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On Connected Co-Independent Hop Domination in Graphs

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Abstract. Let G be a connected graph. A subset S of $V(G)$ is a *connected co-independent hop dominating set* in G if the subgraph induced by S is connected and $V(G)\backslash S$ is an independent set where for each $v \in V(G)\backslash S$, there exists a vertex $u \in S$ such that $d_G(u,v) = 2$. The smallest cardinality of such an S is called the *connected co-independent hop domination number* of G. This paper presents the characterizations of the connected co-independent hop dominating sets in the join, corona and lexicographic product of two graphs. It also discusses the corresponding connected co-independent hop domination numbers of the aforementioned graphs.

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1. Introduction

In the late 1950's and 1960's, the study on domination in graphs was developed, beginning with C. Berge [1] in 1958. There are now many studies involving domination and its variations. One of its variation is the connected co-independent domination number of graphs introduced by B. Gayathri and S. Kaspar in 2010 [3]. Also, connected co-independent domination number in graphs were studied in [2, 6, 12]. Years later, new domination parameter called hop domination in graph is introduced by Natarajan and Ayyaswamy [8]. Hop domination in graphs were also studied in [7, 9–11, 13].

In this study, the researcher defines and establishes a new concept of hop domination called a connected co-independent hop domination and generates some characterizations

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of connected co-independent hop domination in graphs. Connected co-independent hop domination in graphs can have real world applications. For an application, in [5], Desormeaux, Haynes, and Henning inspired their research on these concepts through social networking applications. They considered a factory with a large number of employees and needed to implement a quality assurance checking system of their workers. The factory manager decides to designate an internal committee to do this. In other words, the manager will select some workers to form a quality assurance team to inspect the work of their co-workers. The manager wants to keep this team as small as possible to minimize costs (extra costs for inspectors) and protect privacy (keep the inspectors' identity confidential). To avoid bias, an inspector should neither be close friends nor enemies with any of the workers he/she is responsible for inspecting. To model this situation, a social network graph can be constructed in which each worker is represented by a vertex and an edge between two workers represents possible bias, that is, whether the two workers are close friends or enemies. Ideally, an inspector should not be adjacent to any worker who is being inspected.

In connected co-independent hop domination, every worker will be inspected by the nearest non-biased inspector. That is, an inspector who is a close friend (or an enemy) of a close friend (or enemy) of a worker. This is to save time and effort of locating a particular worker. Also, the inspectors should be acquainted with each other and all non-inspector workers are neither friends nor enemies, that is, they are not adjacent or there is no edge between them. The connected co-independent hop domination number will give the minimum number of inspectors needed.

In this study, we only consider graphs that are finite, simple, undirected and connected. Readers are referred to [4] for elementary Graph Theoretic concepts. An *independent set* S in a graph G is a subset of the vertex-set of G such that no two vertices in S are adjacent in G. The cardinality of a maximum independent set is called the *independence number* of G and is denoted by $\beta(G)$. An independent set $S \subseteq V(G)$ with $|S| = \beta(G)$ is called a β-set of G. A dominating set $D \subseteq V(G)$ is called a *connected co-independent dominating set* of G if D is a connected dominating set of G and $V(G) \setminus D$ is an independent set. The cardinality of such a minimum set D is called a *connected co-independent domination number* of G denoted by $\gamma_{c,coi}(G)$. A connected co-independent dominating set D with $|D| = \gamma_{c,coi}(G)$ is called a $\gamma_{c,coi}$-set of G. Let G be a connected graph. A set $S \subseteq V(G)$ is a *hop dominating set* of G if for every $v \in V(G)\setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set. A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G(u, 2) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G[u, 2] = N_G(u, 2) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G(X, 2) = \bigcup_{u \in X} N_G(u, 2)$. The *closed hop neighborhood* of X in G is the set $N_G[X, 2] = N_G(X, 2) \cup X$. Let G be a graph. A subset S of $V(G)$ is a *strictly co-independent set* of G if $V(G)\setminus S$ is an independent set and $N_G(v) \cap S \neq S$ for all $v \in V(G)\setminus S$. The minimum cardinality of a strictly co-independent set in G, denoted by $sci(G)$ is called the *strictly co-independent number* of G. A strictly co-independent set S with $|S| = sci(G)$ is called an sci-set of G. Let G be a connected graph. A hop dominating set $S \subseteq V(G)$ is a *connected co-independent hop dominating set* of G if $\langle S\rangle$ is connected and $V(G)\backslash S$ is an independent set. The minimum cardinality of a connected co-independent hop dominating set of G, denoted by $\gamma_{ch,coi}(G)$, is called the *connected co-independent hop domination number* of G. A connected co-independent hop dominating set S with $|S| = \gamma_{ch,coi}(G)$ is called a $\gamma_{ch,coi}$-set of G.

2. Preliminary Results

Remark 1. Every connected co-independent hop dominating set in a connected graph G is hop dominating. Hence, $\gamma_h(G) \leq \gamma_{ch,coi}(G)$.

Remark 2. Let G be a connected graph of order n. Then $1 \leq \gamma_{ch,coi}(G) \leq |V(G)|$. Moreover, $\gamma_{ch,coi}(G) = 1$ if and only if $G = K_1$.

Example 1. The equations below give the connected co-independent hop domination number of the path P_n and cycle C_n.

$$\gamma_{ch,coi}(P_n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2, 3 \\ n-2 & \text{if } n \geq 4 \end{cases}$$

$$\gamma_{ch,coi}(C_n) = \begin{cases} 3 & \text{if } n = 3 \\ n-1 & \text{if } n \geq 4 \end{cases}$$

Remark 3. If G is a complete graph, then $\gamma_{ch,coi}(G) = n$.

Theorem 1. Let G be a connected graph of order $n \geq 3$. Then $\gamma_{ch,coi}(G) = 2$ if and only if there exist adjacent vertices x and y of G such that for each $z \in V(G)\backslash\{x, y\}$, $N_G(z) = \{x\}$ or $N_G(z) = \{y\}$ and $z \notin N_G(x) \cap N_G(y)$.

Proof: Suppose $\gamma_{ch,coi}(G) = 2$. Let $S = \{x, y\}$ be a $\gamma_{ch,coi}$-set of G. Since S is connected, $xy \in E(G)$. Let $z \in V(G)\backslash\{x, y\}$. Then $z \notin S$. Since S is a hop dominating set of G, $z \in N_G(x, 2) \cup N_G(y, 2)$. Suppose $z \in N_G(x, 2)$. Then there exist $w \in N_G(z) \cap N_G(x)$. Since $V(G)\backslash S$ is an independent set, $w \in S$. Thus, $w = y$, that is, $N_G(z) = \{y\}$. Similarly, if $z \in N_G(y, 2)$, then $N_G(z) = \{x\}$. Since $z \in N_G(x, 2) \cup N_G(y, 2)$, $z \notin N_G(x) \cap N_G(y)$.

Conversely, suppose that there exist adjacent vertices x and y of G satisfying the given condition. Let $S = \{x, y\}$. Since $xy \in E(G)$, S is connected. Let $z \in V(G)\backslash S$. If $N_G(z) = \{x\}$, then since $xy \in E(G)$, $d_G(y, z) = 2$. While on the other hand, if $N_G(z) = \{y\}$, then $d_G(x, z) = 2$. Thus, S is a hop dominating set of G. Since $N_G(z) = \{x\}$ or $N_G(z) = \{y\}$, $V(G)\backslash S$ is an independent set. Therefore, S is a connected co-independent hop dominating set of G. So, $\gamma_{ch,coi}(G) \leq |S| = 2$. But G is nontrivial. Hence, $\gamma_{ch,coi}(G) \neq 1$ and so $\gamma_{ch,coi}(G) = 2$. □

Theorem 2. Let G be a connected graph of order $n \geq 2$. Then $\gamma_{ch,coi}(G) = n$ if and only if G is complete.

Proof: Suppose $\gamma_{ch,coi}(G) = n$. Suppose that G is not complete. Then there exist distinct vertices $u, v \in V(G)$ such that $d_G(u,v) = 2$. Let $S = V(G) \backslash \{u\}$. Then S is a connected co-independent hop dominating set of G. Therefore, $\gamma_{ch,coi}(G) \leq |S| = n-1$, a contradiction. Thus, G is a complete graph.

Conversely, by Remark 3, $\gamma_{ch,coi}(K_n) = n$. □

3. On Connected Co-Independent Hop Domination in the Join of Graphs

The *join* of two graphs G and H is the graph $G+H$ with vertex set $V(G+H) = V(G) \overset{\bullet}{\cup} V(H)$ and edge set $E(G+H) = E(G) \overset{\bullet}{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 3. Let G and H be any two graphs. Then $S \subseteq V(G+H)$ is a connected co-independent hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$ where one of the following holds:

(i) $S_G = V(G)$ and S_H is a strictly co-independent set of H.

(ii) $S_H = V(H)$ and S_G is a strictly co-independent set of G.

Proof: Suppose S is a connected co-independent hop dominating set of $G+H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Then $S = S_G \cup S_H$. Since S is a hop dominating set of $G+H$, $S_G \neq \varnothing$ and $S_H \neq \varnothing$. Since $V(G+H)\backslash S$ is an independent set, $S_G = V(G)$ or $S_H = V(H)$. Suppose $S_G = V(G)$. Then $V(H)\backslash S_H = V(G+H)\backslash S$ is an independent set. Let $v \in V(H)\backslash S_H$. Then $v \in V(G+H)\backslash S$. Since S is a hop dominating set, there exists $w \in S$ such that $d_{G+H}(v,w) = 2$. Hence, $w \in S_H \backslash N_H(v)$. Thus, $N_H(v) \cap S_H \neq S_H$, showing that S_H is a strictly co-independent set of H. Thus, (i) holds. Similarly, if $S_H = V(H)$, then S_G is a strictly co-independent set of G and (ii) holds.

For the converse, suppose $S = S_G \cup S_H$ where S_G and S_H satisfy the given conditions. Let $v \in V(G+H)\backslash S$. Consider the following cases.

Case 1. $S_G = V(G)$.
Then $v \in V(H)\backslash S_H$. By (i), there exists $w \in S_H \backslash N_H(v)$. Hence, $w \in S$ and $d_{G+H}(v,w) = 2$.

Case 2. $S_H = V(H)$
Then $v \in V(G)\backslash S_G$. By (ii), there exists $u \in S_G \backslash N_G(v)$. Thus, $u \in S$ and $d_{G+H}(u,v) = 2$. Therefore, in either case, S is a hop dominating set of $G+H$.

Since $V(G+H)\backslash S = V(H)\backslash S_H$ if (i) holds or $V(G+H)\backslash S = V(G)\backslash S_G$ if (ii) holds, $V(G+H)\backslash S$ is an independent set. It is clear from the definition of the join of G and H that $\langle S \rangle$ is connected. Therefore, S is a connected co-independent hop dominating set of $G+H$. □

Corollary 1. Let G and H be any two graphs where $|V(G)| = n$ and $|V(H)| = m$. Then $\gamma_{ch,coi}(G + H) = min\{n + sci(H), m + sci(G)\}$.

Proof: Let S be a $\gamma_{ch,coi}$-set of $G + H$. Then S is a connected co-independent hop dominating set of $G + H$. Hence, $S = A \cup B$ whre $A \subseteq V(G)$ and $B \subseteq V(H)$ satisfying conditions (i) or (ii) of Theorem 3. Thus, $\gamma_{ch,coi}(G + H) = |S| = |A| + |B|$.
By condition (i), $|S| = |V(G)| + |B| \geq n + sci(H)$.
By condition (ii), $|S| = |V(H)| + |A| \geq m + sci(G)$.
Hence, $\gamma_{ch,coi}(G + H) = |S| \geq min\{n + sci(H), m + sci(G)\}$.

Next, let X and Y be the minimum strictly co-independent sets of G and H, respectively. Then by Theorem 3, $S = V(G) \cup Y$ or $S = V(H) \cup X$ is a connected co-independent hop dominating set of $G + H$. Thus,

$$\begin{aligned}\gamma_{ch,coi}(G + H) &\leq |S| \\ &= |V(G)| + |Y| \\ &= n + sci(H)\end{aligned}$$

or

$$\begin{aligned}\gamma_{ch,coi}(G + H) &\leq |S| \\ &= |V(H)| + |X| \\ &= m + sci(G)\end{aligned}$$

It follows that, $\gamma_{ch,coi}(G + H) \leq min\{n + sci(H), m + sci(G)\}$.
Therefore, $\gamma_{ch,coi}(G + H) = min\{n + sci(H), m + sci(G)\}$. □

4. On Connected Co-Independent Hop Domination in the Corona of Graphs

The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining every vertex of the ith copy of H to the ith vertex of G. For $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle\{v\}\rangle + H^v, v \in V(G)$.

Theorem 4. Let G be a nontrivial connected graph and H be any graph. A set $S \subseteq V(G \circ H)$ is a connected co-independent hop dominating set of $G \circ H$ if and only if $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$, where $S_v \subseteq V(H^v)$ and $V(H^v) \backslash S_v$ is an independent subset of $V(H^v)$ for each $v \in V(G)$.

Proof: Suppose S is a connected co-independent hop dominating set of $G \circ H$ and let $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Then $S_v \subseteq V(H^v)$. Since $\langle S\rangle$ is connected, $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$. Since $V(G \circ H) \backslash S$ is independent and $V(G \circ H) \backslash S = \bigcup_{v \in V(G)} (V(H^v) \backslash S_v)$,

$V(H^v)\backslash S_v$ is an independent subset of $V(H^v)$, for each $v \in V(G)$.

For the converse, suppose that $S = V(G) \cup (\bigcup_{v\in V(G)} S_v)$ where $S_v \subseteq V(H^v)$ and $V(H^v)\backslash S_v$ is an independent set. Clearly, $\langle S\rangle$ is connected. Let $w \in V(G \circ H)\backslash S$. Then $w \in V(H^v)\backslash S_v$ for some $v \in V(G)$. Since G is nontrivial connected graph, there exists $x \in V(G)$ such that $vx \in E(G)$. Thus, $d_{G\circ H}(w, x) = 2$. This implies that S is a hop dominating set of $G \circ H$. Since $V(G \circ H)\backslash S = \bigcup_{v\in V(G)} (V(H^v)\backslash S_v)$ and $V(H^v)\backslash S_v$ is an independent set for each $v \in V(G)$, $V(G \circ H)\backslash S$ is independent. Therefore, S is a connected co-independent hop dominating set of $G \circ H$. □

Corollary 2. Let G be a nontrivial connected graph of order n and H be any graph of order m. Then $\gamma_{ch,coi}(G \circ H) = n(1 + m - \beta(H))$.

Proof: Let C be a $\gamma_{ch,coi}$-set of $G \circ H$. Then C is a connected co-independent hop dominating set of $G \circ H$. By Theorem 4, $C = V(G)\bigcup(\bigcup_{v\in V(G)} S_v)$ where $V(H^v)\backslash S_v$ is an independent set of H^v for every $v \in V(G)$. Then

$$\begin{aligned}
\gamma_{ch,coi}(G \circ H) = |C| &= |V(G)| + |\bigcup_{v\in V(G)} S_v| \\
&= |V(G)| + \sum_{v\in V(G)} |S_v| \\
&= |V(G)| + \sum_{v\in V(G)} (|V(H^v)| - |V(H^v)\backslash S_v|) \\
&\geq |V(G)| + |V(G)|(|V(H^v)| - \beta(H)) \\
&= n + n(m - \beta(H)) \\
&= n(1 + m - \beta(H)).
\end{aligned}$$

Therefore, $\gamma_{ch,coi}(G \circ H) \geq n(1 + m - \beta(H))$.

Let D be a maximum independent set of H. For each v, let $D_v \subseteq V(H^v)$ such that $\langle D_v\rangle \cong \langle D\rangle$. Let $S_v = V(H^v)\backslash D_v$. Then $C = V(G)\bigcup(\bigcup_{v\in V(G)} S_v)$ is a connected co-independent hop dominating set of $G \circ H$ by Theorem 4. Thus,

$$\begin{aligned}
\gamma_{ch,coi}(G \circ H) \leq |C| &= |V(G) \cup (\bigcup_{v\in V(G)} S_v)| \\
&= |V(G)| + \sum_{v\in V(G)} |S_v| \\
&= |V(G)| + |V(G)|(|V(H^v)| - |D_v|) \\
&= |V(G)| + |V(G)|(|V(H^v)| - \beta(H)) \\
&= n + n(m - \beta(H)) \\
&= n(1 + m - \beta(H)).
\end{aligned}$$

Therefore, $\gamma_{ch,coi}(G \circ H) \leq n(1 + m - \beta(H))$.
Consequently, $\gamma_{ch,coi}(G \circ H) = n(1 + m - \beta(H))$. □

5. On Connected Co-Independent Hop Domination in the Lexicographic Product of Graphs

The *lexicographic product* of two graphs G and H, denoted by $G[H]$, is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ such that $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

Theorem 5. Let G and H be nontrivial connected graphs with $|V(G)| = n$. A subset $C = \bigcup_{x \in S}(\{x\} \times T_x)$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ of $V(G[H])$ is a connected co-independent hop dominating set if and only if

(i) $S = V(G)$.

(ii) For every $x \in V(G)$ such that $T_x \neq V(H)$, $V(H) \backslash T_x$ is an independent set and $T_y = V(H)$ for every $y \in N_G(x)$ where T_x is a hop dominating set of H if $deg_G(x) = n-1$.

Proof: Suppose C is a connected co-independent hop dominating set of $G[H]$ and $S \neq V(G)$. Then, a vertex $v \in V(G) \backslash S$ exists. Thus, $(v, z) \in V(G[H]) \backslash C$ for all $z \in V(H)$. Since H is a nontrivial connected graph, an edge $pq \in E(H)$ exists. Hence, $(v, p), (v, q) \in V(G[H]) \backslash C$ and $(v, p)(v, q) \in E(G[H])$. This contradicts the independence of $V(G[H]) \backslash C$. It follows that $S = V(G)$ and (i) holds. Now, let $x \in V(G)$ such that $T_x \neq V(H)$. We claim that $V(H) \backslash T_x$ is an independent set. Let $u, w \in V(H) \backslash T_x$ where $u \neq w$. Then $(x, u), (x, w) \in V(G[H]) \backslash C$. Since $V(G[H]) \backslash C$ is independent, $(x, u)(x, w) \notin E(G[H])$. Thus, $uw \notin E(H)$. Hence, $V(H) \backslash T_x$ is an independent set. Now, we show that $T_y = V(H)$ for every $y \in N_G(x)$. Suppose $T_y \neq V(H)$. Then there exists $p \in V(H) \backslash T_y$. Thus, $(y, p) \in V(G[H]) \backslash C$. Since $y \in N_G(x)$, $(y, p)(x, q) \in E(G[H])$ for all $q \in V(H)$. This contradicts the independence of $V(G[H]) \backslash C$. Hence, $T_y = V(H)$. Lastly, suppose $deg_G(x) = n - 1$. Then $xa \in E(G)$ for all $a \in V(G) \backslash \{x\}$. Since $V(G) \neq T_x$, a vertex $b \in V(H) \backslash T_x$ exists. Thus, $(x, b) \in V(G[H]) \backslash C$. Since C is a hop dominating set and $(x, b)(a, d) \in E(G[H])$ for all $a \in V(G) \backslash \{x\}$ and $d \in T_a$, there exists $z \in T_x$ such that $d_{G[H]}((x, b), (x, z)) = 2$. Hence, $d_H(b, z) = 2$, showing that $z \in T_x \backslash N_H(b)$. Hence, $N_H(b) \cap T_x \neq T_x$ implying that T_x is strictly co-independent set of H. Thus, *(ii)* holds.

Conversely, suppose $C = \bigcup_{x \in S}(\{x\} \times T_x)$ satisfies conditions (i) and (ii). First, we claim that C is connected in $G[H]$. Let (x, a) and (y, b) be two distinct vertices in C, $(x, a)(y, b) \notin E(G[H])$. Consider the following cases.

Case 1. x=y

Since $(x, a) \neq (y, b)$ and $(x, a)(y, b) \notin E(G[H])$, $a \neq b$ and $ab \notin E(H)$. Since G is a nontrivial connected graph and $S = V(G)$ by (i), there exists $z \in S \cap N_G(x)$. Thus, $(z, w) \in C$ for some $w \in V(H)$. It follows that $[(x, a), (z, w), (y, b)]$ is a path in C.

Case 2. $x \neq y$

Since G is a nontrivial connected graph, there exists an x-y path $[v_1, v_2, ..., v_n]$ where $x = v_1$ and $y = v_n$, $n > 2$. By (i) $v_i \in S$ for all $i \in \{1, 2, ..., n\}$. Let $u_i \in T_{v_i}$, $u_1 = a$ and

$u_n = b$. Then $[(x, a), (v_2, u_2), (v_3, u_3), ..., (y, b)]$ is an (x, a)-(y, b) path in C.

Therefore in either case, C is connected.

Now, let $(u, v), (w, p) \in V(G[H])\backslash C$ where $(u, v) \neq (w, p)$. Consider the following cases.

Case 1. u=w

Since $(u, v), (w, p) \notin C$, $v, p \notin T_u = T_w$ and $v \neq p$. Hence, $v, p \in V(H)\backslash T_u$. Since, $V(H)\backslash T_u$ is independent by (ii), $vp \notin E(H)$. Thus,
$(u, v)(w, p) \notin E(G[H])$.

Case 2. $u \neq w$

Since $v \notin T_u$ and $p \notin T_w$, $T_u \neq V(H)$ and $T_w \neq V(H)$. By (ii), $u \notin N_G(w)$. Thus, $(u, v)(w, p) \notin E(G[H])$.

Therefore, in any case $V(G[H])\backslash C$ is an independent set.

Finally, we show that C is a hop dominating set. Let $(x, y) \in V(G[H])\backslash C$. Then $y \notin T_x$, that is, $V(H) \neq T_x$. Suppose $deg_G(x) = n-1$. By (ii), T_x is a strictly co-independent set of H. Since $y \notin T_x$, $N_H(y) \cap T_x \neq T_x$. This implies that there exists $a \in T_x$ such that $d_H(a, y) = 2$. Hence, $(x, a) \in C$ and $d_{G[H]}((x, y), (x, a)) = 2$. Suppose $deg_G(x) < n - 1$. Then a vertex $z \in V(G)\backslash N_G(x)$ exists. Choose z such that $d_G(x, z) = 2$. Since $S = V(G)$ by (i), there exists $b \in T_z$, that is, $(z, b) \in C$. It follows that $d_{G[H]}((x, y), (z, b)) = 2$. Therefore, C is a hop dominating set of $G[H]$.

Accordingly, C is a connected co-independent hop dominating set of $G[H]$. □

Corollary 3. Let G be any connected noncomplete graph of order m and H be any nontrivial connected graph of order n. Then

$$\gamma_{ch,coi}(G[H]) = m(n - \beta(H)) + r(G)\beta(H),$$

where $r(G) = min\{|D| : V(G)\backslash D \textit{ is an independent set}\}$ and $\beta(H)$ is an independence number of H.

Proof: Let $r(G) = \min\{|D| : V(G)\backslash D$ is an independent set$\}$. Let $D_0 \subseteq V(G)$ such that $V(G)\backslash D_0$ is an independent set and $|D_0| = r(G)$. Let T be a β-set of H. Let $T_x = V(H)\backslash T$ for each $x \in V(G)\backslash D_0$ and let $T_y = V(H)$ for each $y \in D_0$. Then

$$C = \bigcup_{x\in V(G)} [\{x\} \times T_x] = \bigcup_{y\in D_0} (\{y\} \times T_y) \cup \left(\bigcup_{x\in V(G)\backslash D_0} [\{x\} \times T_x] \right)$$

is a connected co-independent hop dominating set of $G[H]$, by Theorem 5. Hence,

$$\begin{aligned}\gamma_{ch,coi}(G[H]) \leq |C| &= nr(G) + (m - r(G))(n - \beta(H)) \\ &= nr(G) + mn - m\beta(H) - nr(G) + r(G)\beta(H) \\ &= mn - m\beta(H) + r(G)\beta(H) \\ &= m(n - \beta(H)) + r(G)\beta(H)\end{aligned}$$

$$\gamma_{c,coi}(G[H]) \leq m(n - \beta(H)) + r(G)\beta(H) \quad .$$

Let $C_0 = \bigcup_{x\in V(G)} [\{x\} \times R_x]$ be a $\gamma_{ch,coi}$-set of $G[H]$. Let $D = \{x \in V(G) : R_x = V(H)\}$. Then $V(G)\backslash D$ is an independent set of G. Then

$$C_0 = \left(\bigcup_{x\in D} R_x\right) \cup \left(\bigcup_{x\in V(G)\backslash D} T_x\right).$$

Moreover,

$$\gamma_{ch,coi}(G[H]) = |C_0| = n|D| + \sum_{x\in V(G)\backslash D} |T_x|.$$

Since $V(H)\backslash T_x$ is an independent set of H for each $x \in V(G)\backslash D$, it follows that $|V(H)\backslash T_x| \leq \beta(H)$ for each $x \in V(G)\backslash D$. Thus, $|V(H)\backslash T_x| = |V(H)| - |T_x| \leq \beta(H)$. Hence $|T_x| \geq n - \beta(H)$. Therefore,

$$\begin{aligned}
\gamma_{ch,coi}(G[H]) = |C_0| &= n|D| + |V(G)\backslash D|\,|T_x| \\
&= n|D| + (|V(G) - |D|)\,|T_x| \\
&\geq n|D| + (m - |D|)(n - \beta(H)) \\
&= n|D| + mn - m\beta(H) - n|D| + |D|\beta(H) \\
&= m(n - \beta(H)) + |D|\beta(H) \\
&= m(n - \beta(H)) + r(G)\beta(H)
\end{aligned}$$

$$\gamma_{c,coi}(G[H]) \geq m(n - \beta(H)) + r(G)\beta(H),$$

Therefore, $\gamma_{c,coi}(G[H]) = m(n - \beta(H)) + r(G)\beta(H)$. □

Corollary 4. Let H be any nontrivial connected graph of order m. Then $\gamma_{ch,coi}(K_n[H]) = m(n-1) + sci(H)$.

Proof: Let T be an *sci*-set of H. Let $v \in V(K_n)$ and $T_v = T$. By Theorem 5, $C = \bigcup_{y\in V(K_n)\backslash\{v\}} (\{y\} \times T_y) \cup (\{v\} \times T_v)$ is a connected co-independent hop dominating set of $K_n[H]$. Since $y \in N_{K_n}(v)$ for each $y \in V(K_n)\backslash\{v\}$, $T_y = V(H)$. Thus,

$$\begin{aligned}
\gamma_{ch,coi}(K_n[H]) &\leq |C| \\
&= (n-1)|T_y| + |T_v| \\
&= (n-1)m + |T| \\
&= (n-1)m + sci(H).
\end{aligned}$$

Let $C_o = \bigcup_{x\in V(K_n)} (\{x\} \times R_x)$ be a $\gamma_{ch,coi}$-set of $K_n[H]$. Since $deg_{K_n}(x) = n-1$ for each $x \in V(K_n)$, by Theorem 5, $R_y = T$ where T is a strictly co-independent set of H for a unique $y \in V(K_n)$ and $R_x = V(H)$ for all $x \in V(K_n)\backslash\{y\}$. Hence, $C_o = (\{y\} \times R_y) \cup (\bigcup_{x\in V(K_n)\backslash\{y\}} (\{x\} \times R_x)$ and

$$\gamma_{ch,coi}(K_n[H]) = |C_o|$$

$$= |R_y| + (n-1)|R_x|$$
$$= |T| + (n-1)|V(H)|$$
$$\geq sci(H) + (n-1)m.$$

Therefore, $\gamma_{ch,coi}(K_n[H]) = m(n-1) + sci(H)$. □

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