G))$ is called a γ_{tph}-set.*

Remark 2.2. *Let G be a connected graph of order $n \geq 4$. Then $\gamma_{tph}(G) \geq 4$. Moreover, for $G = P_4$ or C_4, $V(G)$ is a total perfect hop dominating set of G. For $n \geq 6$, $V(G)$ is a total perfect hop dominating set of G if and only if $|V(G)|$ is even and the vertices of G can be labeled $u_1, u_2, ..., u_{\frac{V(G)}{2}}$, $v_1, v_2, ..., v_{\frac{V(G)}{2}}$ such that $d_G(u_i, v_i) = 2$, $d_G(u_i, u_j) = d_G(v_i, v_j) = d_G(u_i, v_j) = 1$, whenever $i \neq j$.*

Remark 2.3. *Let G be a graph of order n. Then the total perfect hop dominating set of G does not exist if $\gamma(H) = 1$.*

Lemma 2.4. *Let G be a connected graph of order $n \geq 4$. Then $S = \{x_1, x_2, x_3, x_4\}$ is a total perfect hop dominating set of G if $\langle S\rangle \cong P_4 = [x_1, x_2, x_3, x_4]$ and for every $v \in V(G)$ at least one of the following holds.*

(i) $v \in N_G(x_1) \backslash \bigcup_{i \neq 1} N_G(x_i)$ *and* $v \notin N_G(u)$ *for each* $u \in N_G(x_j)$ *where* $j = 3$ *or* 4, *or*

(ii) $v \in N_G(x_4) \backslash \bigcup_{i \neq 4} N_G(x_i)$ *and* $v \notin N_G(u)$ *for each* $u \in N_G(x_j)$ *where* $j = 1$ *or* 2, *or*

(iii) $v \in [N_G(x_1) \cap N_G(x_2)] \backslash \bigcup_{i=3,4} N_G(x_i)$ *and* $v \notin N_G(u)$ *for each* $u \in N_G(x_4)$, *or*

(iv) $v \in [N_G(x_3) \cap N_G(x_4)] \backslash \bigcup_{i=1,2} N_G(x_i)$ *and* $v \notin N_G(u)$ *for each* $u \in N_G(x_1)$, *or*

(v) $v \in \bigcap_{j} N_G(x_j)$ *for exactly three* x_j*'s, or*

(vi) $v \in N_G(x_k, 2) \backslash \bigcup_{i \neq k} N_G(x_i)$ *for* $k = 1$ *or* 4 *and* $deg_G(v) = 1$, *or*

(vii) $v \in N_G(x_1, 2) \backslash \bigcup_{i \neq 1} N_G(x_i)$ *and* $v \in N_G(u)$ *for each* $u \in N_G(x_4, 2) \backslash \bigcup_{j \neq 4} N_G(x_j)$.

Proof. Suppose $S = \{x_1, x_2, x_3, x_4\}$ and $\langle S\rangle \cong P_4 = [x_1, x_2, x_3, x_4]$. Let $v \in V(G)$. If $v \in S$, then by Remark 2.2, $|N_G(v,2) \cap S| = 1$. Suppose that $v \notin S$. If (i) and (ii) hold, then $N_G(v,2) \cap S = \{x_j\}$ for $j = 2$ and $j = 3$, respectively. If (iii) and (iv) hold, then $N_G(v,2) \cap S = \{x_k\}$ for $k = 3$ and $k = 2$, respectively. For condition (v), it can easily be verified that $N_G(v,2) \cap S = \{x_p\}$ where $p \in \{1,2,3,4\}$. If (vi) and (vii) hold, then $N_G(v,2) \cap S = \{x_s\}$ where $s \in \{1,4\}$. Therefore S is a total perfect hop dominating set of G. □

Lemma 2.5. *Let G be a connected graph of order $n \geq 6$. Then $S = \{x_1, x_2, x_3, x_4\}$ is a total perfect hop dominating set of G if $\langle S \rangle \cong P_2 \cup \overline{K}_2$ where $P_2 = [x_2, x_3]$ and $V(\overline{K}_2) = \{x_1, x_4\}$ and the following hold.*

(i) $|N_G(x_2) \cap N_G(x_4)| = 0$ *and* $|N_G(x_1) \cap N_G(x_2)| \neq 0$

(ii) $|N_G(x_1) \cap N_G(x_3)| = 0$ *and* $|N_G(x_3) \cap N_G(x_4)| \neq 0$

(iii) *For every $v \in V(G)$, at least one of the following holds.*

  (a) $v \in [N_G(x_1) \cap N_G(u)] \backslash \bigcup_{k \neq 1} N_G[x_k]$ *for each* $u \in N_G(x_4) \backslash \bigcup_{j \neq 4} N_G[x_j]$, *or*

  (b) $v \in N_G(x_2) \backslash (N_G[x_3] \cup N_G[u])$ *or* $v \in N_G(x_3) \backslash (N_G[x_2] \cup N_G[u])$ *for each* $u \in (N_G(x_i) \cap N_G(x_{i+1}))$ *where* $i \in \{1,3\}$, *or*

  (c) $v \in [N_G(x_2) \cap N_G(u)] \backslash \bigcup_{k \neq 2} N_G(x_k)$ *for each* $u \in N_G(x_3) \backslash \bigcup_{j \neq 3} N_G(x_j)$, *or*

  (d) $v \in [N_G(x_2) \cap N_G(x_4)] \backslash N_G[x_3]$, *or*

  (e) $v \in [N_G(x_1) \cap N_G(x_3)] \backslash N_G[x_2]$, *or*

  (f) $v \in [N_G(x_2) \cap N_G(x_3) \cap N_G(u)] \backslash N_G[x_k]$ *for each* $u \in [N_G(x_k) \cap N_G(x_{k+1})] \backslash N_G[x_4]$ *if* $k = 1$ *or* $u \in [N_G(x_k) \cap N_G(x_{k-1})] \backslash N_G[x_1]$ *if* $k = 4$, *or*

  (g) $v \in [N_G(x_1) \cap N_G(x_2) \cap N_G(u)] \backslash [N_G[x_3] \cup N_G(w)]$ *for each* $u \in [N_G(x_1) \cap N_G(x_2)] \backslash N_G[x_4]$ *and for each* $w \in [N_G(x_3) \cap N_G(x_4)] \backslash N_G[x_4]$, *or*

  (h) $v \in [N_G(x_3) \cap N_G(x_4) \cap N_G(u)] \backslash (N_G[x_2] \cup N_G[w])$ *for each* $u \in [N_G(x_3) \cap N_G(x_4)] \backslash N_G[x_1]$ *and for each* $w \in N_G(x_1) \cap N_G(x_2)$, *or*

  (i) $v \in [N_G(x_1) \cap N_G(u)] \backslash [(\bigcup_{k \neq 1} N_G(x_k)) \cup N_G(w)]$ *for each* $u \in N_G(x_1) \cap N_G(x_2)$ *and for each* $w \in N_G(x_4)$, *or*

  (j) $v \in [N_G(x_4) \cap N_G(u)] \backslash [(\bigcup_{k \neq 4} N_G(x_k)) \cup N_G(w)]$ *for each* $u \in N_G(x_3) \cap N_G(x_4)$, *and for each* $w \in N_G(x_1)$, *or*

  (k) *v satisfies condition (f) and* $v \in N_G(w)$ *for* $w \in N_G(x_2) \backslash \bigcup_{k \neq 2} N_G(x_k)$, *or*

  (l) *v satisfies condition (i) and* $v \in N_G(w)$ *where* $deg_G(w) = 1$, *or*

(m) *v satisfies condition (j) and $v \in N_G(w)$ where $deg_G(w) = 1$, or*

(n) *v satisfies condition (c) and $v \in N_G(w)$ and $u \in N_G(y)$ where $deg_G(w) = deg_G(y) = 1$.*

Proof. Suppoose $S = \{x_1, x_2, x_3, x_4\}$ and $\langle S\rangle \cong P_2 \cup \overline{K}_2$ where $P_2 = [x_2, x_3]$ and $V(\overline{K}_2) = \{x_1, x_4\}$. Let $v \in V(G)$. If $v \in S$, then by (i) and (ii), $|N_G(v, 2) \cap S| = 1$. Suppose that $v \notin S$. If (iii)(a) holds, then $N_G(v, 2) \cap S = \{x_4\}$. If (iii)(b) holds, then $N_G(v, 2) \cap S = \{x_3\}$ for $v \in N_G(x_2)$ and $N_G(v, 2) \cap S = \{x_2\}$ for $v \in N_G(x_3)$. If (iii)(c) holds, then $N_G(v, 2) \cap S = \{x_3\}$. If (iii)(d) and (iii)(e) hold, then $N_G(v, 2) \cap S = \{x_3\}$ for $v \in N_G(x_2) \cap N_G(x_4)$ or $N_G(v, 2) \cap S = \{x_2\}$ for $v \in N_G(x_1) \cap N_G(x_3)$. If (iii)(f) holds, then $N_G(v, 2) \cap S = \{x_1\}$ for $v \in N_G(x_2) \cap N_G(x_3) \cap N_G(u)$ and $u \in N_G(x_1) \cap N_G(x_2)$ or $N_G(v, 2) \cap S = \{x_4\}$ for $v \in N_G(x_2) \cap N_G(x_3) \cap N_G(u)$ and $u \in N_G(x_3) \cap N_G(x_4)$. If (iii)(g) and (iii)(h) hold, then $N_G(v, 2) \cap S = \{x_3\}$ for $v \in N_G(x_1) \cap N_G(x_2) \cap N_G(u)$ or $N_G(v, 2) \cap S = \{x_2\}$ for $v \in N_G(x_3) \cap N_G(x_4) \cap N_G(u)$. If (iii)((i) and (j)) hold, then $N_G(v, 2) \cap S = \{x_2\}$ for $v \in N_G(x_1) \cap N_G(u)$ or $N_G(v, 2) \cap S = \{x_3\}$ for $v \in N_G(x_4) \cap N_G(u)$. If (k) holds, then $N_G(w, 2) \cap S = \{x_2\}$ for $w \in N_G(v) \cap N_G(x_3)$ or $N_G(w, 2) \cap S = \{x_3\}$ for $w \in N_G(v) \cap N_G(x_2)$. If (l) holds, then $N_G(w, 2) \cap S = \{x_1\}$. If (m) holds, then $N_G(w, 2) \cap S = \{x_4\}$. If (n) holds, then $N_G(w, 2) \cap S = \{x_2\}$. Therefore, S is a total perfect hop dominating set of G. □

Lemma 2.6. *Let G be a graph of order $n \geq 4$. Then $S = \{x_1, x_2, x_3, x_4\}$ is not a total perfect hop dominating set of G if the following hold.*

(i) $\langle S\rangle \cong K_2 \cup K_2$ *where* $x_1x_2,\ x_3x_4 \in E(G)$.

(ii) $\langle S\rangle \cong P_3 \cup K_1$.

(iii) $\langle S\rangle \cong \overline{K_4}$.

Proof. If (i) holds and $S = \{x_1, x_2, x_3, x_4\}$ is a total perfect hop dominating set of G, then there exists $v \in \bigcap_j N_G(x_j)$ for $j = 1, 2, 3$ since $N_G(x_1, 2) \cap S \neq \varnothing$ and $N_G(x_2, 2) \cap S \neq \varnothing$. Hence, $d_G(x_3, x_1) = d_G(x_3, x_2) = 2$ contrary to our assumption that S is a total perfect hop dominating set of G. Similarly, there exists $u \in \bigcap_k N_G(x_k)$ for $k = 2, 3, 4$ since $N_G(x_3, 2) \cap S \neq \varnothing$ and $N_G(x_4, 2) \cap S \neq \varnothing$. Hence, $d_G(x_2, x_3) = d_G(x_2, x_4) = 2$ is a contradiction to our assumption that S is a total perfect hop dominating set of G. Similarly, if (ii) and (iii) hold, then S is not a perfect hop dominating set of G. □

Theorem 2.7. *Let G be a connected graph of order greater than 3. Then $\gamma_{tph}(G) = 4$ if and only if $G = P_4$ or $G = C_4$ or $|V(G)| \geq 5$ and there exist vertices x_1, x_2, x_3, x_4 of G such that $\langle\{x_1, x_2, x_3, x_4\}\rangle \cong P_4$ or $\langle\{x_1, x_2, x_3, x_4\}\rangle \cong P_2 \cup \overline{K_2}$ and the conditions given in Lemma 2.4 and Lemma 2.5 are satisfied.*

Proof. Let $\gamma_{tph}(G) = 4$. If $|V(G)| = 4$, then G_4 or C_4. Suppose that $|V(G)| \geq 6$ and $S = \{x_1, x_2, x_3, x_4\}$ be a γ_{tph}-set. Suppose that $\langle\{x_1, x_2, x_3, x_4\}\rangle \ncong P_4$ or $\langle\{x_1, x_2, x_3, x_4\}\rangle \ncong P_2 \cup \overline{K_2}$. Then either $\langle S\rangle \cong [x_1, x_2] \cup [x_3, x_4]$ or $\langle S\rangle \cong [x_1, x_2, x_3] \cup K_1$

where $V(K_1) = \{x_4\}$ or $\langle S\rangle \cong \overline{K_4}$ where $V(\overline{K_4}) = \{x_1, x_2, x_3, x_4\}$. Thus, by Lemma 2.6, S is not a total perfect hop dominating set of G contrary to our assumption.

The converse follows immediately from Lemmas 2.4 and 2.5. □

Corollary 2.8. *Let n, s, and r be positive integers with* $r \geq 0$.

(i) $\gamma_{tph}(P_n) = 4r + 4$ *if* $n = 8r + s$; $4 \leq s \leq 8$

(ii) $\gamma_{tph}(C_n) = \begin{cases} 4, & \text{if } n = 4 \\ 4r+4, & \text{if } n = 8r+8. \end{cases}$

Definition 2.9. A set $S \subseteq V(G)$ is a *total perfect point-wise non-dominating set* of G if for every $v \in V(G)$, there is exactly one vertex $u \in S$ such that $v \notin N_G(u)$. The smallest cardinality of a total perfect point-wise non-dominating set of G, denoted by $tppnd(G)$ is called the *total perfect point-wise non-domination number* of G. Any total perfect point-wise non-dominating set S of G with $|S| = tppnd(G)$ is called a *tppnd*-set.

Remark 2.10. *Let G be a graph of order n. Then the total perfect point-wise non-dominating set of G does not exist if* $\gamma(H) = 1$.

Remark 2.11. *Let G be a graph of order* $n \geq 4$. *Then* $tppnd(G) \geq 2$.

Theorem 2.12. *Let G be a connected graph of order $n \geq 4$. Then $tppnd(G) = 2$ if only if there exist non-adjacent vertices $x, y \in V(G)$ such that $V(G)\backslash\{x, y\} = N_G(x) \cup N_G(y)$ and $N_G(y) \cap N_G(x) = \varnothing$.*

Proof. Suppose $tppnd(G) = 2$. Let $S = \{x, y\}$ be a *tppnd*-set of G. Let $z \in V(G)\backslash\{x, y\}$. Since S is a total perfect point-wise non-dominating set of G, $z \notin N_G(x)$ or $z \notin N_G(y)$ but not both. Hence, $N_G(x) \cup N_G(y)$ and $N_G(y) \cap N_G(x) = \varnothing$.

Conversely, suppose that there exist non-adjacent vertices $x, y \in V(G)$ satisfying the given condition. Let $S = \{x, y\}$ and let $u \in V(G)$. Then either $u \in N_G(x)\backslash N_G(y)$ or $u \in N_G(y)\backslash N_G(x)$. It follows that S is a total perfect point-wise non-dominating set of G. By Remark 2.11, $tppnd(G) = 2$. □

Corollary 2.13. *Let $n \geq 4$ be a positive integer.*

(i) $tppnd(P_n) = 2$ *if* $4 \leq n \leq 6$

(ii) $tppnd(C_n) = 2$ *if* $n = 4, 6$.

Remark 2.14. *Let G be a graph of order $n \geq 4$. If S is a tppnd-set of G, then $|S|$ is even.*

3. Join of Graphs

The *join* $G + H$ of two graphs G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Theorem 3.1. *Let G and H be graphs with $\Delta(G) \neq |V(G)| - 1$ and $\Delta(H) \neq |V(H)| - 1$. A subset S of $V(G + H)$ is a total perfect hop dominating set of $G + H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are total perfect point-wise non-dominating sets of G and H, respectively.*

Proof. Suppose that $S \subseteq V(G + H)$ is a total perfect hop dominating set of $G + H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. If $S_G = \varnothing$, then $S = S_H$. Since $V(G) \subseteq N_{G+H}(S)$, S is not a total perfect hop dominating set of $G + H$, a contradiction to our assumption. Thus, $S_G \neq \varnothing$. Similarly, $S_H \neq \varnothing$. Let $v \in V(G)$. Then there exists a unique vertex $y \in S$ such that $d_{G+H}(y, v) = 2$. So that $y \in S_G$ and $v \notin N_G(y)$. Hence, S_G is a total perfect point-wise non-dominating set of G. Similarly, S_H is a total perfect point-wise non-dominating set of H.

Conversely, suppose $S = S_G \cup S_H$, where S_G and S_H are total perfect point-wise non-dominating sets of G and H, respectively. Let $v \in V(G + H)$. If $v \in V(G)$, then there exists a unique vertex $z \in S_G$ such that $v \notin N_G(z)$. Hence, by definition of $G + H$, $d_{G+H}(z, v) = 2$. Similarly, if $v \in V(H)$, then there exists a unique vertex $z^* \in S_H$ such that $d_{G+H}(z^*, v) = 2$. Therefore, S is a total perfect hop dominating set of $G + H$. □

The next result follows immediately from Theorem 3.1

Corollary 3.2. *Let G and H be graphs with $\Delta(G) \neq |V(G)| - 1$ and $\Delta(H) \neq |V(H)| - 1$. Then, $\gamma_{tph}(G + H) = tppnd(G) + tppnd(H)$. In particular,*

(i) $\gamma_{tph}(P_n + P_m) = 4$ *if* $4 \leq m, n \leq 6$

(ii) $\gamma_{tph}(C_n + C_m) = 4$ *if* $n, m = 4, 6$.

4. Corona of Graphs

The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the *ith* vertex of G to every vertex in the *ith* copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle v \rangle + H^v = v + H^v$.

Definition 4.1. *A set $S \subseteq V(G)$ is a perfect total $(1, 2)^*$-dominating set of G if for every $w \in V(G)$, there is exactly one vertex $x \in S$ such that $wx \in E(G)$ and for every $u \in V(G) \setminus S$, there is exactly one vertex $v \in S$ such that $d_G(u, v) = 2$. The smallest cardinality of a perfect total $(1, 2)^*$-dominating set of G is called the perfect total $(1, 2)^*$-domination number of G, denoted by $\gamma^{*pt}_{1,2}(G)$. A perfect total $(1, 2)^*$-dominating set S of G with cardinality $\gamma^{*pt}_{1,2}(G)$ is called a $\gamma^{*pt}_{1,2}$-set of G.*

Theorem 4.2. *Let G be a connected nontrivial graph whose perfect total $(1,2)^*$-dominating set exists and H a graph with $\gamma(H)=1$. Then $G\circ H$ has a total perfect hop dominating set S if and only if $S = A \cup (\bigcup_{v\in V(G)} S_v)$ where $S_v \subseteq V(H^v)$ for every $v\in V(G)$ and the following conditions are satisfied.*

(i) *$A \subseteq V(G)$ is a perfect total $(1,2)^*$-dominating set of G.*

(ii) *For each $v \in V(G)\backslash A$, $S_u = \varnothing$ for all $u \in N_G(v)$.*

(iii) *For each $v \in A$, $N_G(v,2)\cap S = \varnothing$ and S_w is a γ-set of H for a unique $w \in V(G)\cap N_G(v)$.*

Proof. Suppose S is a total perfect hop dominating set of $G\circ H$ and $A = V(G)\cap S$. Then $A \subseteq V(G)$. Also, S is a perfect hop dominating set of $G\circ H$. Let $x \in V(G)$. If $x \notin A$, then $x \notin C$. Hence, there exists a unique vertex $v \in C$ such that $d_{G\circ H}(x,v) = 2$. We claim that $v\in A$. Suppose that $v \notin A$. Then there exists a vertex $w \in V(G)$ such that $v \in V(H^w)$ and $xw \in E(G)$. If $|V(G)| = 2$, then H is a trivial graph or v is an isolated vertex of H, which is a contradiction to the hypothesis. If $|V(G)| > 2$, then there exist vertices $a \in N_{H^w}(v)\setminus C$ and $b \in A$ such that $d_{G\circ H}(a,b) = 2$. Thus, $wb \in E(G)$ implying that $d_{G\circ H}(x,b) = 2$. This is a contradiction since C is a perfect hop dominating set of $G\circ H$ and $d_{G\circ H}(x,v) = 2 = d_{G\circ H}(x,b)$ where $v,b\in C$. Hence, $v\in A$. This implies that A is a perfect hop dominating set of G. We claim that A is a perfect total dominating set of G. Let $v\in V(G)$ and $a \in V(H^v)$ such that $deg_{H^v}(a) = |V(H)|-1$. Since S is a total perfect hop dominating set of $G\circ H$, a unique vertex $u \in N_G(v)\cap S$ exists. Thus, $u\in A$ implying that A is a perfect total dominating set of G. Hence, (i) holds. Let $v \in V(G)\backslash A$. By (i), there exists a unique vertex $w \in N_G(v,2)\cap A$. Suppose that $S_u \neq \varnothing$ for some $u \in N_G(v)$. Then there exists $a \in S_u$ and $a \in N_{G\circ H}(v,2)\cap S$, contrary to our assumption that S is a total perfect hop dominating set of $G\circ H$. Thus, $S_u = \varnothing$ and (ii) holds. For (iii), let $v\in A$. If $a \in N_G(v,2)\cap S$, then there exists $b \in N_G(v)\cap N_G(a)$. This implies that for all $x \in V(H^b)$, $x \in N_G(v,2)\cap N_G(a,2)$, a contradiction to our assumption for S. Thus, $N_G(v,2)\cap S = \varnothing$. Since S is a total perfect hop dominating set of $G\circ H$, there exists a unique vertex $u \in S\cap N_{G\circ H}(v,2)$. Since $N_G(v,2)\cap S = \varnothing$, $u \in V(H^w)\cap S = S_w$ for a unique $w \in V(G)\cap N_G(v)$. Since $\gamma(H) = 1$, S_w is γ-set of H.

Conversely, suppose that $S = A\cup(\bigcup_{v\in V(G)\backslash A} S_v)$ satisfying conditions (i),(ii) and (iii). Let $v \in V(G\circ H)$. Suppose that $v\in V(G)\backslash A$. Then by (i) and (ii), we are done. If $v\in A$, then by (iii) there exists a unique $w \in N_G(v)\cap V(G)$ such that S_w is a γ-set of H. Hence, there exists a vertex $a \in S_w \cap N_G(v,2)$. Suppose $v \in V(H^w)$ for $w\in V(G)$. By (i), there exists a unique vertex $u \in N_G(w)\cap A$. Hence, $u \in N_{G\circ H}(v,2)$. Therefore S is a total perfect hop dominating set of $G\circ H$. □

Corollary 4.3. *Let G be a connected graph of order $n \geq 4$ whose perfect total $(1,2)^*$-dominating set exists and H a graph with $\gamma(H)=1$. Then $\gamma_{tph}(G\circ H) \leq \gamma^{*pt}_{1,2}(G)+n$.*

Proof. Let $S = A \cup (\bigcup_{v \in V(G)} S_v)$ be a minimum total perfect hop dominating set of $G \circ H$. By Theorem 4.2, A is a $\gamma_{1,2}^{*pt}$-set of G and (ii) and (iii) hold. Then $\gamma_{tph}(G \circ H) = |C| = |A| + \sum_{v \in V(G)} |S_v| \leq |A| + V(G) = \gamma_{1,2}^{*pt}(G) + n$. □

The next result shows that the bound given in Corollary 4.3 is sharp.

Corollary 4.4. *Let H be a graph with $\gamma(H) = 1$. Then the total perfect hop dominating set of $P_2 \circ H$ exists and $\gamma_{tph}(P_2 \circ H) = 4$.*

Proof. Let $P_2 = [x_1, x_2]$. By Theorem 4.2, $S = \{x_1, x_2, a, b\}$, where $a \in V(H^{x_1})$, $b \in V(H^{x_2})$ and $deg_H(a) = deg_H(b) = |V(H)| - 1$ is a total perfect hop dominating set of $P_2 \circ H$. Thus, by Remark 2.2, $\gamma_{tph}(P_2 \circ H) = |S| = 4$. □

Remark 4.5. *The strict inequality in Corollary 4.3 can be attained.*

To illustrate Remark 4.5, consider the graph $P_4 \circ P_3$. It can be verified that $\gamma_{tph}(P_4 \circ P_3) = 4$. However, $\gamma_{1,2}^{*pt}(G) + |V(G)| = 2 + 4 = 6$. Hence, strict inequality is attained.

Corollary 4.6. *Let G be a connected graph of order 3 and H be a graph with $\gamma(H) = 1$. Then the total perfect hop dominating set of $G \circ H$ does not exist.*

Proof. If $|V(G)| = 3$, then $G \cong P_3$ or $G \cong K_3$. Hence, Theorem 4.2 is not satisfied. Therefore, the total perfect hop dominating set of $G \circ H$ does not exist. □

If G is a complete graph K_n, then G has no total perfect hop dominating set. Thus, the next result follows immediately from Theorem 4.2.

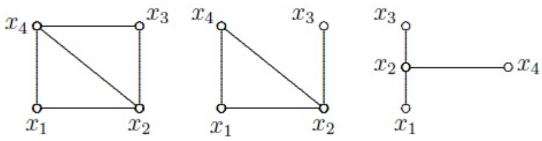
Corollary 4.7. *Let $n \geq 3$ and H a graph with $\gamma(H) = 1$. Then the total perfect hop dominating set of $K_n \circ H$ does not exist.*

Corollary 4.8. *Let H be a connected graph with $\gamma(H) = 1$. Then the total perfect hop dominating set of $C_n \circ H$ for $n \geq 3$ does not exist.*

Proof. Note that if $n \not\equiv 0 \pmod 4$, then C_n does not have a total perfect hop dominating set. If C_n has a total perfect hop dominating set A, then every vertex outside A hops in A and so none of the element in A hops in A. Thus, (iii) in Theorem 4.2 is never satisfied. □

Corollary 4.9. *Let G be a connected graph of order 4 and H be any graph with $\gamma(H) = 1$. Then the total perfect hop dominating set of $G \circ H$ exists if and only if $G \cong P_4$.*

Proof. If $G \cong P_4 = [x_1, x_2, x_3, x_4]$, then $S = \{x_2, x_3, a, b\}$ where $a \in V(H^{x_1})$, $b \in V(H^{x_4})$ and $deg_H(a) = deg_H(b) = |V(H)| - 1$ is a total perfect hop dominating set of $G \circ H$. If $G \cong C_4$ or $G \cong K_4$, then by Corollary 4.8 or Corollary 4.7, respectively, the total perfect hop dominating set of $G \circ H$ does not exist. Suppose $G \notin \{P_4, C_4, K_4\}$. Then G is isomorphic to one of the graphs shown in figure below. By Theorem 4.2 below, it can be verified that the total perfect hop dominating set of $G \circ H$ where G is one of the graphs shown below does not exist.



Therefore the corollary follows. □

Theorem 4.10. *Let H be a graph with $\gamma(H) = 1$. Then the total perfect hop dominating set of $P_n \circ H$ exists if and only if $n = 2$ and $n = 4$. Moreover, $\gamma_{tph}(P_n \circ H) = 4$.*

Proof. By Corollary 4.4 and Corollary 4.9, $P_2 \circ H$ and $P_4 \circ H$ both have total perfect hop dominating set. Let $P_n = [x_1, x_2, ..., x_n]$. Suppose $n \neq 2$ and $n \neq 4$. By Corollary 4.6, $P_3 \circ H$ does not exist. Let $n > 4$ and assume that $P_n \circ H$ has a total perfect hop dominating set S. By Theorem 4.2(i), $x_2 \in S$ and $x_1 \in S$ or $x_3 \in S$ but not both. Suppose $x_1 \in S$. Since $x_3 \notin S$, by Theorem 4.2(iii), $V(H^{x_1}) \cap S \neq \varnothing$. Let $y \in V(H^{x_2}) \cap S$. Then $y \in N_{P_n \circ H}(x_3, 2) \cap S$ and $x_1 \in N_{P_n \circ H}(x_3, 2) \cap S$, contrary to our assumption that S is a total perfect hop dominating set of $P_n \circ H$. Suppose $x_3 \in S$ and $x_1 \notin S$. By Theorem 4.2(i), $x_4 \notin S$. This implies that $e \notin S$ for all $e \in V(H^{x_3})$. Thus, $c \in S$ for a unique vertex $c \in V(H^{x_1})$ where $deg_H(c) = |V(H)| - 1$. Again by Theorem 4.2(i), $x_1 \notin S$ and $x_5 \notin S$. Hence, by Theorem 4.2(ii), $|V(H^{x_4} \cap S)| = 1$. Let $y \in V(H^{x_4}) \cap S$. Then $d_{P_n \circ H}(x_5, x_3) = d_{P_n \circ H}(x_5, y) = 2$, contrary to our assumption that S is a total perfect hop dominating set of $P_n \circ H$. Thus, the total perfect hop dominating set of $P_n \circ H$ for $n > 4$ does not exist. Therefore, the total perfect hop dominating set of $P_n \circ H$ exists if and only if $n = 2$ and $n = 4$. Clearly, $\gamma_{tph}(P_n \circ H) = 4$ for $n = 2$ and $n = 4$. □

Theorem 4.11. *Let G be a non-complete graph with $|V(G)| \geq 3$ and $\gamma(G) = 1$ and H a graph with $\gamma(H) = 1$. Then the total perfect hop dominating set of $G \circ H$ does not exist.*

Proof. Suppose that $G \circ H$ has a total perfect hop dominating set S. Let $y \in V(G)$ with $deg_G(y) = |V(G)| - 1$. By Theorem 4.2(i), there exists a unique vertex $x \in V(G) \cap S$. If $deg_G(z) = |V(H)| - 1$, $y \in S$. If there exists a unique vertex $z \in N_G(x, 2) \cap S$, then $d_{G \circ H}(a, z) = d_{G \circ H}(a, x) = 2$ for $a \in V(H^y)$. If there exists a unique $a \in V(H^y) \cap S$. Then $d_{G \circ H}(z, a) = d_{G \circ H}(z, x) = 2$ where $z \in V(G) \backslash \{x\}$. Suppose $deg_G(x) \geq 2$. Let $u, v \in N_G(x)$ with $u \neq v$. By Theorem 4.2(i), there exists a unique vertex $z \in V(G) \cap N_G(x) \cap S$. If $z = y = u \neq v$, then $d_{G \circ H}(b, y) = d_{G \circ H}(b, x) = 2$ for all $b \in V(H^v)$. If $z = v \neq y$, then $d_{G \circ H}(b, v) = d_{G \circ H}(b, x) = 2$ for all $b \in V(H^y)$. This implies that S is not a total perfect hop dominating set of $G \circ H$. Therefore, the total perfect hop dominating set of $G \circ H$ does not exist. □

5. Lexicographic Product

The *lexicographic product* of two graphs G and H, denoted by $G[H]$, is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1 v_1 \in E(G)$ or $u_1 = v_1$ and $u_2 v_2 \in E(H)$.

Theorem 5.1. *Let G be a nontrivial complete graph and H a nontrivial connected non-complete graph whose total perfect point-wise non-dominating set exists. A subset $C = \bigcup_{x\in S}[\{x\}\times T_x]$ of $V(G[H])$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a total perfect hop dominating set of $G[H]$ if and only if $S = V(G)$ and T_x is a total perfect point-wise non-dominating set of H for each $x \in S$.*

Proof. Let $C = \bigcup_{x\in S}[\{x\} \times T_x]$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$ be a total perfect hop dominating set of $G[H]$. Then C is a perfect hop dominating set of $G[H]$. Suppose $S \neq V(G)$. Let $u \in V(G)\backslash S$. Then $(u, a) \notin C$ for any $a \in V(H)$. Thus, there exists a unique vertex $(y, b) \in C$ such that $d_{G[H]}((u, a),(y, b)) = 2$. Since $u \notin S$ and $y \in S$, $u \neq y$ and $d_G(u, y) = 2$. This implies that $(y, p) \notin C$ for all $p \in V(H)\backslash\{b\}$. Since $\gamma(H) \neq 1$, choose $q \in V(H)\backslash\{b\}$ such that $q \notin N_H(b)$. Then $d_{G[H]}((y, q),(y, b)) = 2$. Pick any $t \in N_H(b)$. Then there exists $z \in S\backslash\{y\}$ such that $d_G(y, z) = 2$. Let $r \in T_z$. Then $d_{G[H]}((y, q),(z, r)) = 2$, a contradiction to the fact that C is a perfect hop dominating set of $G[H]$. Therefore $S = V(G)$. Let $x \in S$. Suppose that $N_G(x, 2) \neq \varnothing$ and $T_x \neq V(H)$. Let $z \in N_G(x, 2)$, $p \in T_z$ and $a \in V(H)\backslash T_x$. Since $(x, a) \notin C$, there is exactly one vertex $(y, b) \in C$ such that $d_{G[H]}((x, a),(y, b)) = 2$. This implies that $x = y$ and $ab \notin E(H)$ or $d_G(x, y) = 2$. Suppose $x = y$ and $ab \notin E(G)$. Then $d_{G[H]}((x, a),(y, b)) = d_{G[H]}((x, a),(z, p)) = 2$ contrary to our assumption that C is a perfect hop dominating set of $G[H]$. On the other hand, suppose that $d_G(x, y) = 2$. If $y \neq z$, then $d_{G[H]}((x, a),(y, b)) = d_{G[H]}((x, a),(z, p)) = 2$. If $y = z$, then $b = p$. Since $\gamma(H) \neq 1$, there exists $q \in V(H)\backslash N_H[p]$. Let $w \in T_x$. Then $d_{G[H]}((z, q),(z, p)) = d_{G[H]}((z, q),(x, w)) = 2$. Since $(z, q) \notin C$ because $|T_x| = 1$, it follows that C is not a perfect hop dominating set of $G[H]$ a contradiction to our assumption for C. Therefore $T_x = V(H)$. Now, let $N_G(x, 2) = \varnothing$ and $a \in V(H)\backslash T_x$. Then $(x, a) \notin C$ and it follows that there is a unique vertex $(y, b) \in C$ such that $d_{G[H]}((x, a),(y, b)) = 2$. Since $N_G(x, 2) = \varnothing$, $x = y$ and $ab \notin E(H)$. This implies that T_x is a perfect point-wise non-dominating set of H. Therefore T_x is a perfect point-wise non-dominating set of H for all $x \in S$. We claim that T_x is a total perfect point-wise non-dominating set of H for all $x \in S$. Let $x \in S$ and $c \in T_x$. Then $(x, c) \in C$. Since C is a total perfect hop dominating set of $G[H]$, there is a unique vertex $(y, d) \in C$ such that $d_{G[H]}((x, c),(y, d)) = 2$. Since G is complete, $x = y$ and $cd \notin E(G)$ This implies that $d \in T_x$ and $cd \notin E(H)$. Therefore, T_x is a total perfect point-wise non-dominating set of H.

Conversely, let $S = V(G)$ and T_x be a total perfect point-wise non-dominating set of H for all $x \in S$. Since every total perfect point-wise non-dominating set is a perfect point-wise non-dominating set, T_x is a perfect point-wise non-dominating set of H for all $x \in S$. Let $(x, a) \notin C$. Since $S = V(G)$, $a \notin T_x$. If $N_G(x, 2) \neq \varnothing$, then we are done since $T_x = V(H)$. If $N_G(x, 2) = \varnothing$, then there exists a unique vertex $b \in T_x$ such that $ab \notin E(H)$. Thus, $(x, b) \in C$ and $d_{G[H]}((x, a),(x, b)) = 2$. Accordingly, C is a perfect hop dominating set of $G[H]$. Let $(x, a) \in C$. Then $x \in S$ and $a \in T_x$. Since T_x is a total perfect point-wise non-dominating set of H, there is a unique vertex $b \in T_x$ such that $ab \notin E(H)$. Since G is a nontrivial complete graph, there exists $y \in V(G) \cap N_G(x)$. Thus,

$$d_{G[H]}((x,a),(x,b)) = d_{G[H]}((x,a),(y,a)) + d_{G[H]}((y,a),(x,b)) = 1 + 1 = 2.$$

Since b is unique, (x,b) is a unique vertex in C. Therefore C is a total perfect hop dominating set of $G[H]$. □

Corollary 5.2. *Let G be a nontrivial complete graph and H a nontrivial connected non-complete graph whose total perfect point-wise non-dominating set exists. Then $\gamma_{tph}(G[H]) = |V(G)| \cdot tppnd(H)$.*

Proof. Let $C = \bigcup_{x\in S}[\{x\} \times T_x]$ be a minimum total perfect hop dominating set of $G[H]$. By Theorem 5.1, $S = V(G)$ and T_x is a minimum total perfect point-wise non-dominating set of H for all $x \in S$. Therefore $\gamma_{tph}(G[H]) = |C| = \sum_{x\in V(G)} |T_x| = |V(G)| \cdot tppnd(H)$ □

Theorem 5.3. *Let G be a nontrivial connected graph whose total perfect hop dominating set exists and H a nontrivial connected graph with $\gamma(H) = 1$. Then a nonempty subset $C = \bigcup_{x\in S}[\{x\} \times T_x]$ of $V(G[H])$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$, is a total perfect hop dominating set of $G[H]$ if and only if S is a total perfect hop dominating set of G and T_x is a γ-set of H.*

Proof. Let $C = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$, be a total perfect hop dominating set of $G[H]$. Then C is a perfect hop dominating set of $G[H]$. We claim that S is a total perfect hop dominating set of G. Let $u \in V(G)$. If $u \notin S$, then $(u,a) \notin C$ for any $a \in V(H)$. Thus, there is exactly one vertex $(v,b) \in C$ such that $d_{G[H]}((u,a),(v,b)) = 2$. Since $u \notin S$ and $v \in S$, $u \neq v$ and $d_G(u,v) = 2$. Suppose $u \in S$. Since G has a total perfect hop dominating set, $N_G(u,2) \neq \varnothing$. Let $z \in N_G(u,2)$. If $z \in S$, then we are done. So suppose that $z \notin S$. Then $|T_u| = 1$, say $T_u = \{p\}$ for some $p \in V(H)$ because C is a perfect hop dominating set of $G[H]$. Let $a \in N_H(p)$. Then there exists a unique $(w,b) \in C \cap N_{G[H]}((u,a),2)$. Since $b \neq p$, $u \neq w$. Thus, $w \in S \cap N_G(u,2)$. Hence, $N_G(u,2) \cap S \neq \varnothing$. Therefore S is a total perfect hop dominating set of G. Now, let $x \in S$. Since S is a total perfect hop dominating set of G, $|T_x| = 1$, say $T_x = \{a\}$. Let $p \in V(H)\backslash T_x$. Suppose $p \notin N_H(a)$. Then $d_{G[H]}((x,p),(x,a)) = 2$. Since S is a total perfect hop dominating set of G, there exists a unique $y \in N_G(x,2) \cap S$. Pick any $c \in T_y$. Then $(y,c) \neq (x,a)$ but $d_{G[H]}((x,p),(y,c)) = 2$. This implies that C is not a perfect hop dominating set of $G[H]$, a contradiction. Therefore, T_x is a γ-set of H.

Conversely, let S be a total perfect hop dominating set of G and T_x is a γ-set of H for every $x \in S$. Let $(x,a) \notin C$. Then either $x \notin S$ or $x \in S$ and $a \notin T_x$. If $x \notin S$, then a unique vertex $y \in S$ exists such that $d_G(x,y) = 2$. Since T_y is a γ-set of H for every $y \in S$, a unique vertex $b \in T_y$ exists such that for all $p \in V(H)\backslash\{b\}$, $p \in N_H(b)$. Then $(y,b) \in C$ and $d_{G[H]}((x,a),(y,b)) = 2$. Suppose $x \in S$ and $a \notin T_x$. Then there is exactly one vertex $z \in S$ such that $d_G(x,z) = 2$. Since T_z is a γ-set of H for every $z \in S$, a unique vertex $c \in T_z$ exists. Hence, $(z,c) \in C$ and $d_{G[H]}((x,a),(z,c)) = 2$. Therefore C is a perfect hop dominating set of $G[H]$. Let $(x,a) \in C$. Since $x \in S$ and S

is a total perfect hop dominating set of G, there exists a unique vertex $y \in S$ such that $d_G(x, y) = 2$. Since T_y is a γ-set of H and $\gamma(H) = 1$, there exists $b \in T_y$. Hence, $(y, b) \in C$ and $d_{G[H]}((x, a), (y, b)) = 2$. Therefore C is a total perfect hop dominating set of $G[H]$. $\square$

Corollary 5.4. *Let G be a nontrivial connected graph whose total perfect hop dominating set exists and H a nontrivial connected graphs with $\gamma(H) = 1$. Then $\gamma_{tph}(G[H]) = \gamma_{tph}(G)$.*

Proof. Let $C = \bigcup_{x \in S} [\{x\} \times T_x]$ be a minimum connected perfect hop dominating set of $G[H]$. Then by Theorem 5.3, S is a minimum total perfect hop dominating set of G and $T_x = \{a\}$ where $a \in V(H)$ such that $deg_H(a) = |V(H)| - 1$. Therefore $\gamma_{tph}(G[H]) = |C| = |S| = \gamma_{tph}(G)$. $\square$

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