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Analytical Study for Certain Ordinary Differential Equations with Variable Coefficients via G_α-Transform

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Abstract. G_α-transform, which is a comprehensive and essential form of Laplace-type integral transforms, has both advantages and limitations. The purpose of this study is to consider the applicable range of G_α-transform in finding solutions of ordinary differential equations with variable coefficients. Finally, several examples are given to demonstrate the effectiveness of these results.

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1. Introduction

The differential equations have played a central role in every aspect of applied mathematics for a very long time, and their importance has increased further with the advent of computers. Several mathematical methods have been applied by various researchers in various fields of science and engineering to obtain the analytical solutions of differential equations, which appeared in the literature [26, 34, 36]. To solve the differential equations, the integral transforms were extensively used. The Laplace transform is one of many integral transforms in applied mathematics and is often used to solve differential equations.

The Laplace transform reduces a linear differential equation to an algebraic equation, which can then be solved using algebra's formal rules. After that, the differential equation can then be solved by applying the inverse Laplace transform [33]. The Laplace transform is beneficial for finding the solution of the diffusion equation in transient flow [8, 35, 43]. In

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addition, many researchers mainly had paid attention to study for theory and applications of Laplace transform, see [9–11, 21, 41] for more details.

The Laplace transform is a well-known fact that it converts a function f of a real variable t to a function F of a complex variable s, which is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt.$$

In addition, if $f(t)$ is a piecewise continuous on $[0, \infty)$ and has an exponential order k, then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > k$.

For $s = 1/u$, the Laplace transform $\mathcal{L}\{f(t)\}$ can be rewritten as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-t/u} f(t)dt.$$

In the last two decades, many integral transforms in the class of Laplace-typed integral transform are introduced, such as Sumudu transform, Elzaki transform, natural transform, Aboodh transform, Mohand transform, G_α-transform, HY-transform, and Kamal transform. These transforms have been used for solving different types of integral equations, ordinary differential equations, partial differential equations, and fractional differential equations, see [1, 3, 15, 22, 37, 42, 45] for more details.

Since the Laplace transform is not suitable for solving some differential equations, in 1993, G. Watugala [44] introduced a new transform, named Sumudu transform, and shown that Sumudu transform has fascinating properties, making it easy to visualize and apply it for finding the solution of ordinary differential equations in control engineering problems. Thus, the Sumudu transform is an ideal transform for control engineering and applied mathematics.

In 2010, H. Eltayeb and A. Kilicman [14] introduced some relationships between Sumudu transform and Laplace transform. They showed that the solution which is given by Laplace transform into a complex domain and given by Sumudu transform into a real domain. Thus, this leads them to consider that if the solution exists by Sumudu transform, then the solution also exists by Laplace transform. Moreover, they showed a strong relationship between Sumudu transform and other integral transforms, see A. Kilicman et al.[13].

Many researchers applied Sumudu transform to solve the system of dynamic equations, partial differential equations with variable coefficient, a semi-infinite string, an integro-differential equation, the fractional neutron transport equation, see [2, 4–6, 12, 20, 23–25, 27, 28] for more details.

The Sumudu transform converts a function f of a real variable t to a function of a complex variable u, which is defined by

$$S\{f(t)\} = \frac{1}{u}\int_0^\infty e^{-t/u} f(t)dt.$$

In addition, if $f(t)$ is a piecewise continuous on $[0, \infty)$ and has an exponential order k, then the Sumudu transform $S\{f(t)\}$ exists for $u < 1/k$.

Elzaki transform is the modified version of Laplace transform and Sumudu transform, which was first introduced by T.M. Elzaki [16] in 2011. Elzaki transform was then presented when Sumudu transform failed to solve some differential equations with variable coefficients [19]. T.M. Elzaki et al. [17, 18] showed that Elzaki transform provides a method for analyzing ordinary differential equations such as linear dynamic systems equation, signals-delay differential equation, and the renewal equation in statistics.

The Elzaki transform converts a function f of a real variable t to a function of a complex variable u, which is defined by

$$E\{f(t)\} = u \int_0^\infty e^{-t/u} f(t)dt.$$

In particular, if $f(t)$ is a piecewise continuous on $t \geq 0$ and has an exponential order k, then the Elzaki transform $E\{f(t)\}$ exists for $u < 1/k$.

Recently, Hj. Kim [29] introduced the intrinsic structure and some properties of G_α-transform, which is defined by

$$F(u) = G_\alpha\{f(t)\} = u^\alpha \int_0^\infty e^{-t/u} f(t)dt,$$

where $\alpha \in \mathbb{Z}$ and u is a complex variable. The G_α-transform can be applied directly to any situation by choosing α appropriately.

In addition, if $f(t)$ is a piecewise continuous on $t \geq 0$ and has an exponential order k, then the G_α-transform $G_\alpha\{f(t)\}$ exists for $u < 1/k$.

The G_α-transform is a Laplace-type integral transform can be reduced to the Laplace transform, Sumudu transform, and Elzaki transform for $\alpha = 0, -1, 1$, respectively.

Moreover, we know that the Laplace transform has a strong point in the transforms of derivatives. If we set $\alpha = -2$, then we obtain a simple tool for transforms of integral, which can be rewritten as

$$G_{-2}\{f(t)\} = \frac{1}{u^2} \int_0^\infty e^{-t/u} f(t)dt,$$

see [30]. Further, Hj. Kim [31] also solved Laguerre's equation by the G_{-2}-transform.

In 2019, S. Sattaso et al. [39] studied the properties of G_α-transform and presented an example that cannot be solved by the Sumudu and Elzaki transforms, but it can be solved by the G_α-transform.

Furthermore, Hj. Kim et al. [38] considered an application of G_α-transform in partial differential equations by using the n-th partial derivatives, and Hj. Kim [7, 32] also considered a proof concerning the Laplace transform of the n-th derivative of any order by mathematical induction and considered a variant of G_α-transform represented by a logarithmic function. The connection of this transform to the convolutional neural network can be found in [40].

In this paper, we give some conditions of certain ordinary differential equations that can be solved by G_α-transform. Furthermore, we include examples to demonstrate the effectiveness of these results.

2. Preliminaries

In this section, we give some basic properties of the G_α-transform, which would appear in this study quite frequently. The proofs of the following properties are given in [29, 39].

Lemma 1. [29] (G_α-transform of derivatives) If $f(t), f'(t), \ldots, f^{(m-1)}(t)$ are continuous and $f^{(m)}(t)$ is a piecewise continuous function on $[0,\infty)$ for $m \in \mathbb{N} \cup \{0\}$ and has an exponential order k for $u < 1/k$, then the following properties hold:

(i) $G_\alpha\{f'(t)\} = \dfrac{F(u)}{u} - u^\alpha f(0);$

(ii) $G_\alpha\{f''(t)\} = \dfrac{F(u)}{u^2} - u^{\alpha-1} f(0) - u^\alpha f'(0);$

(iii) $G_\alpha\{f^{(m)}(t)\} = \dfrac{F(u)}{u^m} - \displaystyle\sum_{k=0}^{m-1} u^{\alpha-m+(k+1)} f^{(k)}(0),$

where $F(u) = G_\alpha\{f(t)\}$.

Lemma 2. [39] (G_α-transform of multiplication by power of t) If $f(t)$ is a piecewise continuous function on $[0,\infty)$ and has an exponential order k for $u < 1/k$, then the following properties hold:

(i) $G_\alpha\{tf(t)\} = u^2F'(u) - \alpha u F(u);$

(ii) $G_\alpha\{t^2 f(t)\} = u^4 F''(u) - 2(\alpha-1)u^3F'(u) + (\alpha-1)\alpha u^2 F(u);$

(iii)
$$\begin{aligned} G_\alpha\{t^n f(t)\} &= u^{2n}F^{(n)}(u) - \binom{n}{1}(\alpha-(n-1))\,u^{2n-1}F^{(n-1)} \\ &\quad + \cdots - \binom{n}{n-1}(\alpha-(n-1))(\alpha-(n-2))\cdots(\alpha-1)u^{n+1}F'(u) \\ &\quad + (\alpha-(n-1))(\alpha-(n-2))\cdots \alpha u^n F(u), \end{aligned}$$

where $F(u) = G_\alpha\{f(t)\}$.

Lemma 3. [39] If $f^{(m)}(t)$ is a piecewise continuous function on $[0,\infty)$ for $m \in \mathbb{N} \cup \{0\}$ and has an exponential order k for $u < 1/k$, then

$$\begin{aligned} G_\alpha\{t^n f^{(m)}(t)\} &= u^{2n}\frac{d^n G_\alpha\{f^{(m)}(t)\}}{du^n} - \binom{n}{1}[\alpha-(n-1)]\,u^{2n-1}\frac{d^{n-1}G_\alpha\{f^{(m)}(t)\}}{du^{n-1}} \\ &\quad + \cdots - \binom{n}{n-1}[\alpha-(n-1)][\alpha-(n-2)]\cdots(\alpha-1)u^{n+1}\frac{dG_\alpha\{f^{(m)}(t)\}}{du} \\ &\quad + [\alpha-(n-1)][\alpha-(n-2)]\cdots \alpha u^n G_\alpha\{f^{(m)}(t)\}. \end{aligned} \tag{1}$$

Lemma 4. [29] If $f(t) = t^n$ for $n \in \mathbb{N} \cup \{0\}$, then

$$G_\alpha\{t^n\} = n! u^{n+\alpha+1}.$$

Remark 1. By using Lemma 3, substituting $n = 1, 2$, and 3 in (1) and derivatives, after some simplification, we obtain

$$\text{(i)}\ G_\alpha\{tf^{(m)}(t)\} = \frac{F'(u)}{u^{m-2}} - (m+\alpha)\frac{F(u)}{u^{m-1}} - \sum_{k=0}^{m-1}(1+k-m)u^{2+k+\alpha-m}f^{(k)}(0);$$

$$\begin{aligned}\text{(ii)}\ G_\alpha\{t^2f^{(m)}(t)\} = {} & \frac{F''(u)}{u^{m-4}} - 2(m+\alpha-1)\frac{F'(u)}{u^{m-3}} + [m(m+1)+2(\alpha-1)m+(\alpha-1)\alpha] \\ & \times\frac{F(u)}{u^{m-2}} - \sum_{k=0}^{m-1}[(\alpha-m+k+1)(\alpha-m+k)-2(\alpha-1)(\alpha-m+k+1) \\ & +(\alpha-1)\alpha]\,u^{3+k+\alpha-m}f^{(k)}(0);\end{aligned}$$

$$\begin{aligned}\text{(iii)}\ G_\alpha\{t^3f^{(m)}(t)\} = {} & \frac{F'''(u)}{u^{m-6}} - 3(m+\alpha-2)\frac{F''(u)}{u^{m-5}} + [3m(m+1)+6(\alpha-2)m \\ & +3(\alpha-2)(\alpha-1)]\,\frac{F'(u)}{u^{m-4}} - [m(m+1)(m+2)+3(\alpha-2)m(m+1) \\ & +3(\alpha-2)(\alpha-1)m+(\alpha-2)(\alpha-1)\alpha]\,\frac{F(u)}{u^{m-3}} \\ & -\sum_{k=0}^{m-1}[(\alpha-m+k+1)(\alpha-m+k)(\alpha-m+k-1) \\ & -3(\alpha-2)(\alpha-m+k+1)(\alpha-m+k)+3(\alpha-2)(\alpha-1)(\alpha-m+k+1) \\ & -(\alpha-2)(\alpha-1)\alpha]\,u^{4+k+\alpha-m}f^{(k)}(0),\end{aligned}$$

where $F(u) = G_\alpha\{f(t)\}$.

3. Main Results

In this section, we show some conditions of certain ordinary differential equations to ensure that those ordinary differential equations can be solved by G_α-transform.

Theorem 1. Consider the m-th order ordinary differential equation of the form

$$\begin{aligned}&\left(a_mt^2+b_mt+c_m\right)y^{(m)}(t) + \left(a_{m-1}t^2+b_{m-1}t+c_{m-1}\right)y^{(m-1)}(t) \\ &+\cdots+\left(a_0t^2+b_0t+c_0\right)y(t) = g(t),\end{aligned} \tag{2}$$

where a_j, b_j, c_j are constants, $j = 0, 1, 2, \ldots, m$ and $g(t)$ is an unknown function. The G_α-transform is a suitable method for solving (2), if the following conditions are satisfies

$$\begin{gathered}c_m = b_m = c_{m-1} = (\alpha-1)\alpha a_0 = 2(\alpha-1)a_0 = 0, \\ [2+2(\alpha-1)+(\alpha-1)\alpha]a_1 - \alpha b_0 = b_{i-1} - 2(\alpha+i-1)a_i = 0\end{gathered}$$

for $i = 1, 2, 3, \ldots, m$, and

$$[i(i+1)+2(\alpha-1)i+(\alpha-1)\alpha]a_i - (i+\alpha-1)b_{i-1} + c_{i-2} = 0$$

for $i = 2, 3, 4, \ldots, m$.

Proof. By using Remark 1(1-2) and taking G_α-transform of both sides to (2), we obtain

$$\begin{aligned}
&\left[\frac{a_m}{u^{m-4}}+\frac{a_{m-1}}{u^{m-5}}+\cdots+\frac{a_1}{u^{-3}}+\frac{a_0}{u^{-4}}\right]F''(u)+\Bigg[-2(\alpha+m-1)\frac{a_m}{u^{m-3}}-2(\alpha+m-2)\frac{a_{m-1}}{u^{m-4}}\\
&-\cdots-2\alpha\frac{a_1}{u^{-2}}-2(\alpha-1)\frac{a_0}{u^{-3}}+\frac{b_m}{u^{m-2}}+\frac{b_{m-1}}{u^{m-3}}+\cdots+\frac{b_1}{u^{-1}}+\frac{b_0}{u^{-2}}\Bigg]F'(u)\\
&+\Bigg[\big(m(m+1)+2(\alpha-1)m+(\alpha-1)\alpha\big)\frac{a_m}{u^{m-2}}+\big((m-1)m+2(\alpha-1)(m-1)\\
&+(\alpha-1)\alpha\big)\frac{a_{m-1}}{u^{m-3}}+\cdots+\big(2+2(\alpha-1)+(\alpha-1)\alpha\big)\frac{a_1}{u^{-1}}+(\alpha-1)\alpha\frac{a_0}{u^{-2}}-(\alpha+m)\frac{b_m}{u^{m-1}}\\
&-(\alpha+m-1)\frac{b_{m-1}}{u^{m-2}}-\cdots-(\alpha+1)b_1-\alpha\frac{b_0}{u^{-1}}+\frac{c_m}{u^m}+\frac{c_{m-1}}{u^{m-1}}+\cdots+\frac{c_1}{u}+c_0\Bigg]F(u)\\
&=G_\alpha\{g(t)\}-q(u), \qquad (3)
\end{aligned}$$

where $q(u)$ be contained in some expressions that are started by summation and do not influence the proof steps.

If the G_α-transform is suitable method for solving (2), then the coefficient of $F(u)$ and $F'(u)$ in (3) should be equal to zero. Thus, if the coefficient of $F(u)=0$, then

$$\begin{aligned}
u^m &\rightarrow c_m=0;\\
u^{m-1} &\rightarrow c_{m-1}-(m+\alpha)b_m=0;\\
u^{m-2} &\rightarrow c_{m-2}-(m+\alpha-1)b_{m-1}+\left(m(m+1)+2(\alpha-1)m+(\alpha-1)\alpha\right)a_m=0;\\
&\vdots\\
u^0 &\rightarrow c_0-(\alpha+1)b_1+\left(6+4(\alpha-1)+(\alpha-1)\alpha\right)a_2=0;\\
u^{-1} &\rightarrow -\alpha b_0+\big(2+2(\alpha-1)+(\alpha-1)\alpha\big)a_1=0;\\
u^{-2} &\rightarrow (\alpha-1)\alpha a_0=0.
\end{aligned}$$

And if the coefficient of $F'(u)=0$, then

$$\begin{aligned}
u^{m-2} &\rightarrow b_m=0\\
u^{m-3} &\rightarrow b_{m-1}-2(m+\alpha-1)a_m=0\\
u^{m-4} &\rightarrow b_{m-2}-2(m+\alpha-2)a_{m-1}=0\\
&\vdots\\
u^{-1} &\rightarrow b_1-2(\alpha+1)a_2=0\\
u^{-2} &\rightarrow b_0-2\alpha a_1=0\\
u^{-3} &\rightarrow 2(\alpha-1)a_0=0.
\end{aligned}$$

In general, we can show that

$$c_m=b_m=c_{m-1}=(\alpha-1)\alpha a_0=2(\alpha-1)a_0=0,$$

$$[2+2(\alpha-1)+(\alpha-1)\alpha]a_1-\alpha b_0=b_{i-1}-2(\alpha+i-1)a_i=0$$

for $i=1,2,3,\ldots,m$, and

$$[i(i+1)+2(\alpha-1)i+(\alpha-1)\alpha]a_i-(i+\alpha-1)b_{i-1}+c_{i-2}=0$$

for $i=2,3,4,\ldots,m$. This completes the proof.

Remark 2. From Theorem 1, if $g(t)=0$, we can just set the coefficient of $F(u)$ equal to zero to reduce conditions. Therefore, the G_α-transform is a suitable method for solving equation (2), if

$$c_m=c_{m-1}-(m+\alpha)b_m=[2+2(\alpha-1)+(\alpha-1)\alpha]a_1-\alpha b_0=(\alpha-1)\alpha a_0=0,$$

and

$$[i(i+1)+2(\alpha-1)i+(\alpha-1)\alpha]a_i-(i+\alpha-1)b_{i-1}+c_{i-2}=0$$

for $i=2,3,4,\ldots,m$.

Theorem 2. Consider the m-th order ordinary differential equation of the form

$$\begin{aligned}\left(a_mt^3+b_mt^2+c_mt+d_m\right)y^{(m)}(t)+\left(a_{m-1}t^3+b_{m-1}t^2+c_{m-1}t+d_{m-1}\right)y^{(m-1)}(t)\\+\cdots+\left(a_0t^3+b_0t^2+c_0t+d_0\right)y(t)=g(t),\end{aligned}\tag{4}$$

where a_j,b_j,c_j,d_j are constants, $j=0,1,2,\ldots,m$ and $g(t)$ is an unknown function. The G_α-transform is a suitable method for solving (4), if the following conditions are satisfies

$$\begin{aligned}d_m=c_m=b_m=d_{m-1}=c_{m-1}=d_{m-2}=0,\\(\alpha-2)(\alpha-1)\alpha a_0=3(\alpha-2)(\alpha-1)a_0=3(\alpha-2)a_0=0,\\\alpha c_0-[2+2(\alpha-1)+(\alpha-1)\alpha]b_1\\+[24+18(\alpha-2)+6(\alpha-2)(\alpha-1)+(\alpha-2)(\alpha-1)\alpha]a_2=0,\\(\alpha-1)\alpha b_0-[6+6(\alpha-2)+3(\alpha-2)(\alpha-1)+(\alpha-2)(\alpha-1)\alpha]a_1=0,\\2(\alpha-1)b_0-[6+6(\alpha-2)+3(\alpha-2)(\alpha-1)]a_1=0,\\d_{i-3}-(\alpha+i-2)c_{i-2}+[(i-1)i+2(\alpha-1)(i-1)+(\alpha-1)\alpha]b_{i-1}\\-[i(i+1)(i+2)+3(\alpha-2)i(i+1)+3(\alpha-2)(\alpha-1)i+(\alpha-2)(\alpha-1)\alpha]a_i=0\end{aligned}$$

for $i=3,4,5,\ldots,m$,

$$c_{i-2}-2(\alpha+i-2)b_{i-1}+[3i(i+1)+6(\alpha-2)i+3(\alpha-2)(\alpha-1)]a_i=0$$

for $i=2,3,4,\ldots,m$, and $b_{i-1}-3(\alpha+i-2)a_i=0$ for $i=1,2,3,\ldots,m$.

Proof. By using Remark 1 and taking G_α-transform of both sides to (4), we obtain

$$\left[\frac{a_m}{u^{m-6}}+\frac{a_{m-1}}{u^{m-7}}+\cdots+\frac{a_1}{u^{-5}}+\frac{a_0}{u^{-6}}\right]F'''(u)+\Big[-3(m+\alpha-2)\frac{a_m}{u^{m-5}}-3(m+\alpha-3)\frac{a_{m-1}}{u^{m-6}}$$

$$
\begin{aligned}
&-\cdots-3(\alpha-1)\frac{a_1}{u^{-4}}-3(\alpha-2)\frac{a_0}{u^{-5}}+\frac{b_m}{u^{m-4}}+\frac{b_{m-1}}{u^{m-5}}+\cdots+\frac{b_1}{u^{-3}}+\frac{b_0}{u^{-4}}\Bigg]F''(u)\\
&+\Bigg[\big(3m(m+1)+6(\alpha-2)m+3(\alpha-2)(\alpha-1)\big)\frac{a_m}{u^{m-4}}+\big(3(m-1)m+6(\alpha-2)(m-1)\\
&+3(\alpha-2)(\alpha-1)\big)\frac{a_{m-1}}{u^{m-5}}+\cdots+\big(6+6(\alpha-2)+3(\alpha-2)(\alpha-1)\big)\frac{a_1}{u^{-3}}+3(\alpha-2)(\alpha-1)\\
&\times\frac{a_0}{u^{-4}}-2(\alpha+m-1)\frac{b_m}{u^{m-3}}-2(\alpha+m-2)\frac{b_{m-1}}{u^{m-4}}-\cdots-2\alpha\frac{b_1}{u^{-2}}-2(\alpha-1)\frac{b_0}{u^{-3}}+\frac{c_m}{u^{m-2}}\\
&+\frac{c_{m-1}}{u^{m-3}}+\cdots+\frac{c_1}{u^{-1}}+\frac{c_0}{u^{-2}}\Bigg]F'(u)+\Bigg[-\big(m(m+1)(m+2)+3(\alpha-2)m(m+1)\\
&+3(\alpha-2)(\alpha-1)m+(\alpha-2)(\alpha-1)\alpha\big)\frac{a_m}{u^{m-3}}-\big((m-1)m(m+1)+3(\alpha-2)(m-1)m\\
&+3(\alpha-2)(\alpha-1)(m-1)+(\alpha-2)(\alpha-1)\alpha\big)\frac{a_{m-1}}{u^{m-4}}-\cdots-\big(6+6(\alpha-2)\\
&+3(\alpha-2)(\alpha-1)+(\alpha-2)(\alpha-1)\alpha\big)\frac{a_1}{u^{-2}}-(\alpha-2)(\alpha-1)\alpha\frac{a_0}{u^{-3}}-(m(m+1)\\
&+2(\alpha-1)m+(\alpha-1)\alpha)\,\frac{b_m}{u^{m-2}}+\big((m-1)m+2(\alpha-1)(m-1)+(\alpha-1)\alpha\big)\frac{b_{m-1}}{u^{m-3}}\\
&+\cdots+\big(2+2(\alpha-1)+(\alpha-1)\alpha\big)\frac{b_1}{u^{-1}}+(\alpha-1)\alpha\frac{b_0}{u^{-2}}-(m+\alpha)\frac{c_m}{u^{m-1}}\\
&-(m+\alpha-1)\frac{c_{m-1}}{u^{m-2}}-\cdots-(\alpha+1)\frac{c_1}{u^{0}}-\alpha\frac{c_0}{u^{-1}}+\frac{d_m}{u^{m}}+\frac{d_{m-1}}{u^{m-1}}+\cdots+\frac{d_1}{u^{1}}+\frac{d_0}{u^{0}}\Bigg]F(u)\\
&=G_\alpha\{g(t)\}-r(u),
\end{aligned}
$$

where $r(u)$ be contained in some expressions that are started by summation and do not influence the proof steps.

By using the previous results, which similar to the Theorem 1, we know that the coefficients of $F(u), F'(u)$ and $F''(u)$ should be equal to zero, by the same process as Theorem 1, we can show that

$$
\begin{aligned}
d_m=c_m=b_m=d_{m-1}=c_{m-1}=d_{m-2}&=0,\\
(\alpha-2)(\alpha-1)\alpha a_0=3(\alpha-2)(\alpha-1)a_0=3(\alpha-2)a_0&=0,\\
\alpha c_0-[2+2(\alpha-1)+(\alpha-1)\alpha]b_1&\\
+[24+18(\alpha-2)+6(\alpha-2)(\alpha-1)+(\alpha-2)(\alpha-1)\alpha]a_2&=0,\\
(\alpha-1)\alpha b_0-[6+6(\alpha-2)+3(\alpha-2)(\alpha-1)+(\alpha-2)(\alpha-1)\alpha]a_1&=0,\\
2(\alpha-1)b_0-[6+6(\alpha-2)+3(\alpha-2)(\alpha-1)]a_1&=0,\\
d_{i-3}-(\alpha+i-2)c_{i-2}+[(i-1)i+2(\alpha-1)(i-1)+(\alpha-1)\alpha]b_{i-1}&\\
-[i(i+1)(i+2)+3(\alpha-2)i(i+1)+3(\alpha-2)(\alpha-1)i+(\alpha-2)(\alpha-1)\alpha]a_i&=0
\end{aligned}
$$

for $i=3,4,5,\ldots,m$,

$$c_{i-2}-2(\alpha+i-2)b_{i-1}+[3i(i+1)+6(\alpha-2)i+3(\alpha-2)(\alpha-1)]a_i=0$$

for $i=2,3,4,\ldots,m$, and $b_{i-1}-3(\alpha+i-2)a_i=0$ for $i=1,2,3,\ldots,m$. The proof is completed.

Remark 3. From Theorem 2, if $g(t) = 0$, we can just set the coefficient of $F(u)$ equal to zero and $F'(u)$ equal to zero to reduce conditions. Therefore, the G_α-transform is a suitable method for solving equation (4), if

$$\begin{aligned}
d_m = c_m = d_{m-1} = c_{m-1} - 2(\alpha + m - 1)b_m = 0,\\
d_{m-2} - (\alpha + m - 1)c_{m-1} + [m(m + 1) + 2(\alpha - 1)m + (\alpha - 1)\alpha]b_m = 0,\\
(\alpha - 2)(\alpha - 1)\alpha a_0 = 3(\alpha - 2)(\alpha - 1)a_0 = 0,\\
\alpha c_0 - [2 + 2(\alpha - 1) + (\alpha - 1)\alpha]b_1+\\
[24 + 18(\alpha - 2) + 6(\alpha - 2)(\alpha - 1) + (\alpha - 2)(\alpha - 1)\alpha]a_2 = 0,\\
(\alpha - 1)\alpha b_0 - [6 + 6(\alpha - 2) + 3(\alpha - 2)(\alpha - 1) + (\alpha - 2)(\alpha - 1)\alpha]a_1 = 0,\\
2(\alpha - 1)b_0 - [6 + 6(\alpha - 2) + 3(\alpha - 2)(\alpha - 1)]a_1 = 0,\\
d_{i-3} - (\alpha + i - 2)c_{i-2} + [(i - 1)i + 2(\alpha - 1)(i - 1) + (\alpha - 1)\alpha]b_{i-1}\\
-[i(i + 1)(i + 2) + 3(\alpha - 2)i(i + 1) + 3(\alpha - 2)(\alpha - 1)i + (\alpha - 2)(\alpha - 1)\alpha]a_i = 0
\end{aligned}$$

for $i = 3, 4, 5, \ldots, m$, and

$$c_{i-2} - 2(\alpha + i - 2)b_{i-1} + [3i(i + 1) + 6(\alpha - 2)i + 3(\alpha - 2)(\alpha - 1)]a_i = 0$$

for $i = 2, 3, 4, \ldots, m$.

4. Examples

In this section, we show the usage of G_α-transform for solving the ordinary differential equations with variable coefficients that according to Theorem 1 and Theorem 2 via some examples.

Example 1. *Consider the ordinary differential equation with variable coefficients of the form*

$$t^2y''(t) + 4ty'(t) + 2y(t) = t^3. \tag{5}$$

From (2) *and* (5)*, we have*

$$a_2 = 1,\ \ b_1 = 4,\ \ c_0 = 2,\ \ a_0 = a_1 = 0,\ \ b_0 = b_2 = 0,\ \ c_1 = c_2 = 0,$$

and we define $\alpha = 1$ to satisfy with the conditions of Theorem 1, so using the G_1-transform leads to find the solution of (5)*. By applying the G_1-transform to* (5) *and using Lemma 3, we obtain*

$$\begin{aligned}
G_1\{t^2y''(t)\} + G_1\{4ty'(t)\} + G_1\{2y(t)\} = G_1\{t^3\}\\
u^2F''(u) - 4uF'(u) + 6F(u) + 4uF'(u) - 8F(u) + 2F(u) = 6u^5\\
F''(u) = 6u^3.
\end{aligned}$$

Then, we have

$$F(u) = \frac{3}{10}u^5 + c_1u + c_2,$$

where c_1 and c_2 are constants. Letting $c_1 = c_2 = 0$, we get $F(u) = \frac{3}{10}u^5$. By using Lemma 4 and the inverse G_1-transform, thus the inverse of u^5 is $\frac{t^3}{6}$, we obtain $y(t) = \frac{1}{20}t^3$ as a solution of (5). It is not difficult to show that $y(t) = \frac{1}{20}t^3$ satisfies (5).

The next example will show that if the conditions do not satisfy Theorem 1, then it is not suitable to solve by this method as the following.

Example 2. *Consider the Legendre differential equation of the form*

$$(1 - t^2)y''(t) - 2ty'(t) = t. \tag{6}$$

From (2) and (6), we have

$$a_2 = -1, \ \ c_2 = 1, \ \ b_1 = -2, \ \ a_0 = a_1 = 0, \ \ b_0 = b_2 = 0, \ \ c_0 = c_1 = 0,$$

and with respect to the conditions in Theorem 1, c_2 should be equal to 0, while c_2 is equal to 1. Therefore, the conditions of Theorem 1 are not satisfied. If we take G_α-transform both sides of (6), we obtain

$$\begin{aligned} G_\alpha\{(1 - t^2)y''(t)\} - G_\alpha\{2ty'(t)\} &= G_\alpha\{t\} \\ -u^2F''(u) + 2\alpha uF'(u) + \left((\alpha - 3)\alpha + \frac{1}{u^2}\right) F(u) &= u^{\alpha+2}. \end{aligned}$$

Observe that (6) changed into a second-order ordinary differential equation with variable coefficients. Thus, using G_α-transform did not lead to finding the solution of (6).

Example 3. *Consider the ordinary differential equation with variable coefficients of the form*

$$t^2y''(t) + 2ty'(t) - 2y(t) = 0. \tag{7}$$

From (2) and (7), we have

$$a_2 = 1, \ \ b_1 = 2, \ \ c_0 = -2, \ \ a_0 = a_1 = 0, \ \ b_0 = b_2 = 0, \ \ c_1 = c_2 = 0,$$

and we define $\alpha = 1$ to satisfy with the conditions of Remark 2, so using the G_1-transform leads to find the solution of (7). By applying the G_1-transform to (7) and using Lemma 3, we obtain

$$\begin{aligned} G_1\{t^2y''(t)\} + G_1\{2ty'(t)\} - G_1\{2y(t)\} &= 0 \\ u^2F''(u) - 4uF'(u) + 6F(u) + 2uF'(u) - 2F(u) - 2F(u) - 2F(u) &= 0 \\ u^2F''(u) - 2uF'(u) &= 0. \end{aligned}$$

Then, we have $\frac{F''(u)}{F'(u)} = \frac{2}{u}$. By integration both sides, we obtain

$$\ln F'(u) = \ln c_1u^2 \ \ \textit{or} \ \ \ F'(u) = c_1u^2,$$

and hence

$$F(u) = \frac{c_1}{3}u^3 + c_2,$$

where c_1 *and* c_2 *are constants. Letting* $c_2 = 0$*, we get* $F(u) = \frac{c_1}{3}u^3$*. By using Lemma 4 and the inverse* G_1*-transform, thus the inverse of* u^3 *is* t*, we obtain* $y(t) = \frac{c_1}{3}t$ *as a solution of* (7)*. It is not difficult to show that* $y(t) = \frac{c_1}{3}t$ *satisfies* (7)*.*

Example 4. *Consider the ordinary differential equation with variable coefficients of the form*

$$t^3y'''(t) + 9t^2y''(t) + 18ty'(t) + 6y(t) = t. \tag{8}$$

From (4) *and* (8)*, we have*

$a_3 = 1,\ b_2 = 9,\ c_1 = 8,\ d_0 = 6,\ a_0 = a_1 = a_2 = 0,\ b_0 = b_1 = b_3 = 0,\ c_0 = c_2 = c_3 = 0,$
$d_1 = d_2 = d_3 = 0,$

and we define $\alpha = 2$ *to satisfy with the conditions of Theorem 2, so using the* G_2*-transform leads to find the solution of* (8)*. By applying the* G_2*-transform to* (8) *and using Lemma 3, we obtain*

$$\begin{aligned} G_2\{t^3y'''(t)\} + G_2\{9t^2y''(t)\} + G_2\{18ty'(t)\} + G_2\{6y(t)\} &= G_2\{t\} \\ u^3F'''(u) - 9u^2F''(u) + 36uF'(u) - 60F(u) + 9u^2F''(u) - 54uF'(u) + 108F(u)& \\ +18uF'(u) - 54F(u) + 6F(u) &= u^4. \end{aligned}$$

Then, we have $F'''(u) = u$*. By integration both sides, we obtain*

$$F(u) = \frac{1}{24}u^4 + \frac{c_1}{2}u^2 + c_2u + c_3,$$

where c_1*,* c_2*, and* c_3 *are constants. Letting* $c_1 = c_2 = c_3 = 0$*, we get* $F(u) = \frac{1}{24}u^4$*. By using Lemma 4 and the inverse* G_2*-transform, thus the inverse of* u^4 *is* t*, we obtain* $y(t) = \frac{1}{24}t$ *as a solution of* (8)*.*

The next example will show that if the conditions do not satisfy Theorem 2, then it is not suitable to solve by this method as the following.

Example 5. *Consider the ordinary differential equation with variable coefficients of the form*

$$(t^3 + t)y'''(t) + 6t^2y''(t) + 6ty'(t) = t^2. \tag{9}$$

From (4) *and* (9)*, we have*

$a_3 = 1,\ c_3 = 1,\ b_2 = 6,\ c_1 = 6,\ a_0 = a_1 = a_2 = 0,\ b_0 = b_1 = b_3 = 0,\ c_0 = c_2 = 0,$
$d_0 = d_1 = d_2 = d_3 = 0,$

and with respect to the conditions in Theorem 2, c_3 should be equal to 0, while c_3 is equal to 1. Therefore, the conditions of Theorem 2 are not satisfied. If we take G_α-transform both sides of (9), we obtain

$$G_\alpha\{(t^3+t)y'''(t)\}+G_\alpha\{6t^2y''(t)\}+G_\alpha\{6ty'(t)\}=G_\alpha\{t^2\}$$

$$u^3F'''(u)-[3+3(\alpha-2)]\,u^2F''(u)+\bigg[18-18(\alpha-2)+3(\alpha-2)(\alpha-1)$$

$$-12(\alpha-1)+\frac{1}{u^2}\bigg]uF'(u)-\bigg[24+36(\alpha-2)+9(\alpha-2)(\alpha-1)$$

$$+(\alpha-2)(\alpha-1)\alpha+(\alpha+3)\frac{1}{u^2}-24(\alpha-1)+6(\alpha+1)\bigg]F(u)=2u^{\alpha+3}.$$

Observe that (9) changed into a third order ordinary differential equation with variable coefficients. Thus, by using G_α-transform did not lead to find the solution of (9).

Example 6. *Consider the ordinary differential equation with variable coefficients of the form*

$$t^3y'''(t)+4t^2y''(t)-2ty'(t)-4y(t)=0. \tag{10}$$

From (4) and (10), we have

$$a_3=1,\ b_2=4,\ c_1=-2,\ d_0=-4,\ a_0=a_1=a_2=0,\ b_0=b_1=b_3=0,$$
$$c_0=c_2=c_3=0,\ d_1=d_2=d_3=0,$$

and we define $\alpha=1$ to satisfy with the conditions of Remark 3, so using the G_1-transform leads to find the solution of (10). By applying the G_1-transform to (10) and using Lemma 3, we obtain

$$G_1\{t^3y'''(t)\}+G_1\{4t^2y''(t)\}-G_1\{2ty'(t)\}-G_1\{4y(t)\}=0$$

$$u^3F'''(u)-9u^2F''(u)+36uF'(u)-60F(u)+3u^2F''(u)-18uF'(u)+36F(u)$$

$$+4u^2F''(u)-16uF'(u)+24F(u)-2uF'(u)+2F(u)+2F(u)-4F(u)=0.$$

Then, we have $\dfrac{F'''(u)}{F''(u)}=\dfrac{2}{u}$. By integration both sides, we obtain

$$\ln F''(u)=\ln c_1u^2 \quad or \quad F''(u)=c_1u^2,$$

and hence

$$F(u)=\frac{c_1}{12}u^4+c_2u+c_3,$$

where c_1, c_2, and c_3 are constants. Letting $c_2=c_3=0$, we get $F(u)=\dfrac{c_1}{12}u^4$. By using Lemma 4 and the inverse G_1-transform, thus the inverse of u^4 is $\dfrac{t^2}{2}$, we obtain $y(t)=\dfrac{c_1}{24}t^2$ as a solution of (10).

Remark 4. We can see that Example 1, 3, and 6 can be solved by G_1-transform, and Example 4 can be solved by G_2-transform, it is clear that Sumudu transform cannot be solved for these ordinary differential equations.

Remark 5. If we choose the suitable value for α and the problem is consistent with the conditions of Theorem 1 or Theorem 2, then we can easily find the solution of the ordinary differential equation. But if the problem is not consistent with the conditions of Theorem 1 or Theorem 2, it will be difficult to find the solution of the ordinary differential equation.

5. Conclusions

We obtained some conditions of certain ordinary differential equations to ensure that it can be solved by G_α-transform. In this regard, we observed that G_α-transform more appropriate than other Laplace-typed integral transforms to solve the ordinary differential equations with variable coefficients by choosing the suitable value for α.

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