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Quantum codes obtained through constacyclic codes over $\mathbb{Z}_3 + \nu\mathbb{Z}_3 + \omega\mathbb{Z}_3 + \nu\omega\mathbb{Z}_3$

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Abstract. This paper is concerned with, structural properties and construction of quantum codes over Z_3 by using constacyclic codes over the finite commutative non-chain ring $\Re = Z_3 + \nu Z_3 + \omega Z_3 + \nu\omega Z_3$ where $\nu^2 = 1$, $\omega^2 = 1$, $\nu\omega = \nu\omega$ and Z_3 is field having 3 elements with characteristic 3. A Gray map is defined between $\Re$ and Z_3^4. The parameters of quantum codes over Z_3 are obtained by decomposing constacyclic codes into cyclic and negacyclic codes over Z_3. As an application, some examples of quantum codes of arbitrary length, are also obtained.

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1. Introduction

Quantum error correction shows an significant role in quantum computing as it is used to correct any errors in quantum data due to decoherence and other quantum noise. Firstly, the existence of quantum error correction code was proven by Shor [14] and individually by Steane [17]. In 1998, Calderbank et. al [3] published a paper in which they developed the theory to construct quantum codes using classical error correction codes. In current years, an essential literature has been established about the quantum error correcting codes. Some authors constructed quantum codes using the Gray image of cyclic codes on some finite rings. For example, a new technique of constructing quantum codes from cyclic codes over finite ring $F_2 + vF_2$ where $v^2 = v$ given by Qian [13]. Kai and Zhu [8] created quantum codes from hermitian self orthogonal codes over F_4 as Gray images of linear and cyclic codes over $F_4 + uF_4$ where $u^2 = 0$. Yin and Ma [18] gave a condition for the existence of quantum codes from cyclic codes over $F_2 + uF_2 + u^2 + F_2$ with Lee

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metric. Some non binary quantum codes are described using classical codes via Gaussian integers by Ozen et. al [11]. Guenda et. al [5] extended the CSS construction to finite commutative Frobenius rings. Dertli et. al [4] received quantum codes from cyclic codes over $F_2 + uF_2 + vF_2 + uvF_2$. Ashraf and Mohammad [1] gave a construction of quantum codes from cyclic codes over$F_3 + vF_3$ where $v^2 = 1$. In 2016, Ozen et al. [12] examined several ternary quantum codes from the cyclic codes over $F_3 + uF_3 + vF_3 + uvF_3$. Very recently, several researchers established a number of new quantum codes via F_p from the classical cyclic and constacyclic codes to which we refer [2, 6, 9–11, 15]. Also, Singh and Mor [16] constructed quantum codes over the finite non-chain ring $\Re = Z_p + \nu Z_p$ where $\nu^2 = \nu$ in 2021.
The remaining paper is arranged as follows, Section 2 contains preliminaries in which some fundamental properties and some essential definitions have been given. Section 3 defines the Gray map from $\Re$ to Z_3^4 and some related properties like self orthogonal, self dual etc. In Section 4, we presented the development of quantum codes through constacyclic codes over the ring $\Re$ which are exemplified in Section 5. Finally, the paper is concluded in the last Section.

2. Preliminaries

Let $\mathbb{Z}_3$ is a finite filed with 3 elements. Now, we first start with a general overview of the ring $\Re = \mathbb{Z}_3 + \nu\mathbb{Z}_3 + \omega\mathbb{Z}_3 + \nu\omega\mathbb{Z}_3$ having characteristic 3 with restrictions $\nu^2 = 1$, $\omega^2 = 1$ and $\nu\omega = \nu\omega$. $\Re$ is a commutative, principal ideal but non-chain finite ring with $3^4 = 81$ elements. The maximal ideal of $\Re$ are

$$< 2 + \nu + 2\omega >, \quad < 2 + \nu + \nu\omega >, \quad < \nu + \omega + 2\nu\omega >, \quad < 2\nu + 2\omega + 2\nu\omega > .$$

Some units of $\Re$ is $1 + \nu + \omega + 2\nu\omega$, $1 + 2\nu + 2\omega + 2\nu\omega$, $\nu\omega$, ν, ω for sake of simplicity we consider ϑ is a unit of $\Re$ and also we note that $\vartheta^{-1} = \vartheta$ for each case.
Let us assume

$$\xi_1 = 1+\nu+\omega+\nu\omega,\ \xi_2 = 1-\nu+\omega-\nu\omega,\ \xi_3 = 1+\nu-\omega-\nu\omega \text{ and } \xi_4 = 1-\nu-\omega+\nu\omega.$$

It is obvious to obtain that $\xi_i^2 = \xi_i$, $\xi_i\xi_j = 0$ and $\sum_{i=1}^{4} \xi_i = 1$ for all $i, j = 1, 2, 3, 4$ and $i \neq j$. Now by chinese remainder theorem, the considered ring can be expressed as

$$\Re = \xi_1\mathbb{Z}_3 \oplus \xi_2\mathbb{Z}_3 \oplus \xi_3\mathbb{Z}_3 \oplus \xi_4\mathbb{Z}_3.$$

Therefore, an arbitrary element $e = e_1 + \nu e_2 + \omega e_3 + \nu\omega e_4$ of $\Re$ where $e_i \in \mathbb{Z}_3$ can be uniquely expressed as

$$e = e_1 + \nu e_2 + \omega e_3 + \nu\omega e_4 = \xi_1 k_1 + \xi_2 k_2 + \xi_3 k_3 + \xi_4 k_4$$

where $k_i \in \mathbb{Z}_3$ for all $i = 1, 2, 3, 4$.
A nonempty subset $\mathcal{K}$ of $\Re^n$ is a linear code over $\Re$ of length n. If $\mathcal{K}$ is an $\Re$-submodule

of $\Re^n$ and the elements of $\mathcal{K}$ are codewords. Let $\mathcal{K}$ be a code over $\Re$ of length n and its polynomial representation be $T(\mathcal{K})$, that is,

$$T(\mathcal{K}) \ = \ \{\sum_{i=0}^{n-1} \chi_i t^i \mid (\chi_0, \chi_1, ..., \chi_{n-1}) \ \in \ \mathcal{K}\}$$

Let Υ, Λ and $\mho$ be the maps from $\Re^n$ to $\Re^n$ defined as

$$\Upsilon(\chi_0, \chi_1, ..., \chi_{n-1}) = (\chi_{n-1}, \chi_0, ..., \chi_{n-2}),$$

$$\Lambda(\chi_0, \chi_1, ..., \chi_{n-1}) = (-\chi_{n-1}, \chi_0, ..., \chi_{n-2}),$$

$$\mho(\chi_0, \chi_1, ..., \chi_{n-1}) = (\vartheta\chi_{n-1}, \chi_0, ..., \chi_{n-2}),$$

respectively. Then $\mathcal{K}$ is a cyclic, negacyclic, ϑ-constacyclic if $\Upsilon(\mathcal{K}) = \mathcal{K}$, $\Lambda(\mathcal{K}) = \mathcal{K}$, $\mho(\mathcal{K}) = \mathcal{K}$ respectively. A code $\mathcal{K}$ over $\Re$ of length n is cyclic, negacyclic and ϑ-constacyclic if and only if $T(\mathcal{K})$ is an ideal of $\Re[t]/ < t^n - 1 >$, $\Re[t]/ < t^n + 1 >$ and $\Re[t]/ < t^n - \vartheta >$ respectively.

For the arbitrary elements $\chi = (\chi_0, \chi_1, ..., \chi_{n-1})$ and $\psi = (\psi_0, \psi_1, ..., \psi_{n-1})$ of $\Re$, the inner product is defined as

$$\chi.\psi \ = \ \sum_{i=0}^{n-1} \chi_i \psi_i.$$

If $\chi.\psi = 0$, then χ and ψ are orthogonal. If $\mathcal{K}$ is a linear code over $\Re$ of length n, then the dual code of $\mathcal{K}$ is defined as

$$\mathcal{K}^{\perp} = \{ \ \chi \ \in \ \Re^n : \ \chi.\psi = 0 \ for \ all \ \psi \in \mathcal{K}\},$$

which is also a linear code over the ring $\Re$ of length n. A code $\mathcal{K}$ is said to be self orthogonal if $\mathcal{K} \subseteq \mathcal{K}^{\perp}$ and said to be self dual if $\mathcal{K} = \mathcal{K}^{\perp}$.

3. Gray Map over $\Re$

The hamming weight $w_H(\chi)$ for any codeword $\chi = (\chi_0, \chi_1, ..., \chi_{n-1}) \in \Re^n$ is defined as the number of all non-zero components in $\chi = (\chi_0, \chi_1, ..., \chi_{n-1})$. The minimum weight of a code $\mathcal{K}$, that is, $w_H(\mathcal{K})$ is the least weight among all of its non zero codewords. The Hamming distance between two codes $\chi = (\chi_0, \chi_1, ..., \chi_{n-1})$ and $\hat{\chi} = (\hat{\chi}_0, \hat{\chi}_1, ..., \hat{\chi}_{n-1})$ of $\Re^n$, denoted by $d_H(\chi, \hat{\chi}) = w_H(\chi - \hat{\chi})$ and is defined as

$$d_H(\chi, \psi) \ = \ \mid \{i \mid \chi_i \neq \psi_i\} \mid .$$

Minimum distance of $\mathcal{K}$, denoted by d_H and is given by minimum distance between the different pairs of codewords of the linear code $\mathcal{K}$. For any codeword $\chi = (\chi_0, \chi_1, ..., \chi_{n-1})$ $\in \Re^n$, the lee weight is defined as $w_L(\chi) = \sum_{i=0}^{n-1} w_L(\chi_i)$ and lee distance of $(\chi, \hat{\chi})$ is given by $d_L(\chi, \hat{\chi}) = w_L(\chi - \hat{\chi}) = \sum_{i=0}^{n-1} w_L(\chi_i - \hat{\chi}_i)$.

Minimum lee distance of $\mathcal{K}$ is denoted by d_L and is given by minimum lee distance of

different pairs of codewords of the linear code $\mathcal{K}$.
The Gray map φ from $\Re$ to $\mathbb{Z}_3^4$, that is, φ: $\Re \rightarrow \mathbb{Z}_3^4$ is defined as

$$\varphi(k \;=\; \xi_1 k_1 + \xi_2 k_2 + \xi_3 k_3 + \xi_4 k_4) = (k_1 + k_2 + k_3 + k_4, k_1 - k_2 + k_3 - k_4, \\ k_1 + k_2 - k_3 - k_4, k_1 - k_2 - k_3 + k_4).$$

Proposition 1. *The Gray map φ is linear and distance preserving isometry map from $(\Re^n, d_L)$ to $(\mathbb{Z}_3^{4n}, d_H)$, where d_L and d_H are the lee distance and hamming distance in $\Re^n$ and $\mathbb{Z}_3^{4n}$ respectively.*

Proof. Let $k_1,\ k_2\ \in\ \Re$ and $\alpha\ \in\ \mathbb{Z}_3$ then

$$\varphi(\alpha k_1 + k_2) \;=\; \alpha\varphi(k_1) + \varphi(k_1)$$

So, φ is linear map.
Now we show that φ is distance preserving map.
By the above definitions, $d_L(\chi, \hat{\chi}) = w_L(\chi - \hat{\chi}) = w_H(\varphi(\chi - \hat{\chi})) = w_H(\varphi(\chi) - \phi(\hat{\chi})) = d_H(\varphi(\chi), \varphi(\hat{\chi}))$.
Hence φ is distance preserving map from $(\Re^n, d_L)$ to $(\mathbb{Z}_3^{4n}, d_H)$.

Proposition 2. *If $\mathcal{K}$ is a linear code over the ring $\Re$ of length n with $|\mathcal{K}| = 3^k$, $d_L(\mathcal{K}) = d$, then $\varphi(\mathcal{K})$ is a ternary linear code having parameters $[4n,\ k,\ d]$.*

Proposition 3. *Let $\mathcal{K}$ be a linear code over the ring $\Re$ of length n. If $\mathcal{K}$ is self orthogonal, then $\varphi(\mathcal{K})$ is also self orthogonal.*

Proof. Let $\mathcal{K}$ be a self orthogonal code and $\eta_1, \eta_2 \in \mathcal{K}$ such that $\eta_1 = \xi_1 a + \xi_2 b + \xi_3 c + \xi_4 d$ and $\eta_2 = \xi_1 a^{'} + \xi_2 b^{'} + \xi_3 c^{'} + \xi_4 d^{'}$ where $a, b, c, d, a^{'}, b^{'}, c^{'}, d^{'} \in \mathbb{Z}_3$ from the definition of self orthogonality, $\eta_1 . \eta_2 = 0$, that is, $\xi_1 a a^{'} + \xi_2 b b^{'} + \xi_3 c c^{'} + \xi_4 d d^{'} = 0$, it follows that $a a^{'} = b b^{'} = c c^{'} = d d^{'} = 0$. Now, applying φ on η_1, η_2, we have $\varphi(\eta_1)$ = (a+b+c+d, a-b+c-d, a+b-c-d, a-b-c+d) and $\varphi(\eta_2) = (a^{'}+b^{'}+c^{'}+d^{'}, a^{'}-b^{'}+c^{'}-d^{'}, a^{'}+b^{'}-c^{'}-d^{'}, a^{'}-b^{'}-c^{'}+d^{'})$ and hence $\varphi(\eta_1).\varphi(\eta_2) = 4a.a^{'} + 4bb' + 4cc' + 4dd' = 0$ that implies $\varphi(\mathcal{K})$ is self orthogonal.

Proposition 4. *[7] Let $\mathcal{K}$ be a linear code over the ring $\Re$ of length n. Then $\varphi(\mathcal{K}^{\perp}) = (\varphi(\mathcal{K}))^{\perp}$. Further, $\mathcal{K}$ is self dual if and only if $\varphi(\mathcal{K})$ is self dual.*

4. Quantum codes obtained through ϑ-constacyclic codes

Let $S_i^{'}s$ be the linear codes for $i \;=\; 1, 2, 3, 4$. we denote

$$S_1 \;\oplus\; S_2 \;\oplus\; S_3 \;\oplus\; S_4 \;=\; \{s_1 + s_2 + s_3 + s_4 \mid s_i \;\in S_i \; for \; i \;=\; 1, 2, 3, 4\}$$

and

$$S_1 \;\otimes\; S_2 \;\otimes\; S_3 \;\otimes\; S_4 \;=\; \{(s_1,\ s_2,\ s_3,\ s_4) \mid s_i \;\in S_i \; for \; i \;=\; 1, 2, 3, 4\}$$

For a linear code $\mathcal{K}$ of length n over $\Re$, we define

$$\mathcal{K}_\infty = \{s_1 + s_2 + s_3 + s_4 \in \mathbb{Z}_3^n \text{ such that } s_1 + s_2\nu + s_3\omega + s_4\nu\omega \in \mathcal{K}\},$$

$$\mathcal{K}_\in = \{s_1 - s_2 + s_3 - s_4 \in \mathbb{Z}_3^n \text{ such that } s_1 + s_2\nu + s_3\omega + s_4\nu\omega \in \mathcal{K}\},$$

$$\mathcal{K}_\ni = \{s_1 + s_2 - s_3 - s_4 \in \mathbb{Z}_3^n \text{ such that } s_1 + s_2\nu + s_3\omega + s_4\nu\omega \in \mathcal{K}\},$$

$$\mathcal{K}_\triangle = \{s_1 - s_2 - s_3 + s_4 \in \mathbb{Z}_3^n \text{ such that } s_1 + s_2\nu + s_3\omega + s_4\nu\omega \in \mathcal{K}\}.$$

Clearly, $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$ and $\mathcal{K}_\triangle$ are the linear codes over $\mathbb{Z}_3$ of length n.

Theorem 5. *[7] Let $\mathcal{K}$ be a linear code over the ring $\Re$ of length n. Then $\varphi(\mathcal{K}) = \mathcal{K}_\infty \otimes \mathcal{K}_\in \otimes \mathcal{K}_\ni \otimes \mathcal{K}_\triangle$ and $|\mathcal{K}| = |\mathcal{K}_\infty||\mathcal{K}_\in||\mathcal{K}_\ni|\mathcal{K}_\triangle|$.*

Corollary 6. *[7] If $\varphi(\mathcal{K}) = \mathcal{K}_\infty \otimes \mathcal{K}_\in \otimes \mathcal{K}_\ni \otimes \mathcal{K}_\triangle$ then $\mathcal{K} = \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle$*

By the help of Theorem 4.1 and Corollary 4.2, we say that the linear code $\mathcal{K}$ can be uniquely expressed as

$$\mathcal{K} = \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle$$

and also

$$|\mathcal{K}| = |\mathcal{K}_\infty||\mathcal{K}_\in||\mathcal{K}_\ni|\mathcal{K}_\triangle|.$$

If G_1, G_2, G_3 and G_4, are the generator matrices of the linear codes $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$, and $\mathcal{K}_\triangle$ respectively. Then, the generator matrix of $\mathcal{K}$ is

$$G = \left[\begin{array}{cccc} \xi_1 G_1 & \xi_2 G_2 & \xi_3 G_3 & \xi_4 G_4 \end{array}\right]^T,$$

and that of $\varphi(\mathcal{K})$ is

$$\varphi(G) = \left[\begin{array}{cccc} \varphi(\xi_1 G_1) & \varphi(\xi_2 G_2) & \varphi(\xi_3 G_3) & \varphi(\xi_4 G_4) \end{array}\right]^T$$

Note: Now, we consider different case of ϑ.

Case 1. $\vartheta = 1 + \nu + \omega + 2\nu\omega$

Theorem 7. *Let $\mathcal{K} = \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle$ be a linear code over the ring $\Re$ of length n where $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$, $\mathcal{K}_\triangle$ are the linear codes over $\mathbb{Z}_3$. Then, $\mathcal{K}$ is a ϑ-constacyclic code over the ring $\Re$ of length n if and only if $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$ are negacyclic codes and $\mathcal{K}_\triangle$ is cyclic code over $\mathbb{Z}_3$ of length n.*

Proof.
Let,

$$\dot{a} = (\dot{a_0}, \dot{a_1}, ..., \dot{a_{n-1}}) \in \mathcal{K}_\infty, \quad \dot{b} = (\dot{b_0}, \dot{b_1}, ..., \dot{b_{n-1}}) \in \mathcal{K}_\in,$$
$$\dot{c} = (\dot{c_0}, \dot{c_1}, ..., \dot{c_{n-1}}) \in \mathcal{K}_\ni, \quad \dot{d} = (\dot{d_0}, \dot{d_1}, ..., \dot{d_{n-1}}) \in \mathcal{K}_\triangle.$$

For an arbitrary element $\zeta_i \in \mathcal{K}$, uniquely expressed as

$$\zeta_i = \xi_1\dot{a}_i + \xi_2\dot{b}_i + \xi_3\dot{c}_i + \xi_4\dot{d}_i,$$

where $\dot{a}_i, \dot{b}_i, \dot{c}_i, \dot{d}_i \in \mathbb{Z}_3$ for $i = 0, 1, ..., n-1$.
Let,

$$\zeta = (\zeta_0, \zeta_1, ..., \zeta_{n-1}) \in \Re^n.$$

First we assume that $\mathcal{K}$ is ϑ-constacyclic code over the ring $\Re$ of length n, then

$$\begin{aligned}
\mho(\zeta) &= ((1+\nu+\omega+2\nu\omega)\zeta_{n-1}, \zeta_0, ..., \zeta_{n-2}) \\
&= (-(1+\nu+\omega+\nu\omega)\dot{a_{n-1}} - (1-\nu+\omega-\nu\omega)\dot{b_{n-1}} - (1+\nu-\omega-\nu\omega)\dot{c_{n-1}} \\
&\quad + (1-\nu-\omega+\nu\omega)\dot{d_{n-1}}, (1+\nu+\omega+\nu\omega)\dot{a_0} + (1-\nu+\omega-\nu\omega)\dot{b_0} + (1+\nu-\omega-\nu\omega)\dot{c_0} \\
&\quad + (1-\nu-\omega+\nu\omega)\dot{d_0}, ..., (1+\nu+\omega+\nu\omega)\dot{a_{n-2}} + (1-\nu+\omega-\nu\omega)\dot{b_{n-2}} \\
&\quad + (1+\nu-\omega-\nu\omega)\dot{c_{n-2}} + (1-\nu-\omega+\nu\omega)\dot{d_{n-2}}) \\
&= (1+\nu+\omega+\nu\omega)\Lambda(\dot{a}) + (1-\nu+\omega-\nu\omega)\Lambda(\dot{b}) + (1+\nu-\omega-\nu\omega)\Lambda(\dot{c}) \\
&\qquad + (1-\nu-\omega+\nu\omega)\Upsilon(\dot{d}) \\
&= \xi_1\Lambda(\dot{a}) + \xi_2\Lambda(\dot{b}) + \xi_3\Lambda(\dot{c}) + \xi_4\Upsilon(\dot{d}),
\end{aligned}$$

which is an element of the linear code $\mathcal{K}$. Therefore, $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$, are negacyclic codes and $\mathcal{K}_\triangle$ is a cyclic code over the ring $\mathbb{Z}_3$ of length n respectively.
Conversely, for any $\zeta = (\zeta_0, \zeta_1, ..., \zeta_{n-1}) \in \mathcal{K}$, where $\zeta_i = \xi_1\dot{a}_i + \xi_2\dot{b}_i + \xi_3\dot{c}_i + \xi_4\dot{d}_i$ and $\dot{a}_i, \dot{b}_i, \dot{c}_i, \dot{d}_i \in \mathbb{Z}_3$ for $i = 0, 1, ..., n-1$. If $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$ are negacyclic codes and $\mathcal{K}_\triangle$ is a cyclic code over $\mathbb{Z}_3$ of length n, then $\Lambda(\dot{a}) \in \mathcal{K}_\infty$, $\Lambda(\dot{b}) \in \mathcal{K}_\in$, $\Lambda(\dot{c}) \in \mathcal{K}_\ni$, $\Upsilon(\dot{d}) \in \mathcal{K}_\triangle$. Hence, we have

$$(1+\nu+\omega+\nu\omega)\Lambda(\dot{a})+(1-\nu+\omega-\nu\omega)\Lambda(\dot{b})+(1+\nu-\omega-\nu\omega)\Lambda(\dot{c})+(1-\nu-\omega+\nu\omega)\Upsilon(\dot{d}) \in \mathcal{K}$$

where it given that

$$\mho(\zeta) = (1+\nu+\omega+\nu\omega)\Lambda(\dot{a})+(1-\nu+\omega-\nu\omega)\Lambda(\dot{b})+(1+\nu-\omega-\nu\omega)\Lambda(\dot{c})+(1-\nu-\omega+\nu\omega)\Upsilon(\dot{d}),$$

which implies that $\mho(\zeta) \in \mathcal{K}$.
Therefore, $\mathcal{K}$ is a ϑ-constacyclic code over the ring $\Re$ of length n .

Case 2. $\vartheta = 1 + 2\nu + 2\omega + 2\nu\omega$

Theorem 8. *Let $\mathcal{K} = \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle$ be a linear code over the ring $\Re$ of length n where $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$, $\mathcal{K}_\triangle$ are the linear codes over $\mathbb{Z}_3$. Then, $\mathcal{K}$ is a ϑ-constacyclic code over the ring $\Re$ of length n if and only if $\mathcal{K}_\infty$ is cyclic code and $\mathcal{K}_\in$, $\mathcal{K}_\ni$, $\mathcal{K}_\triangle$ are negacyclic codes over $\mathbb{Z}_3$ of length n.*

Proof.
Let,

$$\dot{a} = (\dot{a_0}, \dot{a_1}, ..., \dot{a_{n-1}}) \in \mathcal{K}_\infty, \quad \dot{b} = (\dot{b_0}, \dot{b_1}, ..., \dot{b_{n-1}}) \in \mathcal{K}_\in,$$

$$\dot{c} \ = \ (\dot{c_0}, \dot{c_1}, ..., \dot{c_{n-1}}) \ \in \ \mathcal{K}_{\ni}, \qquad \dot{d} \ = \ (\dot{d_0}, \dot{d_1}, ..., \dot{d_{n-1}}) \ \in \ \mathcal{K}_{\triangle}.$$

For an arbitrary element $\zeta_i \in \mathcal{K}$, uniquely expressed as

$$\zeta_i \ = \ \xi_1 \dot{a}_i + \xi_2 \dot{b}_i + \xi_3 \dot{c}_i + \xi_4 \dot{d}_i,$$

where $\dot{a}_i, \dot{b}_i, \dot{c}_i, \dot{d}_i \in \mathbb{Z}_3$ for $i = 0, 1, ..., n-1$.
Let,

$$\zeta \ = \ (\zeta_0, \zeta_1, ..., \zeta_{n-1}) \ \in \ \Re^n.$$

First we assume that $\mathcal{K}$ is ϑ-constacyclic code over the ring $\Re$ of length n, then

$$\begin{aligned}
\mho(\zeta) &= ((1+2\nu+2\omega+2\nu\omega)\zeta_{n-1}, \zeta_0, ..., \zeta_{n-2}) \\
&= ((1+\nu+\omega+\nu\omega)\dot{a_{n-1}} - (1-\nu+\omega-\nu\omega)\dot{b_{n-1}} - (1+\nu-\omega-\nu\omega)\dot{c_{n-1}} \\
&\quad - (1-\nu-\omega+\nu\omega)\dot{d_{n-1}}, (1+\nu+\omega+\nu\omega)\dot{a_0} + (1-\nu+\omega-\nu\omega)\dot{b_0} + (1+\nu-\omega-\nu\omega)\dot{c_0} \\
&\quad + (1-\nu-\omega+\nu\omega)\dot{d_0}, ..., (1+\nu+\omega+\nu\omega)\dot{a_{n-2}} + (1-\nu+\omega-\nu\omega)\dot{b_{n-2}} \\
&\quad + (1+\nu-\omega-\nu\omega)\dot{c_{n-2}} + (1-\nu-\omega+\nu\omega)\dot{d_{n-2}}) \\
&= (1+\nu+\omega+\nu\omega)\Upsilon(\dot{a}) + (1-\nu+\omega-\nu\omega)\Lambda(\dot{b}) + (1+\nu-\omega-\nu\omega)\Lambda(\dot{c}) \\
&\qquad\qquad + (1-\nu-\omega+\nu\omega)\Lambda(\dot{d}) \\
&= \xi_1\Upsilon(\dot{a}) + \xi_2\Lambda(\dot{b}) + \xi_3\Lambda(\dot{c}) + \xi_4\Lambda(\dot{d}),
\end{aligned}$$

which is an element of the linear code $\mathcal{K}$. Therefore, $\mathcal{K}_\infty$ is cyclic code and $\mathcal{K}_\in$, $\mathcal{K}_\ni$, $\mathcal{K}_\triangle$ are negacyclic codes over $\mathbb{Z}_3$ of length n respectively.
Conversely, for any $\zeta = (\zeta_0, \zeta_1, ..., \zeta_{n-1}) \in \mathcal{K}$, where $\zeta_i = \xi_1\dot{a}_i + \xi_2\dot{b}_i + \xi_3\dot{c}_i + \xi_4\dot{d}_i$ and $\dot{a}_i, \dot{b}_i, \dot{c}_i, \dot{d}_i \in \mathbb{Z}_3$ for $i = 0, 1, ..., n-1$. If $\mathcal{K}_\infty$ is cyclic code and $\mathcal{K}_\in$, $\mathcal{K}_\ni$, $\mathcal{K}_\triangle$ are negacyclic codes over $\mathbb{Z}_3$ of length n., then $\Upsilon(\dot{a}) \in \mathcal{K}_\infty$, $\Lambda(\dot{b}) \in \mathcal{K}_\in$, $\Lambda(\dot{c}) \in \mathcal{K}_\ni$, $\Lambda(\dot{d}) \in \mathcal{K}_\triangle$.
Hence, we have

$$(1+\nu+\omega+\nu\omega)\Upsilon(\dot{a}) + (1-\nu+\omega-\nu\omega)\Lambda(\dot{b}) + (1+\nu-\omega-\nu\omega)\Lambda(\dot{c}) + (1-\nu-\omega+\nu\omega)\Lambda(\dot{d}) \ \in \ \mathcal{K}$$

where it given that

$$\mho(\zeta) \ = \ (1+\nu+\omega+\nu\omega)\Upsilon(\dot{a}) + (1-\nu+\omega-\nu\omega)\Lambda(\dot{b}) + (1+\nu-\omega-\nu\omega)\Lambda(\dot{c}) + (1-\nu-\omega+\nu\omega)\Lambda(\dot{d}),$$

which implies that $\mho(\zeta) \ \in \ \mathcal{K}$.
Therefore, $\mathcal{K}$ is a ϑ-constacyclic code over the ring $\Re$ of length n .

Case 3. $\vartheta \ = \ \nu\omega$

Theorem 9. *Let $\mathcal{K} = \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle$ be a linear code over the ring $\Re$ of length n where $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$, $\mathcal{K}_\triangle$ are the linear codes over $\mathbb{Z}_3$. Then, $\mathcal{K}$ is a $\nu\omega$-constacyclic code over the ring $\Re$ of length n if and only if $\mathcal{K}_\infty$, $\mathcal{K}_\triangle$ are cyclic and $\mathcal{K}_\in$, $\mathcal{K}_\ni$ are negacyclic codes over $\mathbb{Z}_3$ of length n.*

Proof.
Let,

$$\dot{a} = (\dot{a_0}, \dot{a_1}, ..., \dot{a_{n-1}}) \in \mathcal{K}_\infty, \quad \dot{b} = (\dot{b_0}, \dot{b_1}, ..., \dot{b_{n-1}}) \in \mathcal{K}_\in,$$
$$\dot{c} = (\dot{c_0}, \dot{c_1}, ..., \dot{c_{n-1}}) \in \mathcal{K}_\ni, \quad \dot{d} = (\dot{d_0}, \dot{d_1}, ..., \dot{d_{n-1}}) \in \mathcal{K}_\triangle.$$

For an arbitrary element $\zeta_i \in \mathcal{K}$, uniquely expressed as

$$\zeta_i = \xi_1 \dot{a}_i + \xi_2 \dot{b}_i + \xi_3 \dot{c}_i + \xi_4 \dot{d}_i,$$

where $\dot{a}_i, \dot{b}_i, \dot{c}_i, \dot{d}_i \in \mathbb{Z}_3$ for $i = 0, 1, ..., n-1$.
Let,

$$\zeta = (\zeta_0, \zeta_1, ..., \zeta_{n-1}) \in \Re^n.$$

First we assume that $\mathcal{K}$ is $\nu\omega$-constacyclic code over the ring $\Re$ of length n, then

$$\begin{aligned}
\mho(\zeta) &= ((\nu\omega)\zeta_{n-1}, \zeta_0, ..., \zeta_{n-2}) \\
&= (\alpha(1+\nu+\omega+\nu\omega)\dot{a_{n-1}} - (1-\nu+\omega-\nu\omega)\dot{b_{n-1}} - (1+\nu-\omega-\nu\omega)\dot{c_{n-1}} \\
&\quad + (1-\nu-\omega+\nu\omega)\dot{d_{n-1}}, (1+\nu+\omega+\nu\omega)\dot{a_0} + (1-\nu+\omega-\nu\omega)\dot{b_0} + (1+\nu-\omega-\nu\omega)\dot{c_0} \\
&\quad + (1-\nu-\omega+\nu\omega)\dot{d_0}, ..., (1+\nu+\omega+\nu\omega)\dot{a_{n-2}} + (1-\nu+\omega-\nu\omega)\dot{b_{n-2}} \\
&\qquad + (1+\nu-\omega-\nu\omega)\dot{c_{n-2}} + (1-\nu-\omega+\nu\omega)\dot{d_{n-2}}) \\
&= (1+\nu+\omega+\nu\omega)\Upsilon(\dot{a}) + (1-\nu+\omega-\nu\omega)\Lambda(\dot{b}) + (1+\nu-\omega-\nu\omega)\Lambda(\dot{c}) \\
&\qquad + (1-\nu-\omega+\nu\omega)\Upsilon(\dot{d}) \\
&= \xi_1\Upsilon(\dot{a}) + \xi_2\Lambda(\dot{b}) + \xi_3\Lambda(\dot{c}) + \xi_4\Upsilon(\dot{d}),
\end{aligned}$$

which is an element of the linear code $\mathcal{K}$. Therefore, $\mathcal{K}_\infty$, $\mathcal{K}_\triangle$ are cyclic and $\mathcal{K}_\in$, $\mathcal{K}_\ni$, are negacyclic codes over the ring $\mathbb{Z}_3$ of length n respectively.
Conversely, for any $\zeta = (\zeta_0, \zeta_1, ..., \zeta_{n-1}) \in \mathcal{K}$, where $\zeta_i = \xi_1 \dot{a}_i + \xi_2 \dot{b}_i + \xi_3 \dot{c}_i + \xi_4 \dot{d}_i$ and $\dot{a}_i, \dot{b}_i, \dot{c}_i, \dot{d}_i \in \mathbb{Z}_3$ for $i = 0, 1, ..., n-1$. If $\mathcal{K}_\infty$, $\mathcal{K}_\triangle$ are cyclic and $\mathcal{K}_\in$, $\mathcal{K}_\ni$, are negacyclic codes over the ring $\mathbb{Z}_p$ of length n, then $\Upsilon(\dot{a}) \in \mathcal{K}_\infty$, $\Lambda(\dot{b}) \in \mathcal{K}_\in$, $\Lambda(\dot{c}) \in \mathcal{K}_\ni$, $\Upsilon(\dot{d}) \in \mathcal{K}_\triangle$. Hence, we have

$$(1+\nu+\omega+\nu\omega)\Upsilon(\dot{a})+(1-\nu+\omega-\nu\omega)\Lambda(\dot{b})+(1+\nu-\omega-\nu\omega)\Lambda(\dot{c})+(1-\nu-\omega+\nu\omega)\Upsilon(\dot{d}) \in \mathcal{K}$$

where it given that

$$\mho(\zeta) = (1+\nu+\omega+\nu\omega)\Upsilon(\dot{a})+(1-\nu+\omega-\nu\omega)\Lambda(\dot{b})+(1+\nu-\omega-\nu\omega)\Lambda(\dot{c})+(1-\nu-\omega+\nu\omega)\Upsilon(\dot{d}),$$

which implies that $\mho(\zeta) \in \mathcal{K}$.
Therefore, $\mathcal{K}$ is a $\nu\omega$-constacyclic code over the ring $\Re$ of length n .

Case 4. $\vartheta = \nu$

Theorem 10. *Let $\mathcal{K} = \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle$ be a linear code over the ring $\Re$ of length n where $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$, $\mathcal{K}_\triangle$ are the linear codes over $\mathbb{Z}_3$. Then, $\mathcal{K}$ is a ν-constacyclic code over the ring $\Re$ of length n if and only if $\mathcal{K}_\infty$, $\mathcal{K}_\ni$ are cyclic and $\mathcal{K}_\in$, $\mathcal{K}_\triangle$ are negacyclic codes over $\mathbb{Z}_3$ of length n.*

Proof. The proof of this theorem is similar to proof of Theorem 4.5.

Case 5. $\vartheta = \omega$

Theorem 11. *Let $\mathcal{K} = \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle$ be a linear code over the ring $\Re$ of length n where $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$, $\mathcal{K}_\triangle$ are the linear codes over $\mathbb{Z}_3$. Then, $\mathcal{K}$ is a ω-constacyclic code over the ring $\Re$ of length n if and only if $\mathcal{K}_\infty$, $\mathcal{K}_\in$ are cyclic and $\mathcal{K}_\ni$, $\mathcal{K}_\triangle$ are negacyclic codes over $\mathbb{Z}_3$ of length n.*

Proof. Proof of the theorem is similar to proof of theorem 4.6.

The following Theorem is Similar to Theorem 14 [7].

Theorem 12. *Let $\mathcal{K}$ be a ϑ-constacyclic code over the ring $\Re$ of length n. Then*

$$\mathcal{K} = <\xi_1 g_1(t), \xi_2 g_2(t), \xi_3 g_3(t), \xi_4 g_4(t)> = <\xi_1 g_1(t) + \xi_2 g_2(t) + \xi_3 g_3(t) + \xi_4 g_4(t)>$$

where $g_i(t)$ are the generator polynomials of $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$ and $\mathcal{K}_\triangle$ for $i = 1, 2, 3, 4$ respectively. Moreover, $|\mathcal{K}| = 3^{4n-\sum_{i=0}^{4} deg(g_i(t))}$

Theorem 13. *Let $\mathcal{K}$ be a ϑ-constacyclic code over the ring $\Re$ of length n. Then $\mathcal{K}^\perp$ is also a ϑ-constacyclic code over the ring $\Re$ of length n. Moreover,*

1. $\mathcal{K}^\perp = \xi_1\mathcal{K}_\infty^\perp \oplus \xi_2\mathcal{K}_\in^\perp \oplus \xi_3\mathcal{K}_\ni^\perp \oplus \xi_4\mathcal{K}_\triangle^\perp$
2. $\mathcal{K}^\perp = <\xi_1 g_1^\star(t), \xi_2 g_2^\star(t), \xi_3 g_3^\star(t), \xi_4 g_4^\star(t)> = <\xi_1 g_1^\star(t) + \xi_2 g_2^\star(t) + \xi_3 g_3^\star(t) + \xi_4 g_4^\star(t)>$
3. $|\mathcal{K}^\perp| = 3^{\sum_{i=1}^{4} deg(g_i(t))}$

where $g_i^\star(t)$ are the reciprocal polynomial of $\frac{x^n+1}{g_1(t)}, \frac{x^n+1}{g_2(t)}, \frac{x^n+1}{g_3(t)}$ and $\frac{x^n-1}{g_4(t)}$ for $i = 1, 2, 3, 4$ respectively.

Lemma 1. *[4] If $\mathcal{K}$ is a cyclic or negacyclic code over the ring $\mathbb{Z}_p$ with a generator polynomial g(t). Then, $\mathcal{K}$ contains its dual code if and only if*

$$x^n - \iota \equiv 0 \ mod(g(t)g^\star(t))$$

where $\iota = \pm 1$.

Case 1. $\vartheta = 1 + \nu + \omega + 2\nu\omega$

Theorem 14. *If $\mathcal{K} = <\xi_1 g_1(t) + \xi_2 g_2(t) + \xi_3 g_3(t) + \xi_4 g_4(t)>$ is a ϑ-constacyclic code over the ring $\Re$ of length n. Then, $\mathcal{K}^\perp \subseteq \mathcal{K}$ if and only if*

$$x^n + 1 \equiv 0 \ mod(g_i(t)g_i^\star(t))$$

and

$$x^n - 1 \equiv 0 \ mod(g_j(t)g_j^\star(t)).$$

for $i = 1, 2, 3$ and $j = 4$.

Proof.

Let $\mathcal{K} = < g(t) > = < \xi_1 g_1(t) + \xi_2 g_2(t) + \xi_3 g_3(t) + \xi_4 g_4(t) >$ be a ϑ-constacyclic code over $\Re$ of length n. Then, $\mathcal{K} = \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle$ where $g_i(t)$ are the generator polynomial of $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$ and $\mathcal{K}_\triangle$ for $i = 1, 2, 3, 4$ respectively.
First we consider

$$x^n + 1 \equiv 0 \; mod(g_i(t)g_i^\star(t))$$

and

$$x^n - 1 \equiv 0 \; mod(g_j(t)g_j^\star(t)).$$

for $i = 1, 2, 3$ and $j = 4$. Then by above lemma, we have

$$\mathcal{K}_\infty^\perp \subseteq \mathcal{K}_\infty, \; \mathcal{K}_\in^\perp \subseteq \mathcal{K}_\in, \mathcal{K}_\ni^\perp \subseteq \mathcal{K}_\ni \text{ and } \mathcal{K}_\triangle^\perp \subseteq \mathcal{K}_\triangle,$$

and therefore

$$\xi_1\mathcal{K}_\infty^\perp \subseteq \xi_1\mathcal{K}_\infty, \; \xi_2\mathcal{K}_\in^\perp \subseteq \xi_2\mathcal{K}_\in, \; \xi_3\mathcal{K}_\ni^\perp \subseteq \xi_3\mathcal{K}_\ni \text{ and } \xi_4\mathcal{K}_\triangle^\perp \subseteq \xi_4\mathcal{K}_\triangle$$

which implies that

$$\xi_1\mathcal{K}_\infty^\perp \oplus \xi_2\mathcal{K}_\in^\perp \oplus \xi_3\mathcal{K}_\ni^\perp \oplus \xi_4\mathcal{K}_\triangle^\perp \subseteq \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle$$

Thus, we have

$$\mathcal{K}^\perp \subseteq \mathcal{K}.$$

Conversely, let us consider

$$\mathcal{K}^\perp \subseteq \mathcal{K},$$

then

$$\xi_1\mathcal{K}_\infty^\perp \oplus \xi_2\mathcal{K}_\in^\perp \oplus \xi_3\mathcal{K}_\ni^\perp \oplus \xi_4\mathcal{K}_\triangle^\perp \subseteq \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle,$$

which implies that

$$\xi_1\mathcal{K}_\infty^\perp \subseteq \xi_1\mathcal{K}_\infty, \; \xi_2\mathcal{K}_\in^\perp \subseteq \xi_2\mathcal{K}_\in, \; \xi_3\mathcal{K}_\ni^\perp \subseteq \xi_3\mathcal{K}_\ni \text{ and } \xi_4\mathcal{K}_\triangle^\perp \subseteq \xi_4\mathcal{K}_\triangle,$$

that implies

$$\mathcal{K}_\infty^\perp \subseteq \mathcal{K}_\infty, \; \mathcal{K}_\in^\perp \subseteq \mathcal{K}_\in, \; \mathcal{K}_\ni^\perp \subseteq \mathcal{K}_\ni \text{ and } \mathcal{K}_\triangle^\perp \subseteq \mathcal{K}_\triangle.$$

Then by above lemma,

$$x^n + 1 \equiv 0 \; mod(g_i(t)g_i^\star(t))$$

and

$$x^n - 1 \equiv 0 \; mod(g_j(t)g_j^\star(t)).$$

for $i = 1, 2, 3$ and $j = 4$.

Case 2. $\vartheta = 1 + \nu + \omega + 2\nu\omega$

Theorem 15. *If* $\mathcal{K} \;=\; <\xi_1 g_1(t)+\xi_2 g_2(t)+\xi_3 g_3(t)+\xi_4 g_4(t)>$ *is a* ϑ*-constacyclic code over the ring* $\Re$ *of length* n*. Then,* $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ *if and only if*

$$x^n - 1 \;\equiv\; 0 \; mod(g_i(t) g_i^{\star}(t))$$

and

$$x^n + 1 \;\equiv\; 0 \; mod(g_j(t) g_j^{\star}(t)).$$

for $i \;=\; 1$ *and* $j \;=\; 2,\; 3,\; 4$.

Proof.
Let $\mathcal{K} = <g(t)> = <\xi_1 g_1(t)+\xi_2 g_2(t)+\xi_3 g_3(t)+\xi_4 g_4(t)>$ be a ϑ-constacyclic code over $\Re$ of length n. Then, $\mathcal{K} \;=\; \xi_1\mathcal{K}_\infty \,\oplus\, \xi_2\mathcal{K}_\in \,\oplus\, \xi_3\mathcal{K}_\ni \,\oplus\, \xi_4\mathcal{K}_\triangle$ where $g_i(t)$ are the generator polynomial of $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$ and $\mathcal{K}_\triangle$ for $i = 1, 2, 3, 4$ respectively.
First we consider

$$x^n - 1 \;\equiv\; 0 \; mod(g_i(t) g_i^{\star}(t))$$

and

$$x^n + 1 \;\equiv\; 0 \; mod(g_j(t) g_j^{\star}(t)).$$

for $i \;=\; 1$ and $j \;=\; 2,\; 3,\; 4$. Then by above lemma, we have

$$\mathcal{K}_\infty^{\perp} \subseteq \mathcal{K}_\infty,\; \mathcal{K}_\in^{\perp} \subseteq \mathcal{K}_\in, \mathcal{K}_\ni^{\perp} \subseteq \mathcal{K}_\ni \text{ and } \mathcal{K}_\triangle^{\perp} \subseteq \mathcal{K}_\triangle,$$

and therefore

$$\xi_1\mathcal{K}_\infty^{\perp} \subseteq \xi_1\mathcal{K}_\infty,\; \xi_2\mathcal{K}_\in^{\perp} \subseteq \xi_2\mathcal{K}_\in,\; \xi_3\mathcal{K}_\ni^{\perp} \subseteq \xi_3\mathcal{K}_\ni \text{ and } \xi_4\mathcal{K}_\triangle^{\perp} \subseteq \xi_4\mathcal{K}_\triangle$$

which implies that

$$\xi_1\mathcal{K}_\infty^{\perp} \oplus \xi_2\mathcal{K}_\in^{\perp} \oplus \xi_3\mathcal{K}_\ni^{\perp} \oplus \xi_4\mathcal{K}_\triangle^{\perp} \subseteq \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle$$

Thus, we have

$$\mathcal{K}^{\perp} \subseteq \mathcal{K}.$$

Conversely, let us consider

$$\mathcal{K}^{\perp} \subseteq \mathcal{K},$$

then

$$\xi_1\mathcal{K}_\infty^{\perp} \oplus \xi_2\mathcal{K}_\in^{\perp} \oplus \xi_3\mathcal{K}_\ni^{\perp} \oplus \xi_4\mathcal{K}_\triangle^{\perp} \subseteq \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle,$$

which implies that

$$\xi_1\mathcal{K}_\infty^{\perp} \subseteq \xi_1\mathcal{K}_\infty,\; \xi_2\mathcal{K}_\in^{\perp} \subseteq \xi_2\mathcal{K}_\in,\; \xi_3\mathcal{K}_\ni^{\perp} \subseteq \xi_3\mathcal{K}_\ni \text{ and } \xi_4\mathcal{K}_\triangle^{\perp} \subseteq \xi_4\mathcal{K}_\triangle,$$

that implies

$$\mathcal{K}_\infty^{\perp} \subseteq \mathcal{K}_\infty,\; \mathcal{K}_\in^{\perp} \subseteq \mathcal{K}_\in,\; \mathcal{K}_\ni^{\perp} \subseteq \mathcal{K}_\ni \text{ and } \mathcal{K}_\triangle^{\perp} \subseteq \mathcal{K}_\triangle.$$

Then by above lemma,

$$x^n - 1 \equiv 0 \; mod(g_i(t)g_i^{\star}(t))$$

and

$$x^n + 1 \equiv 0 \; mod(g_j(t)g_j^{\star}(t)).$$

for $i = 1$ and $j = 2, 3, 4$.

Case 3. $\vartheta = \nu\omega$

Theorem 16. *If* $\mathcal{K} = < \xi_1 g_1(t) + \xi_2 g_2(t) + \xi_3 g_3(t) + \xi_4 g_4(t) >$ *is a* ϑ*-constacyclic code over the ring* $\Re$ *of length* n*. Then,* $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ *if and only if*

$$x^n - 1 \equiv 0 \; mod(g_i(t)g_i^{\star}(t))$$

and

$$x^n + 1 \equiv 0 \; mod(g_j(t)g_j^{\star}(t)).$$

for $i = 1, 4$ *and* $j = 2, 3$.

Proof. Proof of the theorem is similar to proof of Theorem 4.12.

Case 4. $\vartheta = \nu$

Theorem 17. *If* $\mathcal{K} = < \xi_1 g_1(t) + \xi_2 g_2(t) + \xi_3 g_3(t) + \xi_4 g_4(t) >$ *is a* ϑ*-constacyclic code over the ring* $\Re$ *of length* n*. Then,* $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ *if and only if*

$$x^n - 1 \equiv 0 \; mod(g_i(t)g_i^{\star}(t))$$

and

$$x^n + 1 \equiv 0 \; mod(g_j(t)g_j^{\star}(t)).$$

for $i = 1, 3$ *and* $j = 2, 4$.

Proof. Proof of the theorem is similar to proof of theorem 4.13.

Case 5. $\vartheta = \omega$

Theorem 18. *If* $\mathcal{K} = < \xi_1 g_1(t) + \xi_2 g_2(t) + \xi_3 g_3(t) + \xi_4 g_4(t) >$ *is a* ϑ*-constacyclic code over the ring* $\Re$ *of length* n*. Then,* $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ *if and only if*

$$x^n - 1 \equiv 0 \; mod(g_i(t)g_i^{\star}(t))$$

and

$$x^n + 1 \equiv 0 \; mod(g_j(t)g_j^{\star}(t)).$$

for $i = 1, 2$ *and* $j = 3, 4$.

Proof. Proof of the theorem is similar to proof of theorem 4.14.

By above Theorems, we have the following Corollary for each case of ϑ.

Corollary 19. *Let* $\mathcal{K} = \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle$ *be a* ϑ*-constacyclic code over* $\Re$ *of length* n *where* $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$, $\mathcal{K}_\triangle$ *are linear codes of length* n *over the ring* $\mathbb{Z}_3$*. Then,* $\mathcal{K}^\perp \subseteq \mathcal{K}$ *if and only if* $\mathcal{K}_\infty^\perp \subseteq \mathcal{K}_\infty$, $\mathcal{K}_\in^\perp \subseteq \mathcal{K}_\in$, $\mathcal{K}_\ni^\perp \subseteq \mathcal{K}_\ni$ *and* $\mathcal{K}_\triangle^\perp \subseteq \mathcal{K}_\triangle$.

Lemma 2. *[4](CSS Construction) Let* $\mathcal{K}$ *be a linear code over the ring* $\mathbb{Z}_3$ *having parameters* $[n,\ k,\ d]$*. Then a quantum code having parameters* $[[n,\ 2k-n,\ \geq d]]_3$ *can be obtained if* $\mathcal{K}^\perp \subseteq \mathcal{K}$.

The following theorem defines the construction of quantum codes by the use of corollary 4.16 and Lemma 4.17.

Theorem 20. *If* $\mathcal{K} = \xi_1\mathcal{K}_\infty \oplus \xi_2\mathcal{K}_\in \oplus \xi_3\mathcal{K}_\ni \oplus \xi_4\mathcal{K}_\triangle = <\xi_1 g_1(t) + \xi_2 g_2(t) + \xi_3 g_3(t) + \xi_4 g_4(t)>$ *is a* ϑ*-constacyclic code over the ring* $\Re$ *of length* n *where* $g_i(t)$ *are the generator polynomials of* $\mathcal{K}_\infty$, $\mathcal{K}_\in$, $\mathcal{K}_\ni$ *and* $\mathcal{K}_\triangle$ *for* $i = 1,2,3,4$ *respectively. If* $\mathcal{K}_\infty^\perp \subseteq \mathcal{K}_\infty$, $\mathcal{K}_\in^\perp \subseteq \mathcal{K}_\in$, $\mathcal{K}_\ni^\perp \subseteq \mathcal{K}_\ni$ *and* $\mathcal{K}_\triangle^\perp \subseteq \mathcal{K}_\triangle$*, then* $\mathcal{K}^\perp \subseteq \mathcal{K}$ *and there exists a quantum code having parameters* $[4n,\ 2k-4n,\ \geq d_L]_3$ *where* k *is the dimension of linear code* $\varphi(\mathcal{K})$ *and* d_L *is minimum lee distance of a linear code* $\mathcal{K}$.

5. Examples

In this section some examples are provided to illustrate the main result. Here, the quantum codes through ϑ-constacyclic code over the ring $\Re = \mathbb{Z}_3 + \nu\mathbb{Z}_3 + \omega\mathbb{Z}_3 + \nu\omega\mathbb{Z}_3$ where $\nu^2 = 1$, $\omega^2 = 1$ and $\nu\omega = \omega\nu$ are obtains.

Example 1. *In* $\mathbb{Z}_3[t]$, $t^3 - 1 = (t+2)^3$ *and* $t^3 + 1 = (t+1)(t^2 - t + 1)$*. Now, let* $\mathcal{K}$ *be a* $1+\nu+\omega+2\nu\omega$*-constacyclic code over the ring* $\Re = \mathbb{Z}_3 + \nu\mathbb{Z}_3 + \omega\mathbb{Z}_3 + \nu\omega\mathbb{Z}_3$ *where* $\nu^2 = 1$, $\omega^2 = 1$ *and* $\nu\omega = \omega\nu$ *of length 3. Let* $g_1(t) = g_2(t) = g_3(t) = t+1$ *and* $g_4(t) = t^2+t+1$ *then* $g(t) = \xi_1(t+1)+\xi_2(t+1)+\xi_3(t+1)+\xi_4(t^2+t+1)$ *be the generator polynomial of* $\mathcal{K}$*. Since* $g_i(t)g_i^*(t)|t^3+1$ *for* $i = 1,2,3$ *respectively and* $g_4(t)g_4^*(t)|t^3-1$*, then by the use of Theorem 4.11, we get* $\mathcal{K}^\perp \subseteq \mathcal{K}$ *Further* $\varphi(\mathcal{K})$ *is a linear code over the ring* $\mathbb{Z}_3$ *having parameters* $[12,\ 7,\ 3]$*. Then, by the application of Theorem 4.18, we obtain the quantum code having parameters* $[12,\ 2,\ \geq 3]_3$.

Example 2. *In* $\mathbb{Z}_3[t]$, $t^6 - 1 = (t-1)^3(t+1)^3$ *and* $t^6 + 1 = (t^2+1)^3$*. Now, let* $\mathcal{K}$ *be a* $1+2\nu+2\omega+2\nu\omega$*-constacyclic code over the ring* $\Re = \mathbb{Z}_3+\nu\mathbb{Z}_3+\omega\mathbb{Z}_3+\nu\omega\mathbb{Z}_3$ *where* $\nu^2 = 1$, $\omega^2 = 1$ *and* $\nu\omega = \omega\nu$ *of length 6. Let* $g_1(t) = t+1$ *and* $g_2(t) = g_3(t) = g_4(t) = t^2+1$ *then* $g(t) = \xi_1(t+1) + \xi_2(t^2+1) + \xi_3(t^2+1) + \xi_4(t^2+1)$ *be the generator polynomial of* $\mathcal{K}$*. Since* $g_1(t)g_1^*(t)|t^6-1$ *and* $g_i(t)g_i^*(t)|t^6+1$ *for* $i = 2,3,4$ *respectively, then by the use of Theorem 4.12, we get* $\mathcal{K}^\perp \subseteq \mathcal{K}$ *Further* $\varphi(\mathcal{K})$ *is a linear code over the ring* $\mathbb{Z}_3$ *having parameters* $[24,\ 17,\ 3]$*. Then, by the application of Theorem 4.18, we obtain the quantum code having parameters* $[24,\ 10,\ \geq 3]_3$.

Example 3. *In* $\mathbb{Z}_3[t]$, $t^9 - 1 = (t-1)^9$ *and* $t^9 + 1 = (t+1)^9$*. Now, let* $\mathcal{K}$ *be a* $\nu\omega$*-constacyclic code over the ring* $\Re = \mathbb{Z}_3 + \nu\mathbb{Z}_3 + \omega\mathbb{Z}_3 + \nu\omega\mathbb{Z}_3$ *where* $\nu^2 = 1$, $\omega^2 = 1$ *and*

$\nu\omega = \omega\nu$ of length 9. Let $g_1(t) = g_4(t) = t-1$ and $g_2(t) = g_3(t) = t+1$ then $g(t) = \xi_1(t-1)+\xi_2(t+1)+\xi_3(t+1)+\xi_4(t-1)$ be the generator polynomial of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^9-1$ for $i = 1,4$ respectively and $g_j(t)g_j^*(t)|t^9+1$ for $j = 2,3$ respectively, then by the use of Theorem 4.13, we get $\mathcal{K}^\perp \subseteq \mathcal{K}$ Further $\varphi(\mathcal{K})$ is a linear code over the ring $\mathbb{Z}_3$ having parameters $[36,\ 32,\ 2]$. Then, by the application of Theorem 4.18, we obtain the quantum code having parameters $[36,\ 28,\ \geq 2]_3$.

Example 4. *In $\mathbb{Z}_3[t]$, $t^{12}-1 = (t+1)^3(t+2)^3(t^2+1)^3$ and $t^{12}+1 = (t^2+2t+2)^3(t^2+t+2)^3$. Now, let $\mathcal{K}$ be a ν-constacyclic code over $\Re = \mathbb{Z}_3+\nu\mathbb{Z}_3+\omega\mathbb{Z}_3+\nu\omega\mathbb{Z}_3$ where $\nu^2 = 1$, $\omega^2 = 1$ and $\nu\omega = \omega\nu$ and $\nu^2\omega^2 = \nu\omega$ of length 12. Let $g_1(t) = g_3(t) = t+1$ and $g_2(t) = g_4(t) = t^2+t+2$, $g(t) = \xi_1(t+1)+\xi_2(t^2+t+2)+\xi_3(t+1)+\xi_4(t^2+t+2)$ be the generator polynomials of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^{12}-1$ for $i = 1,3$ respectively and $g_j(t)g_j^*(t)|t^{12}+1$ for $j = 2,4$ respectively, then by the use of Theorem 4.14, we get $\mathcal{K}^\perp \subseteq \mathcal{K}$. Further $\varphi(\mathcal{K})$ is a linear code over $\mathbb{Z}_3$ having parameters $[48,\ 43,\ 3]$. Then, by the application of Theorem 4.18, we obtain the quantum code having parameters $[48,\ 38,\ \geq 3]_3$.*

Example 5. *In $\mathbb{Z}_3[t]$, $t^{15}-1 = (t+2)^3(t^4+t^3+t^2+t+1)^3$ and $t^{15}+1 = (t+1)^3(t^4+2t^3+t^2+2t+1)^3$. Now, let $\mathcal{K}$ be a ω-constacyclic code over $\Re = \mathbb{Z}_3+\nu\mathbb{Z}_3+\omega\mathbb{Z}_3+\nu\omega\mathbb{Z}_3$ where $\nu^2 = 1$, $\omega^2 = 1$ and $\nu\omega = \omega\nu$ and $\nu^2\omega^2 = \nu\omega$ of length 15. Let $g_1(t) = g_2(t) = t+2$ and $g_3(t) = g_4(t) = t+1$, $g(t) = \xi_1(t+2)+\xi_2(t+2)+\xi_3(t+1)+\xi_4(t+1)$ be the generator polynomials of $\mathcal{K}$.Since $g_i(t)g_i^*(t)|t^{15}-1$ for $i = 1,2$ respectively and $g_j(t)g_j^*(t)|t^{15}+1$ for $j = 3,4$ respectively, then by the use of Theorem 4.15, we get $\mathcal{K}^\perp \subseteq \mathcal{K}$. Further $\varphi(\mathcal{K})$ is a linear code over $\mathbb{Z}_3$ having parameters $[60,\ 56,\ 2]$. Then, by the application of Theorem 4.18, we obtain the quantum code having parameters $[60,\ 52,\ \geq 2]_3$.*

Example 6. *In $\mathbb{Z}_3[t]$, $t^{20}-1 = (t+1)(t+2)(t^2+1)(t^4+t^3+2t+1)(t^4+t^3+t^2+t+1)(t^4+2t^3+t+1)(t^4+2t^3+t^2+2t+1)$ and $t^{20}+1 = (t^2+t+2)(t^2+2t+2)(t^4+t^2+t+1)(t^4+t^2+2t+1)(t^4+t^3+t^2+1)(t^4+2t^3+t^2+1)$. Now, let $\mathcal{K}$ be a $1+\nu+\omega+2\nu\omega$-constacyclic code over $\Re = \mathbb{Z}_3+\nu\mathbb{Z}_3+\omega\mathbb{Z}_3+\nu\omega\mathbb{Z}_3$ where $\nu^2 = 1$, $\omega^2 = 1$ and $\nu\omega = \omega\nu$ and $\nu^2\omega^2 = \nu\omega$ of length 20. Let $g_1(t) = g_2(t) = g_3(t) = t^2+t+2$ and $g_4(t) = t+2$, $g(t) = \xi_1(t^2+t+2)+\xi_2(t^2+t+2)+\xi_3(t^2+t+2)+\xi_4(t+2)$ be the generator polynomials of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^{20}+1$ for $i = 1,2,3$ respectively and $g_4(t)g_4^*(t)|t^{20}-1$, then by the use of Theorem 4.11, we get $\mathcal{K}^\perp \subseteq \mathcal{K}$. Further $\varphi(\mathcal{K})$ is a linear code over $\mathbb{Z}_3$ having parameters $[80,\ 73,\ 3]$. Then, by the application of Theorem 4.18, we obtain the quantum code having parameters $[80,\ 66,\ \geq 3]_3$.*

6. Conclusion

In this work, we have given a construction of quantum code through ϑ-constacyclic code over the finite non-chain ring $\Re = Z_3+\nu Z_3+\omega Z_3+\nu\omega Z_3$ where $\nu^2 = 1$, $\omega^2 = 1$ and $\nu\omega = \omega\nu$ and $\nu^2\omega^2 = \nu\omega$. We have derived self-orthogonal code over the ring Z_3 as Gray images of linear code over the ring $Z_3+\nu Z_3+\omega Z_3+\nu\omega Z_3$. In particular, the parameters of quantum code over the ring Z_3 are obtained by decomposing ϑ-constacyclic code into cyclic and negacyclic codes over the ring Z_3. For the future scope, one can look at other classes of constacyclic codes over $Z_3 + \nu Z_3 + \omega Z_3 + \nu\omega Z_3$ and $Z_p + \nu Z_p + \omega Z_p + \nu\omega Z_p$.

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