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Finite Groups With Minimal CSS-Subgroups

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Abstract. Let G be a finite group. A subgroup H of G is called SS-quasinormal in G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. A subgroup H of G is called CSS-subgroup in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is SS-quasinormal in G. In this paper, we investigate the influence of minimal CSS-subgroups of G on its structure. Our results improve and generalize several recent results in the literature.

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1. Introduction

All groups considered in this paper are finite. The terminology and notions employed agree with standard usage, as in [2, 5], and G always denotes a finite group.

Following Kegel [9], a subgroup H of G is said to be S-quasinormal in G if H permutes with every Sylow subgroup of G, i.e, $HP = PH$ for any Sylow subgroup P of G. A subgroup H of G is said to be c-normal in G if G has a normal subgroup K such that $G = HK$ and $H \cap K \leqslant H_G$, where $H_G = Core_G(H)$ is the largest normal subgroup of G contained in H (see Wang [18]). Recently, in 2008, Li et al. [12] extended S-quasinormal subgroups of a group G to SS-quasinormal subgroups and they gave the following definition: A subgroup H of G is said to be SS-quasinormal in G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B.

Obviously, every S-quasinormal subgroup is SS-quasinormal. The converse is not true in general. For instance, S_3 is SS-quasinormal subgroup of the symmetric group S_4 but

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not S-quasinormal. More recently, in 2019, Zhao et al. [26] introduced a new subgroup embedding property of a finite group, called CSS-subgroup, which generalize and unify both of c-normality and SS-quasinormality as follows: A subgroup H of G is called CSS-subgroup of G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is SS-quasinormal in G. It is clear that each of c-normality and SS-quasinormality concepts implies CSS-subgroup. The converse does not hold in general (see [26, Examples 1 and 2]).

Over years, many authors studied the influence of minimal subgroups of a finite group on its structure (a subgroup of prime order is called a minimal subgroup). In this context, Buckley [3] got the supersolvability of a group of odd order when all its minimal subgroups are normal. In [17], Shaalan proved that a group G is supersolvable if all subgroups of prime order p or of order 4 (if $p = 2$) of G are S-quasinormal in G. Later on, Wang [18] got the same result of Shaalan [17] just he replaced S-quasinormality by c-normality. By using the SS-quasinormality concept, Li et al. [11] extended these results through the theory of formations and proved that: Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and let G be a group. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup H such that $G/H \in \mathfrak{F}$ and every subgroup of $F^*(H)$ of prime order p or of order 4 (if $p = 2$) is SS-quasinormal in G, where $F^*(H)$ is the generalized Fitting subgroup of H. Also, Wei et al. in [21] used the c-normality concept and obtained the same previous result. For more results in this direction (see [1, 11, 12, 16–18, 20, 21, 24]).

The main purpose of this paper is to improve and extend the above mentioned results by using the recent concept CSS-subgroup. More precisely, we investigate the structure of a finite group G when every subgroup of G of prime order p or of order 4 (if $p = 2$) is CSS-subgroup in G.

2. Basic Definitions and Preliminaries

In this section, we list some definitions and state some known results from the literature which will be used in proving our results.

A class of groups $\mathfrak{F}$ is said to be a formation if $\mathfrak{F}$ is closed under taking epimorphic images and every group G has a smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-residual of G and it is denoted by $G^{\mathfrak{F}}$. A formation $\mathfrak{F}$ is called saturated if it is closed under taking Frattini extensions. Throughout this paper, $\mathfrak{U}$ and $\mathfrak{N}$ will denote the classes of supersolvable groups and nilpotent groups, respectively. It is known that $\mathfrak{U}$ and $\mathfrak{N}$ are saturated formations (see [7, Satz 8.6, p. 713 and Satz 3.7, p. 270]).

A normal subgroup N of a group G is an $\mathfrak{F}$-hypercentral subgroup of G provided N possesses a chain of subgroups $1 = N_0 \trianglelefteq N_1 \trianglelefteq ... \trianglelefteq N_s = N$ such that N_{i+1}/N_i is an $\mathfrak{F}$-central chief factor of G (see [5, p. 387]). The product of all $\mathfrak{F}$-hypercentral subgroups of G is again an $\mathfrak{F}$-hypercentral subgroup, denoted by $Z_{\mathfrak{F}}(G)$, and it is called the $\mathfrak{F}$-hypercenter of G (see [5, IV 6.8]). For the formation $\mathfrak{U}$, the $\mathfrak{U}$-hypercenter of a group G will be denoted by $Z_{\mathfrak{U}}(G)$, that is, $Z_{\mathfrak{U}}(G)$ is the product of all normal subgroups N of G such that each chief factor of G below N has prime order and for the formation $\mathfrak{N}$, the $\mathfrak{N}$-hypercenter of a group G is simply the terminal member $Z_\infty(G)$ of the ascending central series of G. For more details about saturated formations, see [5, IV].

For any group G, the generalized Fitting subgroup $F^*(G)$ is the set of all elements x of G which induce an inner automorphism on every chief factor of G.

Lemma 1. (See [26, Lemma 2.3]) *Let H be CSS-subgroup of G.*

(1) *If $H \leqslant M \leqslant G$, then H is CSS-subgroup of M.*

(2) *Let $N \trianglelefteq G$ and $N \leqslant H$. Then H is CSS-subgroup of G if and only if H/N is CSS-subgroup of G/N.*

(3) *Let π be a set of some primes and N a normal π'-subgroup of G. If H is a π-subgroup of G, then HN/N is CSS-subgroup of G/N.*

Lemma 2. (See [7, Satz 5.4, p. 434 and Satz 5.2, p. 281]) *Let G be a minimal non p-nilpotent group (a non p-nilpotent group all of its proper subgroups are p-nilpotent), where p is a prime.*

(1) *G is a minimal non-nilpotent group.*

(2) *$G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non normal cyclic Sylow q-subgroup of G.*

(3) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(4) *If $p > 2$, then the exponent of P is p and when $p = 2$, the exponent of P is at most 4.*

Lemma 3. (See [17, Theorem 3.2]) *Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. If every subgroup of P of order p or of order 4 (if $p = 2$) is S-quasinormal in G, then G is p-nilpotent.*

Lemma 4. (See [23]) *Let H be a subnormal subgroup of G.*

(1) *If H is a Hall-subgroup of G, then H is normal in G.*

(2) *If H is a π-subgroup of G, then $H \leqslant O_\pi(G)$.*

Lemma 5. (See [11, Lemma 2.2]) *Suppose that P is a p-subgroup of G. Then P is S-quasinormal in G if and only if $P \leqslant O_p(G)$ and P is SS-quasinormal in G.*

Lemma 6. (See [13, Theorem 3.3]) *Suppose that P is a normal p-subgroup of G, where $p > 2$. If every subgroup of P of order p is S-quasinormal in G, then $P \leqslant Z_{\mathfrak{U}}(G)$.*

Lemma 7. (See [22, Theorem 7.7, p. 31]) *Let N be a normal subgroup of G such that $N \leqslant Z_{\mathfrak{U}}(G)$. Then $Z_{\mathfrak{U}}(G/N) = Z_{\mathfrak{U}}(G)/N$.*

Lemma 8. (See [22, Theorem 6.3, p. 220 and Corollary 7.8, p. 33]) *Let P be a normal p-subgroup of G such that $|G/C_G(P)|$ is a power of p. Then $P \leqslant Z_{\mathfrak{U}}(G)$.*

Lemma 9. (See [5, Propositin 3.11, p. 362]) *If* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *are two saturated formations such that* $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$*, then* $Z_{\mathfrak{F}_1}(G) \subseteq Z_{\mathfrak{F}_2}(G)$.

Lemma 10. (See [4]) *Let* K *be a normal subgroup of* G *such that* $G/K \in \mathfrak{F}$*, where* $\mathfrak{F}$ *is a saturated formation. If* $\Omega(P) \leqslant Z_{\mathfrak{F}}(G)$*, where* P *is a Sylow* p*-subgroup of* K*, then* $G/O_{p'}(K) \in \mathfrak{F}$.

Lemma 11. (See [8, X 13] and [14, Lemma 2.3(4)]) *Let* M *be a subgroup of* G.

(1) *If* M *is normal in* G*, then* $F^*(M) \leqslant F^*(G)$.

(2) $F^*(G) \neq 1$ *if* $G \neq 1$.

(3) *If* $F^*(G)$ *is solvable, then* $F^*(G) = F(G)$.

(4) *Suppose* K *is a subgroup of* G *contained in* $Z(G)$*. Then* $F^*(G/K) = F^*(G)/K$.

Lemma 12. (See [10, Corollary 3]) *Let* $\mathfrak{F}$ *be a saturated formation and* G *a group. Suppose that* $C_G(N) \leqslant N \trianglelefteq G$*. Then* $G \in \mathfrak{F}$ *if every cyclic subgroup of* N *of prime order or of order* 4 *is contained in* $Z_{\mathfrak{F}}(G)$.

Lemma 13. (See [15, Lemma 2.8]) *Suppose that* G *is a group and* P *is a normal* p*-subgroup of* G *contained in* $Z_\infty(G)$*. Then* $C_G(P) \geqslant O^p(G)$.

Lemma 14. (See [7, Satz 2.8, p. 420]) *If* P *is a cyclic Sylow* p*-subgroup of* G*, where* p *is the smallest prime dividing* $|G|$*, then* G *is* p*-nilpotent.*

Lemma 15. (See [6, Theorem 3.10, p. 184]) *If* H *is a* p'*-group of automorphisms of the* p*-group* P *with* p *odd which acts trivially on* $\Omega_1(P)$*, then* $H = 1$.

Lemma 16. (See [6, Theorem 2.4, p. 178]) *If* H *is a* p'*-group of automorphisms of the abelian* p*-group* P *which acts trivially on* $\Omega_1(P)$*, then* $H = 1$.

3. Main Results

First we prove:

Theorem 1. *Let* p *be the smallest prime dividing* $|G|$ *and* P *a Sylow* p*-subgroup of* G*. If every subgroup of* P *of prime order* p *or of order* 4 *(if* $p = 2$*) is* CSS*-subgroup of* G*, then* G *is* p*-nilpotent.*

Proof. Assume that the result is false and let G *be a counterexample of minimal order. Let* L *be an arbitrary proper subgroup of* G*. Then every subgroup of* L *of prime order* p *or of order* 4 *(if* $p = 2$*) is* CSS*-subgroup of* G *by the hypothesis. Thus, by Lemma 11, every subgroup of* L *of prime order* p *or of order* 4 *(if* $p = 2$*) is* CSS*-subgroup of* L.

That means L *satisfies the hypothesis of the theorem and so* L *is* p*-nilpotent by the minimal choice of* G*. Hence,* G *is not* p*-nilpotent but all of its proper subgroups are* p*-nilpotent.*

By Lemma 2, G is a minimal non-nilpotent group and so $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G, for some prime $q \neq p$. Furthermore, if $p > 2$, then P is of exponent p and if $p = 2$, P is of exponent at most 4. If every subgroup of P with order p or 4 (if $p = 2$) is S-quasinormal in G, then, by Lemma 3, we get the p-nilpotency of G, a contradiction. Therefore, there exists a subgroup S of P of prime order p or of order 4 (if $p = 2$) such that S is not S-quasinormal in G. By hypothesis, S is CSS-subgroup of G. Then there exists a normal subgroup K of G such that $G = SK$ and $S \cap K$ is SS-quasinormal in G. Assume that $K = G$. It follows that S is SS-quasinormal in G. Since P is normal in G, then S is subnormal in G. Thus, by Lemma 4, $S \leqslant O_p(G)$. Applying Lemma 5, we get S is S-quasinormal in G, a contradiction. Hence, K is a proper normal nilpotent subgroup of G which implies that Q is characteristic in K. Therefore Q is a normal subgroup in G, a final contradiction completing the proof.

Lemma 17. *Let P be a non-trivial normal p-subgroup of G (where $p > 2$). If every minimal subgroup of P is CSS-subgroup of G, then $P \leqslant Z_{\mathfrak{U}}(G)$.*

Proof. *We prove the theorem by induction on $|G| + |P|$. If every minimal subgroup of P is S-quasinormal in G, then by Lemma 6, we get $P \leqslant Z_{\mathfrak{U}}(G)$ and we are done. Thus, we may assume that P has a minimal subgroup L such that L is not S-quasinormal in G. By the hypothesis of the lemma, L is CSS-subgroup of G, i.e., G has a normal subgroup K such that $G = LK$ and $L \cap K$ is SS-quasinormal in G. If $L \cap K \neq 1$, we have $L \cap K = L$. Hence, L is SS-quasinormal in G. Since P is normal in G, then L is subnormal in G. Lemma 4 implies that $L \leqslant O_p(G)$. Applying Lemma 5, L is S-quasinormal in G, a contradiction. Therefore, we may assume $L \cap K = 1$. Then, $P = P \cap G = P \cap LK = L(P \cap K)$ and $P \cap K \trianglelefteq G$. By the hypothesis, every minimal subgroup of the non-trivial normal p-subgroup $P \cap K$ is CSS-subgroup of G. This leads to $P \cap K \leqslant Z_{\mathfrak{U}}(G)$ by induction on $|G| + |P|$. Hence, $P/(P \cap K) \leqslant Z_{\mathfrak{U}}(G/(P \cap K))$ as $P/(P \cap K)$ is a normal subgroup of $G/(P \cap K)$ of order p. But $P \cap K \leqslant Z_{\mathfrak{U}}(G)$, then $Z_{\mathfrak{U}}(G/(P \cap K)) = Z_{\mathfrak{U}}(G)/(P \cap K)$ by Lemma 7. Thus, $P/(P \cap K) \leqslant Z_{\mathfrak{U}}(G)/(P \cap K)$. Now it follows easily that $P \leqslant Z_{\mathfrak{U}}(G)$.*

Immediate consequence of Lemma 17 and Theorem 1, we have the following corollary:

Corollary 1. *Let P be a normal p-subgroup of G. If every subgroup of P of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of G, then $P \leqslant Z_{\mathfrak{U}}(G)$.*

Proof. *Assume that $p > 2$. Then, by Lemma 17, $P \leqslant Z_{\mathfrak{U}}(G)$ and we are done. Hence, consider $p = 2$. Let Q be any Sylow q-subgroup of G, where $q \neq 2$. It is clear that PQ is a subgroup of G. Since every subgroup of P of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of G, then by Lemma 11, every subgroup of P of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of PQ.*

By applying Theorem 1, we have PQ is 2-nilpotent. This implies that $PQ = P \times Q$ and so Q centralizes P. Thus, $O^p(G) \leqslant C_G(P)$ and it follows that $|G/C_G(P)|$ is a power of 2. By Lemma 8, we conclude $P \leqslant Z_{\mathfrak{U}}(G)$.

We now prove:

Theorem 2. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup H in G such that $G/H \in \mathfrak{F}$ and every subgroup of H of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of G.*

Proof. *If $G \in \mathfrak{F}$, then we set $H = 1$ and the result follows. Conversely, assume that the result is false and let G be a counterexample of minimal order. By using Lemma 11 and repeated applications of Theorem 1, the group H has a Sylow tower of supersolvable type which means that H has a normal Sylow p-subgroup P, where p is the largest prime dividing $|H|$. Clearly, P is normal in G and hence $(G/P)/(H/P) \cong G/H \in \mathfrak{F}$. By Lemma 12, every subgroup of H/P of prime order or of order 4 (if $p = 2$) is CSS-subgroup of G/P. Then, by the minimal choice of G, we have $G/P \in \mathfrak{F}$ and so $1 \neq G^{\mathfrak{F}} \leqslant P$. By the hypothesis, every subgroup of $G^{\mathfrak{F}}$ of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of G. Then, by Corollary 1, $G^{\mathfrak{F}} \leqslant Z_{\mathfrak{U}}(G)$. Since $\Omega(G^{\mathfrak{F}}) \leqslant G^{\mathfrak{F}}$ and $Z_{\mathfrak{U}}(G) \leqslant Z_{\mathfrak{F}}(G)$, by Lemma 9, we have $\Omega(G^{\mathfrak{F}}) \leqslant G^{\mathfrak{F}} \leqslant Z_{\mathfrak{U}}(G) \leqslant Z_{\mathfrak{F}}(G)$. Hence, $\Omega(G^{\mathfrak{F}}) \leqslant Z_{\mathfrak{F}}(G)$. Therefore, by Lemma 10, $G \in \mathfrak{F}$, a contradiction.*

The following corollaries are immediate consequences of Theorem 2:

Corollary 2. *Let H be a normal subgroup of G such that G/H is supersolvable. If every subgroup of H of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of G, then G is supersolvable.*

Corollary 3. *Let H be a normal subgroup of G such that $(G/H)'$ is nilpotent. If every subgroup of H of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of G, then G' is nilpotent.*

Corollary 4. *Let G be a group such that every subgroup of G of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of G, then G is supersolvable.*

Now we can prove:

Theorem 3. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup H such that $G/H \in \mathfrak{F}$ and every subgroup of $F^*(H)$ of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of G.*

Proof. *If $G \in \mathfrak{F}$, then we set $H = 1$ and the theorem follows. Now we prove the converse. By the hypothesis and Lemma 11, every subgroup of $F^*(H)$ of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of $F^*(H)$. Corollary 4 implies that $F^*(H)$ is supersolvable. Hence, by Lemma 11, $F^*(H) = F(H)$. Then, by Corollary 1, $O_p(H) \leqslant Z_{\mathfrak{U}}(G)$. Since $Z_{\mathfrak{U}}(G) \leqslant Z_{\mathfrak{F}}(G)$, by Lemma 9, it follows that $O_p(H) \leqslant Z_{\mathfrak{F}}(G)$ and so $F^*(H) = F(H) \leqslant Z_{\mathfrak{F}}(G)$. Applying Lemma 12, we get $G \in \mathfrak{F}$.*

Immediately from Theorem 3, we have the following corollaries:

Corollary 5. *Let H be a normal subgroup G such that G/H is supersolvable. If every subgroup of $F^*(H)$ of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of G, then G is supersolvable.*

Corollary 6. *If every subgroup of $F^*(G)$ of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of G, then G is supersolvable.*

Corollary 7. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if G has a solvable normal subgroup H such that $G/H \in \mathfrak{F}$ and every subgroup of $F(H)$ of prime order p or of order 4 (if $p = 2$) is CSS-subgroup of G.*

We now prove:

Theorem 4. *Let G be a group. If every subgroup of G of prime order is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of G is CSS-subgroup of G or lies in $Z_\infty(G)$, then G is nilpotent.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. Let L be an arbitrary proper subgroup of G and K a cyclic subgroup of L of prime order or of order 4. Then $K \leqslant Z_\infty(G) \cap L \leqslant Z_\infty(L)$. By hypotheses and Lemma 11, K is CSS-subgroup of L. The minimal choice of G implies that L is nilpotent. Since L is an arbitrary proper subgroup of G, we have that G is a minimal non-nilpotent group. Hence, by Lemma 2, $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non normal cyclic Sylow q-subgroup of G, $p \neq q$. Moreover, $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. Now we have:

(1) $p = 2$ and every element of order 4 is CSS-subgroup of G.
Assume that $p > 2$. By Lemma 2, the exponent of P is p. Then, by the hypotheses, $P \leqslant Z_\infty(G)$. Applying Lemma 13, $O^p(G) \leqslant C_G(P)$ which means that $G = PQ = P \times Q$ is nilpotent, a contradiction. If every element of order 4 of G lies in $Z_\infty(G)$, then $P \leqslant Z_\infty(G)$ which means that $G = PQ = P \times Q$ is nilpotent, again contradiction.

(2) For every $x \in P \setminus \Phi(P)$, $|x| = 4$.
Assume that $|x| \neq 4$. Then there exists $x \in P \setminus \Phi(P)$ and $|x| = 2$. Since $P \trianglelefteq G$, we have that $< x^G > \leqslant P$. Then $< x^G > \Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. But as we mentioned above $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. Then $P = < x^G > \Phi(P) = < x^G > \leqslant Z_\infty(G)$. In particular; G is nilpotent, a contradiction.

(3) Finishing the proof.
From 2, every element x in $P \setminus \Phi(P)$ is of order 4. From 1, $< x >$ is CSS-subgroup of G. Then there exists a normal subgroup S of G such that $G = < x > S$ and $< x > \cap S$ is SS-quasinormal in G. Clearly, $P \cap S \trianglelefteq G$. Hence, $(P \cap S)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup $G/\Phi(P)$, it follows that either $P \cap S \leqslant \Phi(P)$ or $P \cap S = P$. Assume first that $P \cap S \leqslant \Phi(P)$. Then $P = P \cap G = P \cap (< x > S) = < x > (P \cap S) = < x > \Phi(P)$. Therefore, $P = < x >$ and this means that P is a cyclic normal Sylow 2-subgroup of G of order 4. By Lemma 14, G is 2-nilpotent and so $G = PQ = P \times Q$ is nilpotent, a contradiction. Thus, assume that $P \cap S = P$. Then $< x > = < x > \cap P = < x > \cap (P \cap S) = (< x > \cap P) \cap S = < x > \cap S$. Hence, $< x >$ is SS-quasinormal in G. $< x > \leqslant P \trianglelefteq G$

implies that $< x >$ is subnormal in G. By Lemma 4, $< x >\leqslant O_2(G)$. Applying Lemma 5, $< x >$ is S-quasinormal in G. Thus, $< x > Q \leqslant G$. If $< x > Q = G$, then $< x >= P$ which implies G is nilpotent, a contradiction. Therefore, $< x > Q < G$ and it follows that $< x > Q$ is nilpotent. Then $< x > Q =< x > \times Q$. Thus, $< x >\leqslant N_G(Q)$ implies that $P \leqslant N_G(Q)$ and so $G = PQ = P \times Q$ is nilpotent, a final contradiction completing the proof.

Theorem 5. *Let H be a normal subgroup of G such that G/H is nilpotent. If every subgroup of H of prime order is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of H is CSS-subgroup of G or lies in $Z_\infty(G)$, then G is nilpotent.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. Let L be an arbitrary proper subgroup of G. Since G/H is nilpotent, we have $L/L \cap H \cong LH/H$ is nilpotent. The element of prime order or of order 4 of $L \cap H$ is contained in $Z_\infty(G) \cap L \leqslant Z_\infty(L)$. By hypotheses and Lemma 11, every cyclic subgroup of order 4 of $L \cap H$ is CSS-subgroup in L. Thus the pair $(L, L \cap H)$ satisfies the hypotheses of the theorem in any case. Then L is nilpotent, that is, G is a minimal non-nilpotent group. Applying Lemma 2, $G = PQ$, where P is normal Sylow p-subgroup of G and Q is non normal cyclic Sylow q-subgroup of G, $p \neq q$. Since G/H and G/P are nilpotent, then $G/P \cap H \leqslant G/P \times G/H$ is nilpotent. Now we deal with:

(1) $P \leqslant H$.
Assume that $p > 2$. Then, by Lemma 2, the exponent of P is p and so $P = P \cap H \leqslant Z_\infty(G)$. Applying Lemma 13, we have $O^p(G) \leqslant C_G(P)$. This implies $G = PQ = P \times Q$ is nilpotent, a contradiction. Thus, we may assume that $p = 2$. Since $P \trianglelefteq G$, it follows that every element of order 2 or 4 of G is contained in P; in particular in H. Thus, every element of order 2 of G lies in $Z_\infty(G)$ and, by hypotheses, every cyclic subgroup of order 4 is CSS-subgroup of G or lies also in $Z_\infty(G)$. Applying similar arguments to those in (2) and (3) of the proof of Theorem 4, we have that G is nilpotent, a contradiction.

(2) $P \nleqslant H$.
Then $P \cap H < P$ and hence $Q(P \cap H) < G$. Therefore, $Q(P \cap H)$ is nilpotent which implies that $Q(P \cap H) = Q \times (P \cap H)$. Moreover, Q is characteristic in $Q(P \cap H)$. Clearly, as $G/P \cap H = (P/P \cap H)(Q(P \cap H)/P \cap H)$ is nilpotent, then $Q(P \cap H)/P \cap H \trianglelefteq G/P \cap H$. Thus $Q(P \cap H) \trianglelefteq G$. Hence $Q \trianglelefteq G$, a contradiction.

Theorem 6. *Let H be a normal subgroup of G such that G/H is nilpotent and every cyclic subgroup of order 4 of $F^*(H)$ is CSS-subgroup of G. Then G is nilpotent if and only if every subgroup of prime order of $F^*(H)$ is contained in $Z_\infty(G)$.*

Proof. If G is nilpotent, then we set $H = 1$ and the result follows. Conversely, assume that the result is false and let G be a counterexample of minimal order. With the same arguments to those in steps (1) and (2) of the proof of Theorem 4.4 in [11], we have:

(1) Every proper normal subgroup of G is nilpotent, and $F(G)$ is the unique maximal normal subgroup of G.

(2) $H = G$, $G' = G$ and $F^*(G) = F(G) < G$.

(3) Let q be a minimal prime divisor of $|F(G)|$ and Q a Sylow q-subgroup of $F(G)$. Then $G/C_G(Q)$ is a q-group.
Since $F^*(G) \neq 1$, then we may assume that q is a minimal prime divisor of $|F(G)|$ and Q is a Sylow q-subgroup of $F(G)$ which is a non-trivial normal subgroup of G. Clearly, from hypotheses, $\Omega_1(Q) \leqslant Z_\infty(G)$. Thus, by Lemma 13, $C_G(\Omega_1(Q)) \geqslant O^q(G)$. If $q > 2$, then, by Lemma 15, $C_G(Q) \geqslant O^q(G)$. This implies that $G/C_G(Q)$ is a q-group. If $q = 2$, let $< x >$ be an arbitrary cyclic subgroup of Q of order 4. By hypotheses, $< x >$ is CSS-subgroup of G. Then, there exists a normal subgroup L of G such that $G =< x > L$ and $< x > \cap L$ is SS-quasinormal in G. If $< x > \cap L = 1$, then L is a proper normal subgroup of G and, by (1), L is nilpotent. It follows that any Sylow p-subgroup of L is normal in G, where p is any prime number such that $p \neq 2$. Therefore, G is nilpotent, a contradiction. Hence we may assume that $< x > \leqslant L$ and $< x >$ is SS-quasinormal in G. Since Q is a normal subgroup of G, it follows that $< x >$ is subnormal in G. Hence, by Lemma 4, $< x > \leqslant O_2(G)$. Applying Lemma 5, $< x >$ is S-quasinormal in G. Now, let P be any Sylow p-subgroup of G, where $p \neq 2$. Therefore $< x > P \leqslant G$. Clearly, as $< x >$ is subnormal in $< x > P$ and $< x >$ is a Sylow 2-subgroup of $< x > P$, we have $< x >$ is normal in $< x > P$. Hence, by Lemma 16, $< x > P$ is nilpotent. It follows that $P \leqslant C_G(< x >)$ and so $O^2(G) \leqslant C_G(< x >)$. This implies that $O^2(G) \leqslant C_G(Q)$ and so $G/C_G(Q)$ is a 2-group.

(4) We have a contradiction.
By 2, $G = G'$ and so $C_G(Q) = G$, $Q \leqslant Z(G)$. By Lemma 11, $F^*(G/Q) = F^*(G)/Q$. Let $\overline{G} = G/Q$. Then, 3 imply that each element $\overline{y}$ of prime order n in $F^*(\overline{G})$ can be viewed as an image in element y of prime order n in $F^*(G)$, For each $n > q$. Thus, by hypotheses, $y \leqslant Z_\infty(G)$. Since $Q \leqslant Z(G)$, then $Z_\infty(G/Q) = Z_\infty(G)/Q$. Hence $\overline{y} \leqslant Z_\infty(G/Q)$. Clearly, $F^*(G/Q)$ does not have an element of order 2. This means that $\overline{G}$ satisfies the hypotheses of the theorem. Then $\overline{G} = G/Q$ is nilpotent by our choice of G and so G is nilpotent which yields the desired contradiction.

4. Some Applications

As it was mentioned in the introduction each of c-normality and SS-quasinormality subgroups implies CSS-subgroups. Therefore the following results are direct consequences of our results.

Corollary 8. ([1, Lemma 3.1]) *Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of a group G. If every subgroup of P of prime order or of order 4 (if $p = 2$) is c-normal in G, then G is p-nilpotent.*

Corollary 9. ([18, Theorem 4.2]) *Let G be a group such that every subgroup of G of prime order or of order 4 (if $p = 2$) is c-normal in G, then G is supersolvable.*

Corollary 10. ([1, Theorem 3.2] and [16, Theorem 3.9]) *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. $G \in \mathfrak{F}$ if and only if there exists a normal subgroup H in G such that $G/H \in \mathfrak{F}$ and every subgroup of H of prime order or of order 4 (if $p = 2$) is c-normal in G.*

Corollary 11. ([1, Theorem 3.6] and [24, Theorem 3]) *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. $G \in \mathfrak{F}$ if and only if there exists a normal solvable subgroup H in G such that $G/H \in \mathfrak{F}$ and every subgroup of $F(H)$ of prime order or of order 4 (if $p = 2$) is c-normal in G.*

Corollary 12. ([21, Theorem 3.2]) *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. If G has a normal subgroup H such that $G/H \in \mathfrak{F}$ and every subgroup of $F^*(H)$ of prime order or of order 4 is c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 13. ([19, Theorem 3.1]) *Let H be a normal subgroup of a group G such that G/H is nilpotent and every cyclic subgroup of order 4 of $F^*(H)$ is c-normal in G, then G is nilpotent if and only if every subgroup of prime order of $F^*(H)$ is contained in the hypercenter $Z_\infty(G)$ of G.*

Corollary 14. *Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of a group G. If every subgroup of P of prime order or of order 4 (if $p = 2$) is SS-quasinormal in G, then G is p-nilpotent.*

Corollary 15. ([11, Theorem 3.4]) *Let G be a group such that every subgroup of G of prime order or of order 4 (if $p = 2$) is SS-quasinormal in G, then G is supersolvable.*

Corollary 16. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. $G \in \mathfrak{F}$ if and only if there exists a normal subgroup H in G such that $G/H \in \mathfrak{F}$ and every subgroup of H of prime order or of order 4 (if $p = 2$) is SS-quasinormal in G.*

Corollary 17. ([11, Theorem 3.5]) *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. $G \in \mathfrak{F}$ if and only if there exists a normal solvable subgroup H in G such that $G/H \in \mathfrak{F}$ and every subgroup of $F(H)$ of prime order or of order 4 (if $p = 2$) is SS-quasinormal in G.*

Corollary 18. ([11, Theorem 3.6]) *Let G be a group. If G has a normal subgroup H such that G/H is supersolvable and every subgroup of $F^*(H)$ of prime order or of order 4 is SS-quasinormal in G, then G is supersolvable.*

Corollary 19. ([11, Theorem 3.7]) *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup H such that $G/H \in \mathfrak{F}$ and every subgroup of $F^*(H)$ of prime order or of order 4 is SS-quasinormal in G.*

Corollary 20. ([11, Theorem 4.1]) *Let G be a group. If every subgroup of G of prime order is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of G is SS-quasinormal in G or lies in $Z_\infty(G)$, then G is nilpotent.*

Corollary 21. ([11, Theorem 4.2]) *Let H be a normal subgroup of a group G such that G/H is nilpotent. If every subgroup of H of prime order is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of H is SS-quasinormal in G or lies in $Z_\infty(G)$, then G is nilpotent.*

Corollary 22. ([11, Theorem 4.4]) *Let H be a normal subgroup of a group G such that G/H is nilpotent and every cyclic subgroup of order 4 of $F^*(H)$ is SS-quasinormal in G, then G is nilpotent if and only if every subgroup of prime order of $F^*(H)$ is contained in the hypercenter $Z_\infty(G)$ of G.*

Based on the results that have been achieved in this paper and [25, 26], the following questions arise:

Question 1. *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime dividing $|G|$. Assume that all maximal subgroups of P are CSS-subgroups of G. Is G p-nilpotent?*

Question 2. *Assume that all maximal subgroups of every Sylow subgroup of a group G are CSS-subgroups of G. Is G supersolvable?*

Question 3. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and H a normal subgroup of G such that $G/H \in \mathfrak{F}$. Assume that every non-cyclic Sylow subgroup P of H has a subgroup D with $1 < |D| < |P|$ such that every subgroup of P of order $|D|$ (and 4 if $|D| = 2$) is CSS-subgroup of G. Is $G \in \mathfrak{F}$?*

Question 4. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and H a normal subgroup of G such that $G/H \in \mathfrak{F}$. Assume that every non-cyclic Sylow subgroup P of $F^*(H)$ has a subgroup D with $1 < |D| < |P|$ such that every subgroup of P of order $|D|$ (and 4 if $|D| = 2$) is CSS-subgroup of G. Is $G \in \mathfrak{F}$?*

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