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Existence and uniqueness of solutions for the nonlinear fractional differential equations with two-point and integral boundary conditions

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Abstract. In this paper the existence and uniqueness of solutions to the fractional differential equations with two-point and integral boundary conditions is investigated. The Green function is constructed, and the problem under consideration is reduced to the equivalent integral equation. Existence and uniqueness of a solution to this problem is analyzed using the Banach the contraction mapping principle and Krasnoselskiis fixed point theorem.

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1. Introduction

In recent years, the theory of the fractional differential equations has played a very important role in a new branch of applied mathematics, which has been utilized for mathematical models in engineering, physics, chemistry, signal analysis, etc. For details and applications, we refer the reader to the classical reference texts such as [1-6]. Fractional differential equations are considered as a valuable tool to model many real world problems. Boundary value problems for such differential equations represent an important class of applied analysis. Most of the studied fractional differential equations by taking Caputo or RiemannLiouville derivatives. Engineers and scientists have developed some new models that involve fractional differential equations for which the Riemann Liouville derivative is not considered appropriate. Therefore, certain modifications were introduced to avoid the difficulties and some new types of fractional order derivative operators were introduced in the literature by authors like Caputo, Hadamard, and Erdely Kober, etc.

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Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems presenting both theoretical and practical importance. They include two, three, multipoint and nonlocal boundary value problems as special cases. Integral boundary value problems occur in the mathematical modeling of variety of physics processes and have recently received considerable attention. For some recent works on the boundary value problems with integral boundary conditions we refer to [7-15, 17-19, 21-25] and the references cited therein.

In this paper, we study existence and uniqueness of nonlinear fractional differential equations of the type

$$^{c}D_{0+}^{\alpha}x\left(t\right)=f\left(t,x\left(t\right)\right),\ \ for\ t\in\left[0,T\right], \tag{1}$$

subject to two-point and integral boundary conditions

$$Ax\left(0\right)+\int\limits_{0}^{T}n\left(t\right)x\left(t\right)dt+Bx\left(T\right)=C, \tag{2}$$

where $0<\alpha<1, {}^{c}D_{0+}^{\alpha}$ is the Caputo fractional derivatives, $A,B\in R^{n\times n}$ and $n\left(t\right):\left[0,T\right]\rightarrow R^{n\times n}$ are given matrices and $\det\left(A+\int\limits_{0}^{T}n\left(t\right)dt+B\right)\neq 0$.

The paper is organized as follows. In Section 2, we give some notations, recall some concepts, and introduce a concept of a continuous solution for our problem. In Section 3, we give two main results: the first result based on the Banach contraction principle and the second result based on the Krasnoselskiis fixed point theorem.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. By $C\left(\left[0,T\right];R^{n}\right)$ we denote the Banach space of all continuous functions from $[0,T]$ into R^{n} with the norm $\|x\|=\max\left\{\left|x\left(t\right)\right|:t\in\left[0,T\right]\right\}$, where $\left|\cdot\right|$ norm in R^{n} .

Definition 1. *The RiemannLiouville fractional integral of order $\alpha>0$ of a continuous function $y:[0,\infty)\rightarrow R$, is defined by*

$$\left(J^{\alpha}y\right)\left(x\right)=\frac{1}{\Gamma\left(\alpha\right)}\int\limits_{0}^{x}\left(x-s\right)^{\alpha-1}y\left(s\right)ds,$$

provided the right-hand side exists on $\left(0,\infty\right)$, where $\Gamma(\cdot)$ is the Gamma function defined for any complex number z as

$$\Gamma\left(z\right)=\int\limits_{0}^{\infty}t^{z-1}e^{-t}dt.$$

Definition 2. *The (right-sided) Riemann-Liouville fractional derivative is defined by*

$$^{RL}D^{\alpha}y = (D^{\alpha}y)(x) = \frac{d}{dx^n}\left(J^{n-\alpha}y\right)(x), x > 0,$$

where $n = [\alpha] + 1, [\alpha]$ *, denotes the integer part of the real number* α*, provided the right-hand side is point-wise defined on* $(0, \infty)$*.*

The Riemann-Liouville fractional derivative is left-inverse (but not right-inverse) of the Riemann-Liouville fractional integral, which is a natural generalization of the Cauchy formula for the n-fold primitive of a function y. As to the initial value problems for fractional differential equations with fractional derivatives in the Riemann-Liouville sense, they should be given as (bounded) initial values of the fractional integral $J^{n-\alpha}$ and of its integer derivatives of order $k = 1, 2, ..., n-1$.

Definition 3. *The Caputo fractional derivative of order* $\alpha > 0$ *of a continuous function* y*, is defined by*

$$^{C}D^{\alpha}_{0+}y = (D^{\alpha}y)(x) = \left(J^{n-\alpha}y^{(n)}\right)(x), n-1 < \alpha \leq n, x > 0,$$

provided the right-hand side is point-wise defined on (a, ∞)*.*

Obviously, this definition allows one to consider the initial-value problems for the fractional differential equations with initial conditions that are expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order.

Remark 1. *[20] Under natural conditions on* $y(x)$*, the Caputo fractional derivative becomes the conventional integer order derivative of the function* $y(x)$ *as* $\alpha \to n$*.*

Remark 2. *[20] Let* $\alpha, \beta > 0$ *and* $n = [\alpha] + 1$*; then the following relations hold:*

$$^{c}D^{\alpha}_{0+}t^{\beta} = \tfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-\alpha}, \beta > n,$$
$$^{c}D^{\alpha}_{0+}t^{k} = 0, k = 0, 1, ..., n-1.$$

Lemma 1. *[20] For* $\alpha > 0, y(t) \in C([0,T]) \cap L_1([0,T])$ *the homogeneous fractional differential equation*

$$^{c}D^{\alpha}_{0+}y(t) = 0,$$

has a solution

$$y(t) = c_0 + c_1 t + c_2 t^2 ... + c_{n-1}t^{n-1},$$

where $c_i \in R, i = 1, 2, ..., n-1$ *and* $n = [\alpha] + 1$*.*

Lemma 2. *[20] Assume that* $y(t) \in C([0,T]) \cap L_1([0,T])$*, with derivative of order* n *that belongs to* $C([0,T]) \cap L_1([0,T])$*. Then*

$$I^{\alpha}_{0+}\,^{c}D^{\alpha}_{0+}y(t) = y(t) + c_0 + c_1 t + c_2 t^2 ... + c_{n-1}t^{n-1},$$

where $c_i \in R, i = 1, 2, ..., n-1$ *and* $n = [\alpha] + 1$*.*

Lemma 3. *[20] Let $p, q \geq 0, f \in L_1([0,T])$. Then*

$$I_{0+}^{p} I_{0+}^{q} f(t) = I_{0+}^{p+q} f(t) = I_{0+}^{q} I_{0+}^{p} f(t)$$

is satisfied almost everywhere on $[0,T]$. Moreover, if $f \in C([0,T])$, then (14) is true for all $t \in [0,T]$.

Lemma 4. *[20] If $\alpha > 0, f \in C([0,T])$, then ${}^{c}D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t)$ for all $t \in [0,T]$.*

We have the following result which is useful in what follows.

Theorem 1. *Let $y \in C([0,T]; R^n)$. Then the unique solution of the linear boundary value problem*

$$\begin{cases} {}^{c}D_{0+}^{\alpha} x(t) = y(t), \\ Ax(0) + \int\limits_0^T n(t)x(t)dt + Bx(T) = C, \end{cases} \tag{3}$$

is given by

$$x(t) = N^{-1}C + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} y(s)\, ds$$

$$-\frac{N^{-1}}{\Gamma(\alpha)} \int\limits_0^T n(t) \int\limits_0^t (t-s)^{\alpha-1} y(s)\, ds dt - \frac{N^{-1}B}{\Gamma(\alpha)} \int\limits_0^T (T-s)^{\alpha-1} y(s)\, ds, \tag{4}$$

where

$$N = \left(A + \int\limits_0^T n(t)dt + B \right).$$

Proof. Assume that x is a solution of boundary value problem (3). Then we have

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} y(s)\, ds, \ t \in [0,T],$$

where $x(0)$ is still an arbitrary constant vector.

For determining $x(0)$ we use the boundary value condition $Ax(0) + \int\limits_0^T n(t)\, x(t)\, dt$ $+Bx(T) = C$:

$$C = Ax(0) + \int\limits_0^T n(t)\, x(t)\, dt + Bx(T) = \left(A + \int\limits_0^T n(t)\, dt + B \right) x(0)$$

$$+\frac{1}{\Gamma(\alpha)} \int\limits_0^T n(t) \int\limits_0^t (t-s)^{\alpha-1} y(s)\, ds dt + \frac{B}{\Gamma(\alpha)} \int\limits_0^T (T-s)^{\alpha-1} y(s)\, ds.$$

From here we get

$$x(0) = N^{-1}C - \frac{N^{-1}}{\Gamma(\alpha)}\int_0^T n(t)\int_0^t (t-s)^{\alpha-1}y(s)\,dsdt + \frac{N^{-1}B}{\Gamma(\alpha)}\int_0^T (T-s)\,y(s)\,ds,$$

and consequently for all $t \in [0,T]$ (4) is true.

Lemma 5. *(Krasnoselskiis fixed point theorem, [16]) Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that*

(a) $Ax + By \in M$ whenever $x, y \in M$;

(b)A is compact and continuous;

(c) B is a contraction mapping.

Then there exists $z \in M$ such that $z = Az + Bz$.

3. Main results

In this section, the theorems on uniqueness and existence of a solution for boundary value problem (1)-(2) is given. For the forthcoming analysis we impose suitable conditions on the functions involved in boundary value problem (1), (2). We assume the following conditions are set:

(H1) The function $f : [0,T] \times R^n \to R^n$ is continuous and satisfies the following Lipschitz condition

$$\|f(t,x) - f(t,y)\| \le L\|x-y\|, x, y \in R^n, t \in [0,T], L > 0.$$

(H2)

$$\|f(t,x)\| \le G, \text{ for all } x \in R^n, t \in [0,T], G \ge 0.$$

Theorem 2. *Assume that $f : [0,T] \times R^n \to R^n$ is jointly continuous and satisfies (H1) and (H2). If*

$$\left[\frac{L\|N^{-1}\|\,\|n\|\,T^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{L\|N^{-1}B\|\,T^{\alpha}}{\Gamma(\alpha+1)}\right] < 1, \tag{5}$$

then fractional differential equation (1) with boundary conditions (2) has at least one solution on $[0,T]$.

Proof. Consider $B_r = \{x \in C([0,T]; R^n) : \|x\| \le r\}$, where

$$r \ge \frac{GT^\alpha}{\Gamma(\alpha+1)} + \|N^{-1}C\| + \frac{G\|n\|\,\|N^{-1}\|\,T^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{G\|N^{-1}B\|\,T^{\alpha}}{\Gamma(\alpha+1)}.$$

Define two mappings A_1 and A_2 on B_r by

$$(A_1x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s))\,ds, \tag{6}$$

$$(A_2x)(t) = N^{-1}\left(C - \frac{1}{\Gamma\alpha)}\int\limits_0^T n(t)\int\limits_0^t (t-s)^{\alpha-1} f(s,x(s))\,dsdt\right.$$

$$\left.-\frac{B}{\Gamma(\alpha)}\int\limits_0^T (T-s)^{\alpha-1} f(s,x(s))ds\right). \tag{7}$$

For $x, y \in B_r$ by (H2), we obtain

$$\|(A_1x)(t) + (A_2y)(t)\| \le \frac{G}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\,ds + \left\|N^{-1}C\right\|$$

$$+\frac{G\|n\|\left\|N^{-1}\right\|}{\Gamma}(\alpha)\int\limits_0^T\int\limits_0^t (t-s)^{\alpha-1}\,dsdt + \frac{\left\|N^{-1}B\right\|G}{\Gamma(\alpha)}\int\limits_0^T (T-s)^{\alpha-1}\,ds$$

$$\le \frac{GT^\alpha}{\Gamma(\alpha+1)} + \left\|N^{-1}C\right\| + \frac{G\|n\|\left\|N^{-1}\right\|T^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{G\left\|N^{-1}B\right\|T^\alpha}{\Gamma(\alpha+1)} \le r.$$

This shows that $A_1x + A_2y \in B_r$. Therefore, condition (a) of Lemma 5 holds. It is claimed that A_1 is compact and continuous. Continuity of f implies that the operator (6) is continuous. $(A_1x)(t)$ is uniformly bounded on B_r as

$$\|A_1x\| \le \frac{GT^\alpha}{\Gamma(\alpha+1)}.$$

Since f is bounded on the compact set $[0,T]\times B_r$, let $\sup\limits_{[0T]\times B_r}\|f(t,x)\| = M_f$. Then, for $t_1, t_2 \in [0,T], t_1 < t_2$ we get

$$\|(A_1x)(t_2) - (A_1x)(t_1)\|$$

$$= \frac{1}{\Gamma(\alpha)}\left\|\int\limits_0^{t_1}\left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right)f(s,x(s))\,ds + \int\limits_{t_1}^{t_2}(t_2-s)^{\alpha-1} f(s,x(s))\,ds\right\|$$

$$\le \frac{M_f}{\Gamma(\alpha)}\left(\frac{t_2^\alpha}{\alpha} - \frac{t_1^\alpha}{\alpha}\right),$$

which is independent of x and tends to zero as $t_2 \to t_1$. Therefore, A_1 is relatively compact on B_r. By Arzela Ascolis Theorem, A_1 is compact on B_r.

For $x, y \in B_r$ and $t \in [0,T]$, by (H1), we have

$$\|(A_2x)(t) - (A_2y)(t)\|$$

$$\le \frac{1}{\Gamma(\alpha)}\left\|N^{-1}\int\limits_0^T n(t)\int\limits_0^t (t-s)^{\alpha-1}(f(s,x(s)) - f(s,y(s)))\,dsdt\right\|$$

$$+\frac{1}{\Gamma(\alpha)}\left\|N^{-1}B\int_{0}^{T}(T-s)^{\alpha-1}\left(f\left(s,x\left(s\right)\right)-f\left(s,y\left(s\right)\right)\right)ds\right\|$$

$$\leq\left[\frac{L\left\|N^{-1}\right\|\|n\|\,T^{\alpha+1}}{\Gamma(\alpha+1)}+\frac{L\left\|N^{-1}B\right\|T^{\alpha}}{\Gamma(\alpha+1)}\right]\|x-y\|$$

It follows from (5) that the operator (7) is a contraction mapping. Thus, by Krasnoselskiis fixed point theorem, (1) -(2) has at least one solution.

Theorem 3. *Assume that $f:[0,T]\times R^n\to R^n$ is a continuous function satisfying the assumption (H1). Then the problems (1)- (2) has a unique solution on $[0,T]$ if*

$$L\Lambda<1,$$

where Λ is given by

$$\Lambda=\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{\left\|N^{-1}\right\|\|n\|\,T^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{\left\|N^{-1}B\right\|T^{\alpha}}{\Gamma(\alpha+1)}.$$

Proof. Define a mapping $F:C\left([0,T];R^n\right)\to\left([0,T];R^n\right)$ by

$$(Fx)(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}f\left(s,x\left(s\right)\right)ds$$

$$+N^{-1}\left(C-\frac{1}{\Gamma(\alpha)}\int_{0}^{T}n(t)\int_{0}^{t}(t-s)^{\alpha-1}f\left(s,x\left(s\right)\right)dsdt\right.$$

$$\left.-\frac{B}{\Gamma(\alpha)}\int_{0}^{T}(T-s)^{\alpha-1}f\left(s,x\left(s\right)\right)ds\right). \tag{8}$$

Let us first show that $FB_r\subset B_r$, where is the operator defined by (8) and $r\geq\frac{M_fL}{1-L\Lambda}$ with $M_f=\sup\limits_{t\in[0,T]}|f(t,0)|$, $\Lambda=\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{\left\|N^{-1}\right\|\|n\|T^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{\left\|N^{-1}B\right\|T^{\alpha}}{\Gamma(\alpha+1)}$. Then, in view of the assumptions (H1) and (H2), we have

$$|f(t,x)|\leq|f(t,x)-f(t,0)|+|f(t,0)|\leq L|x|+M_f\leq Lr+M_f.$$

For any $x\in B_r$, we have

$$\|Fx\|=\sup_{t\in[0,T]}|Fx(t)|\leq\left\|N^{-1}C\right\|$$

$$+\left(Lr+M_{f}\right)\left\{\frac{T^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{\left\|N^{-1}\right\|\left\|n\right\|T^{\alpha+1}}{\Gamma\left(\alpha+2\right)}+\frac{\left\|N^{-1}B\right\|T^{\alpha}}{\Gamma\left(\alpha+1\right)}\right\}\leq r,$$

which implies that $FB_r \subset B_r$. Next, for $x, y \in C\left([0,T]\,;R^n\right)$ and for each $t \in [0,T]$, we obtain

$$\left\|Fx-Fy\right\|\leq\sup_{[0,T]}\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-s\right)^{\alpha-1}\left|f\left(s,x\left(s\right)\right)-f\left(s,y\left(s\right)\right)\right|ds$$

$$+\frac{\left\|N^{-1}\right\|\left\|n\right\|}{\Gamma\left(\alpha\right)}\int_{0}^{T}\int_{0}^{t}\left(t-s\right)^{\alpha-1}\left|f\left(s,x\left(s\right)\right)-f\left(s,y\left(s\right)\right)\right|ds$$

$$+\frac{\left\|N^{-1}B\right\|}{\Gamma\left(\alpha\right)}\int_{0}^{T}\left(T-s\right)^{\alpha-1}\left|f\left(s,x\left(s\right)\right)-f\left(s,y\left(s\right)\right)\right|ds$$

$$\leq L\left\{\frac{T^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{\left\|N^{-1}\right\|\left\|n\right\|T^{\alpha+1}}{\Gamma\left(\alpha+2\right)}+\frac{\left\|N^{-1}B\right\|T^{\alpha}}{\Gamma\left(\alpha+1\right)}\right\}\left\|x-y\right\|=L\Lambda\left\|x-y\right\|.$$

Since $L\Lambda < 1$ the operator F is a contraction. By Banach contraction mapping principle the operator F has a unique fixed point, which means that the problem (1) and (2) has a unique solution for on $[0,T]$.

References

[1] R.P. Agarwal, M.Benchohra, S.Hamani. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Applicandae Mathematicae,* 109(3): 973-1033, 2010.

[2] R.P. Agarwal, M. Benchohra, S.Hamani. Boundary value problems for fractional differential equations. *Georgian Mathematical Journal,* 16(3): 401-411, 2009.

[3] B. Ahmad, S Ntouyas, A. Alsaedi. *Fractional order differential systems involving right Caputo and left RiemannLiouville fractional derivatives with nonlocal coupled conditions.* Bound. Value Probl. 2019, Article ID 109 (2019)

[4] B. Ahmad and J. J. Nieto. *Existence results for nonlinear boundary value problems of fractional integro-differential equations with integral boundary conditions,* Boundary Value Problems, vol. 2009, Article ID 708576, 11 pages, 2009.

[5] J. Allison and N. Kosmatov. Multi-point boundary value problems of fractional order, *Communications in Applied Analysis,* 12: 451458, 2008.

[6] D. Araya and C. Lizama, Almost automorphic mild solutions to fractional differential equations, *Nonlinear Analysis: Theory, Methods & Applications,* 69(11): 36923705, 2008.

[7] A. Ashyralyev, Y. A. Sharifov. Existence and uniqueness of solutions for the system of nonlinear fractional differential equations with nonlocal and integral boundary conditions. *Abstract and Applied Analysis* Vol. 2012. Id 594802, 2012.

[8] B. Bonilla, M. Rivero, L. Rodrguez-Germ, and J. J. Trujillo. Fractional differential equations as alternative models to nonlinear differential equations, *Applied Mathematics and Computation,* 187(1): 7988, 2007.

[9] A. Cabada, Z. Hamdi. Nonlinear fractional differential equations with integral boundary value conditions. *Appl. Math. Comput.* 228: 251257, 2014.

[10] Y.-K. Chang and J. J. Nieto. Some new existence results for fractional differential inclusions with boundary conditions, *Mathematical and Computer Modelling,* 49(3-4): 605609, 2009.

[11] V. Daftardar-Gejji and S. Bhalekar. Boundary value problems for multi-term fractional differential equations, *Journal of Mathematical Analysis and Applications,* 345(2): 754765, 2008.

[12] P. W. Eloe and B. Ahmad. Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions, *Applied Mathematics Letters,* 18(5): 521527, 2005.

[13] V. Gafiychuk, B. Datsko, and V. Meleshko. Mathematical modeling of time fractional reaction-diffusion systems, *Journal of Computational and Applied Mathematics,* 220(1-2): 215225, 2008.

[14] J.R Graef, L. Kong Q. Kong, M. Wang. Fractional boundary value problems with integral boundary conditions. *Appl. Anal.* 92: 20082020 (2013)

[15] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations, vol. 204 of North-Holland Mathematics Studies,* Elsevier Science, Amsterdam, The Netherlands, 2006.

[16] M.A. Krasnoselskii, Two remarks on the method of successive approximations. *Usp. Mat. Nauk* 10, 123127 (1955)

[17] M.J. Mardanov, Y.A. Sharifov, K.E. Ismayilova, S.A.Zamanova. Existence and Uniqueness of Solutions for the System of First-order Nonlinear Differential Equations with Three-point and Integral Boundary Conditions, *European journal of pure and applied mathematics* 12(3): 2019, 756-770

[18] M.J. Mardanov, Y.A. Sharifov, H.N. Aliyev, R.A. Sardarova. Existence and Uniqueness of Solutions for the First Order Non-linear Differential Equations with Multi-Point Boundary Conditions, *European journal of pure and applied mathematics* 13(3): 2020, 414-426.

[19] M.J. Mardanov , Y.A.Sharifov, Y.S.Gasimov , C.Cattani. Non-Linear First-Order Differential Boundary Problems with Multipoint and Integral Conditions. *Fractal and Fractional,* 5(1):1-13, 2021.

[20] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives:* Theory and Applications, Gordon and Breach Science, Yverdon, Switzerland, 1993.

[21] Y.A. Sharifov Existence and uniqueness of solutions for the system of nonlinear fractional differential equations with nonlocal boundary conditions. *Proceedings of IMM of NAS of Azerbaijan,* XXXVI (XLIV): 125-134, 2012.

[22] Y.A. Sharifov, F.M. Zeynally, S.M. Zeynally, Existence and uniqueness of solutions for nonlinear fractional differential equations with two-point boundary conditions, *Advanced Mathematical Models Applications* 3(1):54-62, 2018.

[23] X. Zhang, L. Liu, Y. Wu, Y Zou. Existence and uniqueness of solutions for systems of fractional differential equations with Riemann-Stieltjes integral boundary condition. *Adv. Differ. Equ.* 2018, Article ID 204 (2018).

[24] Y. Zou, G. He, The existence of solutions to integral boundary value problems of fractional differential equations at resonance. *J. Funct. Spaces* 2017, Article ID 2785937 (2017).

[25] S.S. Yusubov, Boundary value problems for hyperbolic equations with a Caputo fractional derivative. *Advanced Mathematical Models, Applications,* 5(2): 192-204, 2020.