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An application of finite groups to Hopf algebras

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Abstract. Kaplansky's famous conjectures about generalizing results from groups to Hopf algebras inspired many mathematicians to try to find solusions for them. Recently, Cohen and Westreich in [8] and [10] have generalized the concepts of nilpotency and solvability of groups to Hopf algebras under certain conditions and proved interesting results. In this article, we follow their work and give a detailed example by considering a finite group G and an algebraically closed field K. In more details, we construct the group Hopf algebra $H = KG$ and examine its properties to see what of the properties of the original finite group can be carried out in the case of H.

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1. Introduction

Finite group theory has been remarkably enriched in the last few decades by putting more attention on the classification of finite simple groups. The most important structure theorem for finite groups is the Jordan–Holder Theorem, which states that any finite group is built up from finite simple groups. The importance of this structure is that the properties of the subgroups of a given finite group G suggest substantial information about the group G itself such as the nilpotence and the solvability of G, see [3], [1],[5],[4],[21],[16] and [15]. In particular, having the property of simplicity of the group G can be deduced by investigating its subgroups.

Finite group properties such as simplicity, solvability, nilpotency, supersolvability, etc. have been an active area of reseach and been investigated by many mathematicians as idivisual classes or under specific formations, see for example [14] and [26].

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In the present work we limit our attention on the concepts of solvability and nilpotency of finite groups and semisimple Hopf algebras. Recall that a finite group G is said to be solvable if it has a series of normal subgroups $\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd ...... \lhd G_n = G$ such that G_i/G_{i-1} is abelian where $0 \leq i \leq n$, (see [23]). In [26], Wang proved that G is solvable if and only if M is c-normal in G for every maximal subgroup M of G. A group G is called σ-primary if $\mid G \mid$ is a σ-primary number. A group G is said to be σ-solvable if every chief factor of G is σ-primary [2].

In 1870, Benjamin Pierce was the first who introduced the term nilpotent in the context of his work on the classification of algebras. In algebras, an element x of a ring R is said to be nilpotent if there exists some positive integer n such that $x^n = 0$. In group theory, a group G is said to be nilpotent if it has an invariant series $\{e\} = G_0 \unlhd G_1 \unlhd G_2 \unlhd .... \unlhd G_n = G$ such that $G_{i+1}/G_i \leqslant Z(G/G_i)$, (see [23]).

Generalizing the notions of nilpotency and solvability of groups to nilpotency and solvability of semisimple Hopf algebras was shown to be available by giving several criteria for Hopf algebras to be so, see [7],[8],[9],[10].

It was noticed that chains of normal left coideal subalgebras of a Hopf algebra H play a similar role to chains of normal subgroups of a group. Consecuently, replacing normal subgroups with normal left coideal subalgebras can give a satisfactory intrinsic definition of nilpotent Hopf algebras.

Generalizing solvability is more difficult as there is no obvious analogue of Hopf quotients of left coideal subalgebras L over N where $N \subset L$ is a left coideal subalgebra normal in L but not necessarily in H. Whatsoever, this definition is undesirable as it conflicts with what is expected from group theory. Commutative or nilpotent Hopf algebras are not always solvable in that sense [11]. In [10], it was suggested a concrete definition for solvability of semisimple Hopf algebras named Hopf solvability. When $H = KG$, then Hopf solvability is equivalent to G being a solvable group. It was proved that commutative or nilpotent semisimple Hopf algebras are always solvable Hopf algebras.

In [8] and [10] Cohen and Westreich introduced the concepts of nilpotent and solvable Hopf algebras under certain conditions and they proved many interesting related results.

In this article, we construct the group Hopf algebra $H = KG$, for a finite group G and an algebraically closed field K, and examine its properties to see what of the properties of the original finite group can be carried out in the case of H.

2. Preliminaries

Recall that if H is a bialgebra. A subset $N \subseteq H$ is a sub-bialgebra of H if N is a sub-algebra i.e ($1_H \in N$ and for all $a, b \in N$ we have $ab \in N$) and N is a sub-coalgebra i.e ($\Delta(N) \subset N \otimes N$). If H is Hopf algebra with antipode S and $S(N) \subseteq N$ then N is said to be a Hopf subalgebra. Also, it is called a left coideal subalgebras if it is subalgebras and $\Delta(N) \subseteq H \otimes N$ (see[17] and [22]).

The following definitions and results are needed for our work.

Definition 1. *[17] The center of an algebra H is the set $Z \subseteq H$ of elements that commute with the whole algebra, i.e*

$$Z = \{z \in H |\ zh = hz\ \forall\ h \in H\}.$$

Definition 2. *[22] Let H be any Hopf algebra. The left adjoint action of H on itself is given by*

$$(ad_l h)(g) = \sum h_1 g S(h_2),\ \forall\ h, g \in H$$

The right adjoint action of H on itself is given by

$$(ad_r h)(g) = \sum S(h_1) g h_2,\ \forall\ h, g \in H$$

A Hopf subalgebra N of H is said to be normal if $(ad_l H)(N) \subseteq N$ and $(ad_r H)(N) \subseteq N$.

Definition 3. *[6] A left coideal subalgebra of H is called normal if N is closed under the left adjoint action of H on itself. Similarly, a right coideal subalgebra of H is called normal if N is closed under the right adjoint action of H on itself.*

H is called simple if it contains no proper normal Hopf subalgebras [13] .

Theorem 1. *[20] A group algebra KG of an arbitrary group G over a filed K is simple if and only if G has no non-trivial finite normal subgroup.*

Definition 4. *[22] Let H be a Hopf algebra. A left integral in H is an element $\Lambda \in H$ such that $h\Lambda = \epsilon(h)\Lambda$, for all $h \in H$; Similarly, a right integral in H is an element $\Lambda \in H$ such that $\Lambda h = \epsilon(h)\Lambda$, for all $h \in H$.*

Definition 5. *[19] An element $h \in H$ is said to be cocommutative if $\sum h_1 \otimes h_2 = \sum h_2 \otimes h_1$.*

Lemma 1. *[10]*

Let N be a left coideal subalgebra of H with an integral Λ_N. Then:

(i) N is a Hopf subalgebra if and only if Λ_N is cocommutative.

(ii) N is normal in H if and only if $\Lambda_N \in Z(H)$.

In the next two theorems we present Maschke's Theorem in the group algebras case and in the general finite dimensional Hopf algebras case.

Theorem 2. *[18] A group algebra of a finite group is semisimple if and only if the characteristic of the field does not divide the order of the group.*

Theorem 3. *[24] A finite dimensional Hopf algebra H is semisimple as an algebra if and only if $\epsilon(\Lambda) \neq 0$.*

In [11], the left adjoint action of H on itself was denoted by $\underset{ad}{\cdot}$ that is, for all $a, h \in H$,

$$h \underset{ad}{\cdot} a = \sum h_1 a S(h_2)$$

Definition 6. *[10] Let H be a semisimple Hopf algebra. A chain of left coideal subalgebras of H, $N_0 \subset N_1 \subset \ldots \subset N_t$ is a solvable series if, for all $0 \leq i \leq t-1$,*

(i) $\Lambda_{N_i} \in Z(N_{i+1})$ *, the center of N_{i+1}, where Λ_{N_i} is the integral of N_i.*

(ii) For all $a, b \in N_{i+1}$,

$$(a \underset{ad}{\cdot} b)\Lambda_{N_i} = \langle \epsilon, a \rangle \, b\Lambda_{N_i}.$$

The Hopf algebra H is said to be solvable if it has a solvable series so that $N_0 = K$ and $N_t = H$.

Remark 1. *[10] Commutative Hopf algebras are solvable by the definition with $K \subset H$ as a solvable series.*

Lemma 2. *[10] If N is a left coideal subalgebra of H, then the following hold:*

(i) $H\Lambda_N \cong H \diagup HN^+$ as left H-modules via $\pi_{|H\Lambda_N}$, where π is the natural H-module projection from H to $H \diagup HN^+$, where $N^+ = N \cap ker(\epsilon)$.

(ii) $H\Lambda_N \cong H \diagup HN^+$ as right H-modules and

(iii) If N is also normal in H, then $\pi_{|H\Lambda_N}$ is an algebra isomorphism as well.

Definition 7. *[8] A semisimple Hopf algebra H is nilpotent if the ascending central series*

$$K \subseteq Z_1 \subseteq Z_2 \subseteq \ldots.$$

satisfies $Z_m = H$ for some $m \geqslant 1$. The smallest such m is called the index of nilpotency of H.

Remark 2. *[8] Let $H = KG$, G be finite group. Then $\tilde{Z}(H) = KZ_G$, where Z_G is the center of the group G and $\tilde{Z}(H) = \{h \in H; \sum h_1 \otimes h_2 x S h_3 = h \otimes x \quad \text{for all } x \in H\}$ is the Hopf center of H.*

Proposition 1. *[10] A semisimple Hopf algebra H is nilpotent if and only if it has a series of normal left coideal subalgebras*

$$K = N_0 \subset N_1 \subset \ldots \subset N_t = H$$

such that

$$N_{i+1}\Lambda_{N_i} \subset Z(H\Lambda_{N_i}),$$

for all $1 \leq i \leq t$, where $Z(H\Lambda_{N_i})$ is the center of $H\Lambda_{N_i}$.

Proof. We assume that H is nilpotent with $K = Z_0 \subset Z_1 \subset \ldots \subset Z_t = H$. Let Λ_{Z_i} be the integral of Z_i. By Lemma 1.4, π_i is an algebra isomorphism between $H\Lambda_{Z_i}$ and H_i. As $\pi_i(Z_{i+1}) \subset \tilde{Z}(H_i) \subset Z(H_i)$, we get $Z_{i+1}\Lambda_{Z_i}$ is central in $H\Lambda_{Z_i}$.

Conversely, we assume that $K = N_0 \subset N_1 \subset \ldots \subset N_t = H$ satisfies the condition in Definition 1. Let $K \subseteq Z_1 \subseteq \ldots$ be an ascending central series for H. Then by the assumption we get $N_1 \subset Z(H)$. As N_1 is a normal left coideal subagebra of H, we have $N_1 \subset \tilde{Z}(H) = Z_1$. If $N_i \subset Z_i$, then by induction we show that $N_{i+1} \subset Z_{i+1}$. By the definition of Z_i, we have $\pi_i(N_i) \subseteq \pi_i(Z_i) = K$. As by the assumption $N_{i+1}\Lambda_{N_i} \subset Z(H\Lambda_i)$ and as $\pi_i(\Lambda_{N_i}) = 1$, we get

$$\pi_i(N_{i+1}) = \pi_i(N_{i+1}\Lambda_{N_i}) \subset \pi_i(Z(H\Lambda_{N_i})) = Z(\pi_i(H)) = Z(H_i).$$

But $\pi_i(N_{i+1})$ is a normal left coideal subalgebra of H_i contained in its center, hence $\pi_i(N_{i+1}) \subset \tilde{Z}(H_i)$. This leads to $\pi_{i+1}(N_{i+1}) = K$ and thus $N_{i+1} \subseteq Z_{i+1}$. As $N_t = H$ we get $Z_t = H$. Therefore, H is nilpotent.

Corollary 1. *[10] Semisimple nilpotent Hopf algebras are solvable.*

Proof. We assume that H has a series as in Definition 1. As each N_i is normal in H, we obtain $\Lambda_{N_i} \in Z(H)$, which satisfies (i) in Definition 6 of solvability. Next, as Λ_{N_i} is a central idemoptent of H, it follows by assumption that $\sum a_1 b(Sa_2)\Lambda_{N_i} = \langle \epsilon, a\rangle b\Lambda_{N_i}$ for all $a \in H$, $b \in N_{i+1}1$. This leads to (ii) of the definition of solvability.

3. From finite groups to Hopf algebras

3.1. The finite group C_3

We first consider the finite group $G = C_3 = \langle x | x^3 = 1\rangle = \{1, x, x^2\}$ and recall some of its properties. This group has no proper subgroup in particular it has no proper normal subgroup. So it is a simple group. Furthermore, since $| C_3 |= 3$, it is a cyclic group, hence it is abelian and consequently solvable and nilpotent.

3.2. The group algebra $\mathcal{H} = \mathbb{R}C_3$

The group algebra. Let G be a (multiplicative) group, and KG the associated group algebra. This is a K-vector space with basis $\{g | g \in G\}$. So its elements are of the form $\sum_{g\in G} r_g\, g$ where $(r_g)_{g\in G}$ a family of elements from K [12].

KG is an algebra with multiplication given by $\mu(g \otimes h) = gh$ and a unit map given by $\lambda(r) = r1$ for all $g, h \in G$ and $r \in K$ [25]. KG is a coalgebra with comultiplication given by $\Delta(g) = g \otimes g$ and a counit map given by $\epsilon(g) = 1$ for all $g \in G$. Moreover, KG is a Hopf algebra with an antipode map given by $S(g) = g^{-1}$ for all $g \in G$ [9].

We now construct a group algebra $\mathcal{H} = \mathbb{R}C_3$ considering the finite group C_3 with basis $\{1, x, x^2\}$ where $\mathbb{R}$ is the field of real numbers. In the following table the product map $\mu : \mathcal{H} \otimes \mathcal{H} \longmapsto \mathcal{H}$ which is defined by $\mu(g_1 \otimes g_2) = g_1 g_2$ is applied and calculated for all $g_1, g_2 \in G$:

$\mu(g_1 \otimes g_2)$	1	x	x^2
1	1	x	x^2
x	x	x^2	1
x^2	x^2	1	x

Table 1: $\mu(g_1 \otimes g_2)$.

The unit map $\lambda : \mathbb{R} \longmapsto \mathcal{H}$ is defined by $\lambda(r) = 1$. The maps μ and λ satisfy the required conditions that are:

(i) The associative property:

$$\mu(I \otimes \mu)(g \otimes (g_1 \otimes g_2)) = \mu(\mu \otimes I)((g \otimes g_1) \otimes g_2),$$

for all $g, g_1, g_2 \in G$. For example,
if $g = x$, $g_1 = x$ and $g_2 = x$, then $\mu(I \otimes \mu)(g \otimes (g_1 \otimes g_2)) = \mu(x \otimes x^2) = 1$. On the other hand, $\mu(\mu \otimes I)((g \otimes g_1) \otimes g_2) = \mu(x^2 \otimes x) = 1$.
Also, if $g = x$, $g_1 = x$ and $g_2 = x^2$, then $\mu(I \otimes \mu)(g \otimes (g_1 \otimes g_2)) = \mu(x \otimes 1) = x$. On the other hand, $\mu(\mu \otimes I)((g \otimes g_1) \otimes g_2) = \mu(x^2 \otimes x^2) = x$.
By using the same way, all the different choices of elements were checked and we present all of them in the following table:

g	g_1	g_1	$\mu\big((I\otimes\mu)g\otimes(g_1\otimes g_2)\big)$	$\mu\big((\mu\otimes I)(g\otimes g_1)\otimes g_2\big)$
1	1	1	1	1
1	1	x	x	x
1	1	x^2	x^2	x^2
1	x	1	x	x
1	x^2	1	x^2	x^2
x^2	x^2	1	x	x
x^2	x^2	x	x^2	x^2
x^2	x^2	x^2	1	1
x^2	1	x^2	x^2	x^2
x^2	x	x^2	x^2	x^2
x	x	1	x^2	x^2
x	x	x	1	1
x	x	x^2	x	x
x	1	x	x^2	x^2
x	x^2	x	x	x

Table 2: The associative property

(ii) The unit property:

$$\mu(I\otimes\lambda)(g\otimes r)=\mu(\lambda\otimes I)(r\otimes g),$$

for all $g\in G$ and $r\in\mathbb{R}$ which we do as follows knowing $\lambda(r)=1$: If $g=1$, then $\mu(I\otimes\lambda)(1\otimes r)=\mu(1\otimes 1)=1$. On the other hand, $\mu(\lambda\otimes I)(r\otimes 1)=\mu(1\otimes 1)=1$. If $g=x$, then $\mu(I\otimes\lambda)(x\otimes r)=\mu(x\otimes 1)=x$. On the other hand, $\mu(\lambda\otimes I)(r\otimes x)=\mu(1\otimes x)=x$. If $g=x^2$, then $\mu(I\otimes\lambda)(x^2\otimes r)=\mu(x^2\otimes 1)=x^2$. On the other hand, $\mu(\lambda\otimes I)(r\otimes x^2)=\mu(1\otimes x^2)=x^2$.

Thus, $(\mathcal{H},\mu,\lambda)$ is an $\mathbb{R}$-algebra.

Next, to show that $(\mathcal{H},\Delta,\epsilon)$ is an $\mathbb{R}$-coalgebra, we define the coproduct map $\Delta:\mathcal{H}\longmapsto\mathcal{H}\otimes\mathcal{H}$ by $\Delta(g)=g\otimes g$ and the counit map $\epsilon:\mathcal{H}\longmapsto\mathbb{R}$ by $\epsilon(g)=1$, for all $g\in G$. Δ and ϵ that satisfy the following required conditions:

(i) The coassociative property: $(I\otimes\Delta)\Delta(g)=(\Delta\otimes I)\Delta(g)$ for all $g\in G$. Indeed if $g=1$, then $(I\otimes\Delta)\Delta(1)=(I\otimes\Delta)(1\otimes 1)=1\otimes 1\otimes 1$. Also, $(\Delta\otimes I)\Delta(1)=(\Delta\otimes I)(1\otimes 1)=1\otimes 1\otimes 1$. If $g=x$, then $(I\otimes\Delta)\Delta(x)=(I\otimes\Delta)(x\otimes x)=x\otimes x\otimes x$. Also, $(\Delta\otimes I)\Delta(x)=(\Delta\otimes I)(x\otimes x)=x\otimes x\otimes x$. If $g=x^2$, then $(I\otimes\Delta)\Delta(x^2)=(I\otimes\Delta)(x^2\otimes x^2)=x^2\otimes x^2\otimes x^2$. Also, $(\Delta\otimes I)\Delta(x^2)=(\Delta\otimes I)(x^2\otimes x^2)=x^2\otimes x^2\otimes x^2$.

(ii) The counit property :$(\epsilon\otimes I)\Delta(g)=1\otimes g$ and $(I\otimes\epsilon)\Delta(g)=g\otimes 1$ for all $g\in G$. Indeed, if $g=1$, then $(\epsilon\otimes I)\Delta(1)=(\epsilon\otimes I)(1\otimes 1)=1\otimes 1$ and $(I\otimes\epsilon)\Delta(1)=(I\otimes\epsilon)(1\otimes 1)=1\otimes 1$. If $g=x$, then $(\epsilon\otimes I)\Delta(x)=(\epsilon\otimes I)(x\otimes x)=1\otimes x$ and $(I\otimes\epsilon)\Delta(x)=(I\otimes\epsilon)(x\otimes x)=x\otimes 1$. If $g=x^2$, then $(\epsilon\otimes I)\Delta(x^2)=(\epsilon\otimes I)(x^2\otimes x^2)=1\otimes x^2$ and $(I\otimes\epsilon)\Delta(x^2)=(I\otimes\epsilon)(x^2\otimes x^2)=x^2\otimes 1$.

So, $(\mathcal{H}, \Delta, \epsilon)$ is an $\mathbb{R}$-coalgebra.

Now, to show that $(\mathcal{H}, \mu, \lambda, \Delta, \epsilon)$ is an $\mathbb{R}$-bialgebra, as we have already shown that $(\mathcal{H}, \mu, \lambda)$ is an $\mathbb{R}$-algebra and $(\mathcal{H}, \Delta, \epsilon)$ is an $\mathbb{R}$-coalgebra, we only need to show that Δ and ϵ are algebra morphisms, that is for $g, g_1 \in G$ we have

$$\Delta(\mu(g \otimes g_1) = \Delta(g) \cdot \Delta(g_1),$$

and

$$\epsilon(\mu(g \otimes g_1)) = \epsilon(g)\epsilon(g_1),$$

where the multiplication $\cdot$ on $\mathcal{H} \otimes \mathcal{H}$ is just the usual multiplication on the tensor products

$$(g \otimes g_1) \cdot (g^{'} \otimes g_1^{'}) = \mu(g \otimes g^{'}) \otimes \mu(g_1 \otimes g_1^{'}).$$

We check this properties as follows: If $g = x$ and $g_1 = 1$, then

$$\Delta(\mu(x \otimes 1)) = \Delta(x) = x \otimes x.$$

On the other hand,

$$\Delta(x) \cdot \Delta(1) = (x \otimes x) \cdot (1 \otimes 1) = x \otimes x.$$

So, $\Delta(\mu(x \otimes 1)) = \Delta(x) \cdot \Delta(1)$. Also,

$$\epsilon(\mu(x \otimes 1)) = \epsilon(x) = 1.$$

On the other hand,

$$\epsilon(x)\epsilon(1) = (1)\,(1) = 1.$$

So, $\epsilon(\mu(x \otimes 1)) = \epsilon(x)\epsilon(1)$. If $g = x$ and $g_1 = x$, then

$$\Delta(\mu(x \otimes x)) = \Delta(x^2) = x^2 \otimes x^2.$$

On the other hand,

$$\Delta(x) \cdot \Delta(x) = (x \otimes x) \cdot (x \otimes x) = x^2 \otimes x^2.$$

So, $\Delta(\mu(x \otimes x)) = \Delta(x) \cdot \Delta(x)$. Also,

$$\epsilon(\mu(x \otimes x)) = \epsilon(x^2) = 1.$$

On the other hand,

$$\epsilon(x)\epsilon(x) = (1)\,(1) = 1.$$

So, $\epsilon(\mu(x \otimes x)) = \epsilon(x)\epsilon(x)$. If $g = x$ and $g_1 = x^2$, then

$$\Delta(\mu(x \otimes x^2)) = \Delta(1) = 1 \otimes 1.$$

On the other hand,

$$\Delta(x) \cdot \Delta(x^2) = (x \otimes x) \cdot (x^2 \otimes x^2) = 1 \otimes 1.$$

So, $\Delta(\mu(x \otimes x^2)) = \Delta(x) \cdot \Delta(x^2)$. Also,

$$\epsilon(\mu(x \otimes x^2)) = \epsilon(1) = 1.$$

On the other hand,

$$\epsilon(x)\epsilon(x^2) = (1)\,(1) = 1.$$

So, $\epsilon(\mu(x \otimes x^2)) = \epsilon(x)\epsilon(x^2)$. If $g = 1$ and $g_1 = 1$, then

$$\Delta(\mu(1 \otimes 1)) = \Delta(1) = 1 \otimes 1.$$

On the other hand,

$$\Delta(1) \cdot \Delta(1) = (1 \otimes 1) \cdot (1 \otimes 1) = 1 \otimes 1.$$

So, $\Delta(\mu(1 \otimes 1)) = \Delta(1) \cdot \Delta(1)$. Also,

$$\epsilon(\mu(1 \otimes 1)) = \epsilon(1) = 1.$$

On the other hand,

$$\epsilon(1)\epsilon(1) = (1)\,(1) = 1.$$

So, $\epsilon(\mu(1 \otimes 1)) = \epsilon(1)\epsilon(1)$. If $g = 1$ and $g_1 = x$, then

$$\Delta(\mu(1 \otimes x)) = \Delta(x) = x \otimes x.$$

On the other hand,

$$\Delta(1) \cdot \Delta(x) = (1 \otimes 1) \cdot (x \otimes x) = x \otimes x.$$

So, $\Delta(\mu(1 \otimes x)) = \Delta(1) \cdot \Delta(x)$. Also,

$$\epsilon(\mu(1 \otimes x)) = \epsilon(x) = 1.$$

On the other hand,

$$\epsilon(1)\epsilon(x) = (1)\,(1) = 1.$$

So, $\epsilon(\mu(1 \otimes x) = \epsilon(1)\epsilon(x)$ If $g = 1$ and $g_1 = x^2$, then

$$\Delta(\mu(1 \otimes x^2)) = \Delta(x^2) = x^2 \otimes x^2.$$

On the other hand,

$$\Delta(1) \cdot \Delta(x^2) = (1 \otimes 1) \cdot (x^2 \otimes x^2) = x^2 \otimes x^2.$$

So, $\Delta(\mu(1 \otimes x^2)) = \Delta(1) \cdot \Delta(x^2)$. Also,

$$\epsilon(\mu(1 \otimes x^2)) = \epsilon(x^2) = 1.$$

On the other hand,

$$\epsilon(1)\epsilon(x^2) = (1)\,(1) = 1.$$

So, $\epsilon(\mu(1 \otimes x^2) = \epsilon(1)\epsilon(x^2)$. If $g = x^2$ and $g_1 = 1$, then

$$\Delta(\mu(x^2 \otimes 1)) = \Delta(x^2) = x^2 \otimes x^2.$$

On the other hand,

$$\Delta(x^2) \cdot \Delta(1) = (x^2 \otimes x^2) \cdot (1 \otimes 1) = x^2 \otimes x^2.$$

So, $\Delta(\mu(x^2 \otimes 1)) = \Delta(x^2) \cdot \Delta(1)$. Also,

$$\epsilon(\mu(x^2 \otimes 1)) = \epsilon(x^2) = 1.$$

On the other hand,

$$\epsilon(x^2)\epsilon(1) = (1)\,(1) = 1.$$

So, $\epsilon(\mu(x^2 \otimes 1) = \epsilon(x^2)\epsilon(1)$. If $g = x^2$ and $g_1 = x$, then

$$\Delta(\mu(x^2 \otimes x)) = \Delta(1) = 1 \otimes 1.$$

On the other hand,

$$\Delta(x^2) \cdot \Delta(x) = (x^2 \otimes x^2) \cdot (x \otimes x) = 1 \otimes 1.$$

So, $\Delta(\mu(x^2 \otimes x)) = \Delta(x^2) \cdot \Delta(x)$. Also,

$$\epsilon(\mu(x^2 \otimes x)) = \epsilon(1) = 1.$$

On the other hand,

$$\epsilon(x^2)\epsilon(x) = (1)\,(1) = 1.$$

So, $\epsilon(\mu(x^2 \otimes x) = \epsilon(x^2)\epsilon(x)$. If $g = x^2$ and $g_1 = x^2$, then

$$\Delta(\mu(x^2 \otimes x^2)) = \Delta(x) = x \otimes x.$$

On the other hand,

$$\Delta(x^2) \cdot \Delta(x^2) = (x^2 \otimes x^2) \cdot (x^2 \otimes x^2) = x \otimes x.$$

So, $\Delta(\mu(x^2 \otimes x^2)) = \Delta(x^2) \cdot \Delta(x^2)$. Also,

$$\epsilon(\mu(x^2 \otimes x^2)) = \epsilon(x) = 1.$$

On the other hand,

$$\epsilon(x^2)\epsilon(x^2) = (1)\,(1) = 1.$$

So, $\epsilon(\mu(x^2 \otimes x^2) = \epsilon(x^2)\epsilon(x^2)$. Thus, $(\mathcal{H}, \mu, \lambda, \Delta, \epsilon)$ is an $\mathbb{R}$-bialgebra.

Finally, to show that $(\mathcal{H}, \mu, \lambda, \Delta, \epsilon, S)$ is an $\mathbb{R}$-Hopf algebra, we define the antipode map $S : \mathcal{H} \longmapsto \mathcal{H}$, for all $g \in G$, by $S(g) = g^{-1}$ and we check that S satisfies the antipode property, i.e. $\mu(I \otimes S)\Delta(g) = \mu(S \otimes I)\Delta(g)$ for all $g \in G$ which we do as follows: If $g = 1$, then $\mu(I \otimes S)\Delta(1) = \mu(I \otimes S)(1 \otimes 1) = 1$. On the other hand, $\mu(S \otimes I)\Delta(1) = \mu(S \otimes I)(1 \otimes 1) = 1$. If $g = x$, then $\mu(I \otimes S)\Delta(x) = \mu(I \otimes S)(x \otimes x) = \mu(x \otimes x^2) = 1$. On the other hand, $\mu(S \otimes I)\Delta(x) = \mu(S \otimes I)(x \otimes x) = \mu(x^2 \otimes x) = 1$. If $g = x^2$, then $\mu(I \otimes S)\Delta(x) = \mu(I \otimes S)(x^2 \otimes x^2) = \mu(x^2 \otimes x) = 1$. On the other hand, $\mu(S \otimes I)\Delta(x^2) = \mu(S \otimes I)(x^2 \otimes x^2) = \mu(x \otimes x^2) = 1$. Therefore, $(\mathcal{H}, \mu, \lambda, \Delta, \epsilon, S)$ is an $\mathbb{R}$-Hopf algebra.

3.3. Commutativity and cocommutativity of the Hopf algebra $\mathcal{H}$

It is easy to verify that each element in $\mathcal{H}$ commutes with all the other element of $\mathcal{H}$. So $\mathcal{H}$ is commutative as an algebra. Thus, $\mathcal{H}$ is a commutative Hopf Algebra.

Now, to check the cocommutativity of $\mathcal{H}$ we need to show that it is cocommutative as a coalgebra, i.e. $\tau(\Delta(g)) = \Delta(g)$ for all $g \in C_3$ which we do as follows: If $g = 1$, then

$$\tau(\Delta(1)) = \tau(1 \otimes 1) = 1 \otimes 1 = \Delta(1).$$

If $g = x$, then

$$\tau(\Delta(x)) = \tau(x \otimes x) = x \otimes x = \Delta(x).$$

If $g = x^2$, then

$$\tau(\Delta(x^2)) = \tau(x^2 \otimes x^2) = x^2 \otimes x^2 = \Delta(x^2).$$

Therefore, $\mathcal{H}$ is a cocommutative Hopf algebra.

3.4. The Normal Hopf subalgebras of $\mathcal{H}$

Since C_3 is simple, it follows that $\mathcal{H}$ has no proper subalgebras. Moreover, Since the only Hopf subalgebras of $\mathcal{H}$ are the trivial ones, thus $\mathcal{H}$ has no proper normal Hopf subalgebra. Therefore, $\mathcal{H}$ is simple.

3.5. Semi-simplicity of the Hopf algebra $\mathcal{H}$

To check the semi-simplicity of the Hopf algebra $\mathcal{H}$ we need first to recall that the center of a Hopf algebra $\mathcal{H}$ is defined by

$$Z(\mathcal{H}) = \{h \in \mathcal{H}; hh_1 = h_1 h \ for \ all \ h_1 \in \mathcal{H}\}.$$

In our example, since each element in $\mathcal{H}$ commutes with all the element of $\mathcal{H}$, then $Z(\mathcal{H}) = \mathcal{H}$.

In our example $|C_3| = 3$ is not divisible by the characteristic of the field $\mathbb{R}$ which is zero. So, by theorem 2, $\mathcal{H}$ is semisimple Hopf algebra.

3.6. Nilpotency and solvability of the Hopf algebra $\mathcal{H}$

In this subsection we discuss whether $\mathcal{H}$ is a nilpotent or a solvable Hopf algebra. Since $\mathcal{H}$ is commutative Hopf algebra with $\mathbb{R} \subset \mathcal{H}$ as a solvable series, it follows that $\mathcal{H}$ is solvable by using Remark 1. Furthermore, since $\mathcal{H}$ has central series $\mathbb{R} \subseteq Z_1$, where $Z_1 = \tilde{Z}(\mathcal{H})$, hence by using Remark 2 we get $\tilde{Z}(\mathcal{H}) = \mathbb{R}Z_{C_3} = \mathbb{R}C_3$. Thus, $Z_1 = \mathbb{R}C_3$. Therefore, by Definition 7, $\mathcal{H}$ is nilpotent.

4. Conclusion

We noticed from our example that all the properties of the original finite group C_3 have been carried out in the case of its group Hopf algebra $\mathcal{H} = \mathbb{R}C_3$ such as commutativity, simplicity, nilpotency and solvability. In general, for a finite group G, the normal Hopf subalgebras of KG are of the form KG_i where G_i is a normal subgroup of G and K is an algebraically closed field of characteristic zero. In particular, KG is simple as a Hopf algebra if and only if G is a simple group. Furthermore, if KG is simple, then it has no a proper quotient Hopf algebra [13].

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