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Upper and lower almost weak (τ_1, τ_2)-continuity

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Abstract. The purpose of the present paper is to introduce the notions of upper and lower almost weakly (τ_1, τ_2)-continuous multifunctions. Several characterizations of upper and lower almost weakly (τ_1, τ_2)-continuous multifunctions are investigated.

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1. Introduction

Topology as a field of mathematics is concerned with all questions directly or indirectly related to continuity. Continuity of functions in topological spaces has been investigated by many mathematicians. This concept has been extended to the setting multifunctions and has been generalized by weaker forms of open sets. Semi-open sets [18], preopen sets [19], α-open sets [20] and β-open sets [10] play an important role in the researching of generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of weak forms of continuity for functions and multifunctions. In 1961, Levine [17] introduced the concept of weakly continuous functions in topological spaces. Husain [11] introduced the concept of almost continuous functions. Janković [12] defined almost weakly continuous functions as a generalization of both weakly continuous functions due to Levine [17] and almost continuous functions in the sense of Husain [11]. Noiri and Popa [21, 25] investigated further characterizations of almost weakly continuous functions. Smithson [27] and Popa [23, 24] extended independently these concepts to multifunctions by introducing and characterizing the notions of almost continuous multifunctions and weakly continuous multifunctions. Ekici and Park [9] introduced and studied upper and lower almost γ-continuous multifunctions as a generalization of some types of continuous multifunctions including almost continuity, almost α-continuity, almost precontinuity, almost quasi-continuity and γ-continuity. The concept

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of bitopological spaces was first introduced by Kelly [14]. Şenel and Çağman [8] extended the notion of bitopological spaces to soft bitopological spaces. Şenel [7] presented the concept of soft bitopological Hausdorff spaces and introduced some new notions in soft bitopological spaces such as SBT points, SBT continuous functions and SBT homeomorphisms. Khedr et al. [15] investigated the notions of β-open sets and β-continuity in bitopological spaces. In 2020, Laprom et al. [16] introduced and investigated the notions of $\beta(\tau_1, \tau_2)$-continuous multifunctions and almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. In this paper, we introduce the concepts of upper and lower almost weakly (τ_1, τ_2)-continuous multifunctions. Furthermore, several characterizations of upper and lower almost weakly (τ_1, τ_2)-continuous multifunctions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let (X, τ_1, τ_2) be a bitopological space and let A be a subset of X. The closure of A and the interior of A with respect to the topology τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-semi-open [5] (resp. $\tau_1\tau_2$-regular open [2], $\tau_1\tau_2$-regular closed [6], $\tau_1\tau_2$-preopen [13]) if $A \subseteq \tau_1$-Cl$(\tau_2$-Int$(A))$ (resp. $A = \tau_1$-Int$(\tau_2$-Cl$(A))$, $A = \tau_1$-Cl$(\tau_2$-Int$(A))$, $A \subseteq \tau_1$-Int$(\tau_2$-Cl$(A)))$. The complement of $\tau_1\tau_2$-semi-open (resp. $\tau_1\tau_2$-preopen) set is said to be $\tau_1\tau_2$-semi-closed (resp. $\tau_1\tau_2$-preclosed). The $\tau_1\tau_2$-semi-closure [5] (resp. $\tau_1\tau_2$-preclosure [15]) of A is defined by the intersection of $\tau_1\tau_2$-semi-closed (resp. $\tau_1\tau_2$-preclosed) sets containing A and is denoted by $\tau_1\tau_2$-sCl(A) (resp. $\tau_1\tau_2$-pCl(A)). The $\tau_1\tau_2$-semi-interior [5] (resp. $\tau_1\tau_2$-preinterior [22]) of A is defined by the union of $\tau_1\tau_2$-semi-open (resp. $\tau_1\tau_2$-preopen) sets contained in A and is denoted by $\tau_1\tau_2$-sInt(A) (resp. $\tau_1\tau_2$-pInt(A)).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [3], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$.

Lemma 1. [22] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties are hold:*

(1) $\tau_1\tau_2$-pInt(A) is $\tau_1\tau_2$-preopen.

(2) $\tau_1\tau_2$-pCl(A) is $\tau_1\tau_2$-preclosed.

Lemma 2. [22] *For a subset A of a bitopological space (X, τ_1, τ_2), $x \in \tau_1\tau_2$-pCl(A) if and only if $U \cap A \neq \emptyset$ for every $\tau_1\tau_2$-preopen set U containing x.*

Lemma 3. [22] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties are hold:*

(1) $X - \tau_1\tau_2\text{-}pInt(A) = \tau_1\tau_2\text{-}pCl(X - A)$.

(2) $X - \tau_1\tau_2\text{-}pCl(A) = \tau_1\tau_2\text{-}pInt(X - A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-closed* [5] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is said to be $\tau_1\tau_2$*-open*. The intersection of all $\tau_1\tau_2$-closed sets containing A is called the $\tau_1\tau_2$*-closure* [5] of A and denoted by $\tau_1\tau_2\text{-Cl}(A)$. The union of all $\tau_1\tau_2$-open sets contained in A is called the $\tau_1\tau_2$*-interior* [5] of A and denoted by $\tau_1\tau_2\text{-Int}(A)$. A subset N of a bitopological space (X, τ_1, τ_2) is said to be a $\tau_1\tau_2$*-neighbourhood* [5] (resp. $\tau_1\tau_2$*-preneighbourhood* [5]) of $x \in X$ if there exists a $\tau_1\tau_2$-open (resp. $\tau_1\tau_2$-preopen) set V of (X, τ_1, τ_2) such that $x \in V \subseteq N$.

Lemma 4. [5] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ *and* $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) *If* $A \subseteq B$, *then* $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ *is* $\tau_1\tau_2$*-closed.*

(4) A *is* $\tau_1\tau_2$*-closed if and only if* $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

Lemma 5. [1] *For a subset A of a topological space (X, τ), the following properties hold:*

(1) $Cl(A) \cap G \subseteq Cl(A \cap G)$ *for every open set* G.

(2) $Int(A \cup F) \subseteq Int(A) \cup F$ *for every closed set* F.

Lemma 6. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $\tau_1\tau_2\text{-}pCl(A) = A \cup \tau_1\text{-}Cl(\tau_2\text{-}Int(A))$.

(2) $\tau_1\tau_2\text{-}pInt(A) = A \cap \tau_1\text{-}Int(\tau_2\text{-}Cl(A))$.

Proof. (1) To begin with, observe that

$$\begin{aligned}
\tau_1\text{-Cl}(\tau_2\text{-Int}(A \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(A)))) &\subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(A) \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(A))) \\
&= \tau_1\text{-Cl}(\tau_1\text{-Cl}(\tau_2\text{-Int}(A))) \\
&= \tau_1\text{-Cl}(\tau_2\text{-Int}(A)) \\
&\subseteq A \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(A))
\end{aligned}$$

by Lemma 5(2). Hence, $A \cup \tau_1$-Cl(τ_2-Int(A)) is $\tau_1\tau_2$-preclosed and thus

$$\tau_1\tau_2\text{-pCl}(A) \subseteq A \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(A)).$$

On the other hand, since $\tau_1\tau_2$-pCl(A) is $\tau_1\tau_2$-preclosed, we have

$$\tau_1\text{-Cl}(\tau_2\text{-Int}(A)) \subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-pCl}(A))) \subseteq \tau_1\tau_2\text{-pCl}(A)$$

and so $A \cup \tau_1$-Cl(τ_2-Int(A)) $\subseteq \tau_1\tau_2$-pCl(A). Consequently, we obtain

$$A \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(A)) = \tau_1\tau_2\text{-pCl}(A).$$

(2) This follows from (1).

3. Characterizations

In this section, we introduce the notions of upper and lower almost weakly (τ_1, τ_2)-continuous multifunctions. Moreover, some characterizations of upper and lower almost weakly (τ_1, τ_2)-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:*

(1) upper almost weakly (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, $x \in \tau_1$-Int(τ_2-Cl($F^+(\sigma_1\sigma_2$-Cl(V))));

(2) lower almost weakly (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, $x \in \tau_1$-Int(τ_2-Cl($F^-(\sigma_1\sigma_2$-Cl(V)))).

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost weakly (τ_1, τ_2)-continuous;

(2) $F^+(V) \subseteq \tau_1$-Int(τ_2-Cl($F^+(\sigma_1\sigma_2$-Cl(V)))) for every $\sigma_1\sigma_2$-open set V of Y;

(3) τ_1-Cl(τ_2-Int($F^-(V)$)) $\subseteq F^-(\sigma_1\sigma_2$- Cl($V$)) for every $\sigma_1\sigma_2$-open set V of Y;

(4) $\tau_1\tau_2$-pCl($F^-(V)$) $\subseteq F^-(\sigma_1\sigma_2$-Cl($V$)) for every $\sigma_1\sigma_2$-open set V of Y;

(5) $F^+(V) \subseteq \tau_1\tau_2$-pInt($F^+(\sigma_1\sigma_2$-Cl($V$))) for every $\sigma_1\sigma_2$-open set V of Y;

(6) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-preopen set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl($V$).

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by (1), we have $x \in \tau_1$-Int(τ_2-Cl($F^+(\sigma_1\sigma_2$-Cl(V)))). Therefore, $F^+(V) \subseteq \tau_1$-Int(τ_2-Cl($F^+(\sigma_1\sigma_2$-Cl(V)))).

(2) ⇒ (3): Let V be any $\sigma_1\sigma_2$-open set of Y. Since $Y - \sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-open and by (2),

$$\begin{aligned} X - F^-(\sigma_1\sigma_2\text{-Cl}(V)) &= F^+(Y - \sigma_1\sigma_2\text{-Cl}(V)) \\ &\subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))))) \\ &\subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(F^+(Y - V))) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(X - F^-(V))) \\ &= X - \tau_1\text{-Cl}(\tau_2\text{-Int}(F^-(V))). \end{aligned}$$

Thus, $\tau_1\text{-Cl}(\tau_2\text{-Int}(F^-(V))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. By (3) and Lemma 6(1),

$$\tau_1\tau_2\text{-pCl}(F^-(V)) = F^-(V) \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(4) ⇒ (5): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - \sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-open and by (4),

$$\begin{aligned} X - \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) &= \tau_1\tau_2\text{-pCl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= \tau_1\tau_2\text{-pCl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^-(Y - V) \\ &= X - F^+(V). \end{aligned}$$

Thus, $F^+(V) \subseteq \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(5) ⇒ (6): Let $x \in X$ and let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (5), $x \in F^+(V) \subseteq \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and there exists a $\tau_1\tau_2$-preopen set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$.

(6) ⇒ (1): Let $x \in X$ and let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (6), there exists a $\tau_1\tau_2$-preopen set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$; hence $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $x \in U \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(U)) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Cl}(V))))$. This shows that F is upper almost weakly (τ_1, τ_2)-continuous.

Theorem 2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost weakly (τ_1, τ_2)-continuous;

(2) $F^-(V) \subseteq \tau_1\text{-}Int(\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\tau_1\text{-}Cl(\tau_2\text{-}Int(F^+(V))) \subseteq F^+(\sigma_1\sigma_2\text{- }Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $\tau_1\tau_2\text{-}pCl(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $F^-(V) \subseteq \tau_1\tau_2\text{-}pInt(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(6) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-preopen set U of X containing x such that $F(z) \cap \sigma_1\sigma_2\text{-}Cl(V) \neq \emptyset$ for each $z \in U$.

Proof. The proof is similar to that of Theorem 1.

Theorem 3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost weakly (τ_1, τ_2)-continuous;

(2) $\tau_1\text{-}Cl(\tau_2\text{-}Int(F^-(\sigma_1\sigma_2\text{-}Int(H)))) \subseteq F^-(H)$ for every $\sigma_1\sigma_2$-closed set H of Y;

(3) $\tau_1\tau_2\text{-}pCl(F^-(\sigma_1\sigma_2\text{-}Int(H))) \subseteq F^-(H)$ for every $\sigma_1\sigma_2$-closed set H of Y;

(4) $\tau_1\tau_2\text{-}pCl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}pInt(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset B of Y.

Proof. (1) ⇒ (2): Let H be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - H$ is $\sigma_1\sigma_2$-open in Y, by Theorem 1,

$$\begin{aligned} X - F^-(H) &= F^+(Y - H) \\ &\subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Cl}(Y - H)))) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(F^+(Y - \sigma_1\sigma_2\text{-Int}(H)))) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(X - F^-(\sigma_1\sigma_2\text{-Int}(H)))) \\ &= X - \tau_1\text{-Cl}(\tau_2\text{-Int}(F^-(\sigma_1\sigma_2\text{-Int}(H)))) \end{aligned}$$

and hence $\tau_1\text{-Cl}(\tau_2\text{-Int}(F^-(\sigma_1\sigma_2\text{-Int}(H)))) \subseteq F^-(H)$.

(2) ⇒ (3): Let H be any $\sigma_1\sigma_2$-closed set of Y. By Lemma 6(1), we have

$$\begin{aligned} \tau_1\tau_2\text{-pCl}(F^-(\sigma_1\sigma_2\text{-Int}(H))) &= F^-(\sigma_1\sigma_2\text{-Int}(H)) \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(F^-(\sigma_1\sigma_2\text{-Int}(H)))) \\ &\subseteq F^-(H). \end{aligned}$$

(3) ⇒ (4): This is obvious.

(4) ⇒ (5): Let B be any subset of Y. By (4), we have

$$\begin{aligned} X - \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) &= \tau_1\tau_2\text{-pCl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= \tau_1\tau_2\text{-pCl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= \tau_1\tau_2\text{-pCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(B)). \end{aligned}$$

Thus, $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$.

(5) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. By (5), we have

$$F^+(V) \subseteq \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$$

and hence F is upper almost weakly (τ_1, τ_2)-continuous by Theorem 1.

Theorem 4. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost weakly (τ_1, τ_2)-continuous;

(2) τ_1-Cl(τ_2-Int($F^+(\sigma_1\sigma_2$-Int(H)))) $\subseteq F^+(H)$ for every $\sigma_1\sigma_2$-closed set H of Y;

(3) $\tau_1\tau_2$-pCl($F^+(\sigma_1\sigma_2$-Int(H))) $\subseteq F^+(H)$ for every $\sigma_1\sigma_2$-closed set H of Y;

(4) $\tau_1\tau_2$-pCl($F^+(\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl(B)))) $\subseteq F^+(\sigma_1\sigma_2$-Cl($B$)) for every subset B of Y;

(5) $F^-(\sigma_1\sigma_2$-Int(B)) $\subseteq \tau_1\tau_2$-pInt($F^-(\sigma_1\sigma_2$-Cl($\sigma_1\sigma_2$-Int(B)))) for every subset B of Y.

Proof. The proof is similar to that of Theorem 3.

Definition 2. [28] *Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called $(\tau_1, \tau_2)\theta$-cluster point of A if $\tau_1\tau_2$-Cl(U) $\cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-closure of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A).*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$*-closed* [28] if $A = (\tau_1, \tau_2)\theta$-Cl$(A)$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets contained in A is called the $(\tau_1, \tau_2)\theta$*-interior* [28] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 7. [28] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_2\tau_2$-open in X, then $\tau_1\tau_2$-Cl(A) $= (\tau_1, \tau_2)\theta$-Cl(A).

(2) $(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.

Definition 3. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-closed* [28] *(resp. $(\tau_1, \tau_2)p$-open* [4]*) if $A = \tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A)) (resp. $A \subseteq \tau_1\tau_2$-Int($\tau_1\tau_2$-Cl(A))).*

Theorem 5. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost weakly (τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-pCl($F^-(\sigma_1\sigma_2$-Int($(\sigma_1, \sigma_2)\theta$-Cl($B$)))) $\subseteq F^-((\sigma_1, \sigma_2)\theta$-Cl($B$)) for every subset B of Y;

(3) $\tau_1\tau_2$-pCl($F^-(\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl(V)))) $\subseteq F^-(\sigma_1\sigma_2$-Cl($V$)) for every $\sigma_1\sigma_2$-open set V of Y;

(4) $\tau_1\tau_2$-pCl($F^-(\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl(V)))) $\subseteq F^-(\sigma_1\sigma_2$-Cl($V$)) for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(5) $\tau_1\tau_2$*-pCl*$(F^-(\sigma_1\sigma_2$*-Int*$(H))) \subseteq F^-(H)$ *for every* $(\sigma_1, \sigma_2)r$*-closed set* H *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Let $x \in X - F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Then, $x \in F^+(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ and $(\sigma_1, \sigma_2)\theta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y. By Theorem 1, there exists a $\tau_1\tau_2$-preopen set U of X containing x such that

$$\begin{aligned} U &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B))) \\ &= F^+(Y - \sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))) \\ &= X - F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))). \end{aligned}$$

Thus, $U \cap F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))) = \emptyset$ and hence

$$x \in X - \tau_1\tau_2\text{-pCl}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))).$$

Therefore, $\tau_1\tau_2\text{-pCl}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$.

(2) $\Rightarrow$ (3): The proof is obvious since $(\sigma_1, \sigma_2)\theta\text{-Cl}(V) = \sigma_1\sigma_2\text{-Cl}(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and by (3),

$$\begin{aligned} &\tau_1\tau_2\text{-pCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= \tau_1\tau_2\text{-pCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Int}(V)))) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(V)). \end{aligned}$$

(4) $\Rightarrow$ (5): Let H be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, $\sigma_1\sigma_2\text{-Int}(H)$ is $(\sigma_1, \sigma_2)p$-open in Y and by (4),

$$\begin{aligned} \tau_1\tau_2\text{-pCl}(F^-(\sigma_1\sigma_2\text{-Int}(H))) &= \tau_1\tau_2\text{-pCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(H))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(H))) \\ &= F^-(H). \end{aligned}$$

(5) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)r$-closed in Y and by (5),

$$\tau_1\tau_2\text{-pCl}(F^-(V)) \subseteq \tau_1\tau_2\text{-pCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

It follows from Theorem 1 that F is upper almost weakly (τ_1, τ_2)-continuous.

Theorem 6. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower almost weakly* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2$*-pCl*$(F^+(\sigma_1\sigma_2$*-Int*$((\sigma_1, \sigma_2)\theta$*-Cl*$(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta$*-Cl*$(B))$ *for every subset* B *of* Y*;*

(3) $\tau_1\tau_2$-$pCl(F^+(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V)))) \subseteq F^+(\sigma_1\sigma_2$-$Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(4) $\tau_1\tau_2$-$pCl(F^+(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V)))) \subseteq F^+(\sigma_1\sigma_2$-$Cl(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(5) $\tau_1\tau_2$-$pCl(F^+(\sigma_1\sigma_2$-$Int(H))) \subseteq F^+(H)$ *for every* $(\sigma_1, \sigma_2)r$*-closed set* H *of* Y*.*

Proof. The proof is similar to that of Theorem 5.

In order to obtain further characterizations of upper and lower almost weakly (τ_1, τ_2)-continuous multifunctions, we recall some definitions. For a multifunction

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2),$$

by $\mathrm{Cl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ [5] (resp. $\mathrm{pCl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$) we denote a multifunction defined as follows: $\mathrm{Cl}F_{\circledast}(x) = \sigma_1\sigma_2\text{-}\mathrm{Cl}(F(x))$ (resp. $\mathrm{pCl}F_{\circledast}(x) = \sigma_1\sigma_2\text{-}\mathrm{pCl}(F(x))$) for each $x \in X$.

Definition 4. [5] *A subset* A *of a bitopological space* (X, τ_1, τ_2) *is said to be:*

(1) $\tau_1\tau_2$*-paracompact if every cover of* A *by* $\tau_1\tau_2$*-open sets of* X *is refined by a cover of* A *which consists of* $\tau_1\tau_2$*-open sets of* X *and is* $\tau_1\tau_2$*-locally finite in* X*;*

(2) $\tau_1\tau_2$*-regular if for each* $x \in A$ *and each* $\tau_1\tau_2$*-open set* U *of* X *containing* x*, there exists a* $\tau_1\tau_2$*-open set* V *of* X *such that* $x \in V \subseteq \tau_1\tau_2$-$Cl(V) \subseteq U$.

Lemma 8. [5] *If* A *is a* $\tau_1\tau_2$*-regular* $\tau_1\tau_2$*-paracompact set of a bitopological space* (X, τ_1, τ_2) *and* U *is a* $\tau_1\tau_2$*-open neighbourhood of* A*, then there exists a* $\tau_1\tau_2$*-open set* V *of* X *such that* $A \subseteq V \subseteq \tau_1\tau_2$-$Cl(V) \subseteq U$.

Lemma 9. [5] *If* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is a multifunction such that* $F(x)$ *is* $\tau_1\tau_2$*-regular and* $\tau_1\tau_2$*-paracompact for each* $x \in X$*, then* $ClF^+_{\circledast}(V) = pClF^+_{\circledast}(V) = F^+(V)$ *for each* $\sigma_1\sigma_2$*-open set* V *of* Y*.*

Theorem 7. *Let* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a multifunction such that* $F(x)$ *is* $\sigma_1\sigma_2$*-paracompact and* $\sigma_1\sigma_2$*-regular for each* $x \in X$*. Then the following properties are equivalent:*

(1) F *is upper almost weakly* (τ_1, τ_2)*-continuous;*

(2) $pClF_{\circledast}$ *is upper almost weakly* (τ_1, τ_2)*-continuous;*

(3) $ClF_{\circledast}$ *is upper almost weakly* (τ_1, τ_2)*-continuous.*

Proof. We put $G = \mathrm{Cl}F_{\circledast}$ or $\mathrm{pCl}F_{\circledast}$ in the sequel. Suppose that F is upper almost weakly (τ_1, τ_2)-continuous. Let $x \in X$ and let V be any $\sigma_1\sigma_2$-open set of Y containing $G(x)$. By Lemma 9, we have $x \in G^+(V) = F^+(V)$ and hence, there exists a $\tau_1\tau_2$-open set U containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(V)$. Since $F(z)$ is $\sigma_1\sigma_2$-paracompact and

$\sigma_1\sigma_2$-regular for each $z \in U$, by Lemma 8, there exists a $\tau_1\tau_2$-open set W such that $F(z) \subseteq W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$; hence $G(z) \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ for each $z \in U$. Thus, $G(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ and hence G is upper almost weakly (τ_1, τ_2)-continuous.

Conversely, suppose that G is upper almost weakly (τ_1, τ_2)-continuous. Let $x \in X$ and let V be any $\sigma_1\sigma_2$-open set of Y containing $G(x)$. By Lemma 9, we have $x \in F^+(V) = G^+(V)$ and hence $G(x) \subseteq V$. There exists a $\tau_1\tau_2$-open set U containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, $U \subseteq G^+(V) = F^+(V)$ and so $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that F is upper almost weakly (τ_1, τ_2)-continuous.

Lemma 10. [5] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, $ClF^-_{\circledast}(V) = pClF^-_{\circledast}(V) = F^-(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Theorem 8. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost weakly (τ_1, τ_2)-continuous;

(2) $pClF_{\circledast}$ is lower almost weakly (τ_1, τ_2)-continuous;

(3) $ClF_{\circledast}$ is lower almost weakly (τ_1, τ_2)-continuous.

Proof. By using Lemma 10 this can be shown similarly to that of Theorem 7.

The $\tau_1\tau_2$-*prefrontier* [5] of a subset A of a bitopological space (X, τ_1, τ_2), denoted by $\tau_1\tau_2\text{-pfr}(A)$, is defined by

$$\begin{aligned}\tau_1\tau_2\text{-pfr}(A) &= \tau_1\tau_2\text{-pCl}(A) \cap \tau_1\tau_2\text{-pCl}(X - A)\\ &= \tau_1\tau_2\text{-pCl}(A) - \tau_1\tau_2\text{-pInt}(A).\end{aligned}$$

Theorem 9. *The set of all points x of X at which a multifunction*

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not upper almost weakly (τ_1, τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-prefrontier of the upper inverse images of the $\sigma_1\sigma_2$-closure of $\sigma_1\sigma_2$-open sets containing $F(x)$.

Proof. Let $x \in X$ at which F is not upper almost weakly (τ_1, τ_2)-continuous. There exists a $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. Thus,

$$x \in \tau_1\tau_2\text{-pCl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(V))) = X - \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V))).$$

Since $x \in F^+(V)$, we have $x \in \tau_1\tau_2\text{-pCl}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence

$$x \in \tau_1\tau_2\text{-pfr}(F^+(\sigma_1\sigma_2\text{-Cl}(V))).$$

Conversely, if F is upper almost weakly (τ_1, τ_2)-continuous, then for any $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ there exists a $\tau_1\tau_2$-open set U containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$; hence $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $x \in \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. This contradicts with the fact that $x \in \tau_1\tau_2\text{-pfr}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. Thus, F is not upper almost weakly (τ_1, τ_2)-continuous at x.

Theorem 10. *The set of all points x of X at which a multifunction*

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not lower almost weakly (τ_1, τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-prefrontier of the lower inverse images of $\sigma_1\sigma_2$-closure of $\sigma_1\sigma_2$-open sets meeting $F(x)$.

Proof. The proof is similar to that of Theorem 9.

Lemma 11. [26] *The following hold for a multifunction $F : X \to Y$:*

(i) $G_F^+(A \times B) = A \cap F^+(B)$,

(ii) $G_F^-(A \times B) = A \cap F^-(B)$,

for any subsets $A \subseteq X$ and $B \subseteq Y$.

Lemma 12. *Let (X, τ_1, τ_2) be a bitopological space. If A is $\tau_1\tau_2$-preopen and B is $\tau_1\tau_2$-open in X, then $A \cap B$ is $\tau_1\tau_2$-preopen.*

Proof. Suppose that A is $\tau_1\tau_2$-preopen and B is $\tau_1\tau_2$-open in X. Then,

$$A \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A))$$

and $B = \tau_1\text{-Int}(B) = \tau_2\text{-Int}(B)$. By Lemma 5(1),

$$A \cap B \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cap B = \tau_1\text{-Int}(\tau_2\text{-Cl}(A) \cap B) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A \cap B)).$$

Thus, $A \cap B$ is $\tau_1\tau_2$-preopen.

Definition 5. [5] *A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-compact if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover.*

By ρ_i, we denote the product topology $\tau_i \times \sigma_i$ for $i = 1, 2$.

Theorem 11. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-compact for each $x \in X$. Then F is upper almost weakly (τ_1, τ_2)-continuous if and only if $G_F : (X, \tau_1, \tau_2) \to (X \times Y, \rho_1, \rho_2)$ is upper almost weakly (τ_1, τ_2)-continuous.*

Proof. Suppose that $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper almost weakly (τ_1, τ_2)-continuous. Let $x \in X$ and let W be any $\rho_1\rho_2$-open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist $\tau_1\tau_2$-open set $U(y)$ of X and $\sigma_1\sigma_2$-open set $V(y)$ of Y such that $(x, y) \in U(y) \times V(y) \subseteq W$. The family $\{V(y) \mid y \in F(x)\}$ is a $\sigma_1\sigma_2$-open cover of $F(x)$ and there exists a finite number of points, say, $y_1, y_2, ..., y_n$ in $F(x)$ such that $F(x) \subseteq \cup\{V(y_i) \mid 1 \leq i \leq n\}$. Put $U = \cap\{U(y_i) \mid i = 1, 2, ..., n\}$ and

$$V = \cup\{V(y_i) \mid i = 1, 2, ..., n\}.$$

Then, U is $\tau_1\tau_2$-open in X and V is $\sigma_1\sigma_2$-open in Y such that $\{x\} \times F(x) \subseteq U \times V \subseteq W$. Since F is upper almost weakly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-preopen set G containing x such that $F(G) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. By Lemma 11, $U \cap G \subseteq U \cap F^+(\sigma_1\sigma_2\text{-Cl}(V)) = G_F^+(U \times \sigma_1\sigma_2\text{-Cl}(V)) \subseteq G_F^+(\sigma_1\sigma_2\text{-Cl}(W))$. By Lemma 12, $U \cap G$ is $\tau_1\tau_2$-preopen in X containing x and $G_F(U \cap G) \subseteq \sigma_1\sigma_2\text{-Cl}(W)$. This shows that G_F is upper almost weakly (τ_1, τ_2)-continuous.

Conversely, suppose that $G_F : (X, \tau_1, \tau_2) \to (X \times Y, \rho_1, \rho_2)$ is upper almost weakly (τ_1, τ_2)-continuous. Let $x \in X$ and let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Since $X \times V$ is $\rho_1\rho_2$-open in $X \times Y$ and $G_F(x) \subseteq X \times Y$, by Theorem 1, there exists a $\tau_1\tau_2$-preopen set U containing x such that $G_F(U) \subseteq X \times \sigma_1\sigma_2\text{-Cl}(V)$. Therefore, by Lemma 11, $U \subseteq G_F^+(X \times \sigma_1\sigma_2\text{-Cl}(V)) = F^+(\sigma_1\sigma_2\text{-Cl}(V))$ and hence $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that F is upper almost weakly (τ_1, τ_2)-continuous.

Theorem 12. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost weakly (τ_1, τ_2)-continuous if and only if $G_F : (X, \tau_1, \tau_2) \to (X \times Y, \rho_1, \rho_2)$ is lower almost weakly (τ_1, τ_2)-continuous.*

Proof. Suppose that $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost weakly (τ_1, τ_2)-continuous. Let $x \in X$ and let W be any $\rho_1\rho_2$-open set of $X \times Y$ such that $G_F(x) \cap W \neq \emptyset$. Then, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subseteq W$ for some $\tau_1\tau_2$-open set U of X and $\sigma_1\sigma_2$-open set V of Y. Since F is lower almost weakly (τ_1, τ_2)-continuous and $y \in F(x) \cap V$, there exists a $\tau_1\tau_2$-preopen set G of X containing x such that $F(z) \cap \sigma_1\sigma_2\text{-Cl}(V) \neq \emptyset$ for each $z \in G$; hence $G \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$. By Lemma 11, we have $U \cap G \subseteq U \cap F^-(\sigma_1\sigma_2\text{-Cl}(V)) = G_F^-(U \times \sigma_1\sigma_2\text{-Cl}(V)) \subseteq G_F^-(\sigma_1\sigma_2\text{-Cl}(W))$. Moreover, $U \cap G$ is a $\tau_1\tau_2$-preopen set containing x and hence G_F is lower almost weakly (τ_1, τ_2)-continuous.

Conversely, suppose that $G_F : (X, \tau_1, \tau_2) \to (X \times Y, \rho_1, \rho_2)$ is lower almost weakly (τ_1, τ_2)-continuous. Let $x \in X$ and let V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \emptyset$. Then, $X \times Y$ is $\rho_1\rho_2$-open and

$$G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset.$$

There exists a $\tau_1\tau_2$-preopen set U containing x such that $G_F(z) \cap \rho_1\rho_2\text{-Cl}((X \times V)) \neq \emptyset$ for each $z \in U$. By Lemma 11, $U \subseteq G_F^-(\rho_1\rho_2\text{-Cl}(X \times V)) = F^-(\sigma_1\sigma_2\text{-Cl}(V))$. This shows that F is lower almost weakly (τ_1, τ_2)-continuous.

4. Conclusion

The concepts of openness and continuity are extensively developed and used in many fields of applications such as data mining, computational topology for geometric design and molecular design, information systems, digital topology and computer graphics. Continuity of functions and multifunctions in topological spaces and bitopological spaces have been researched by many mathematicians. Several investigations related to open sets have been published and various forms of continuity types have been introduced. This paper deals with the notions of upper and lower almost weakly (τ_1, τ_2)-continuous multifunctions. Some characterizations of upper and lower almost weakly (τ_1, τ_2)-continuous multifunctions are established. The ideas and results of this paper may motivate further research.

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