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On the Fourier Transform Related to the Diamond Klein–Gordon Kernel

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Abstract. In this article, we study the fundamental solution of the operator

$$\left((\diamondsuit + m^2)\left(\frac{\triangle^2 + \boxdot^2}{2}\right)\right)^k$$

iterated k-times, which is defined by (10), where m is a non-negative real number, and k is a non-negative integer. After that, we study the Fourier transform of the operator $\left((\diamondsuit + m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k \delta$, where δ is the Dirac delta function.

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Key Words and Phrases: Diamond Klein–Gordon kernel; Diamond operator; Laplace operator; Fourier transform; wave equation

Introduction

The operator $\diamond^k$ has been first introduced by Kananthai [5], is named as the diamond operator iterated k-times, and is defined by

$$\diamondsuit^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k, \quad p+q=n, \tag{1}$$

where n is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and k is a non-negative integer. The operator $\diamondsuit^k$ can be expressed in the form $\diamondsuit^k = \boxdot^k \triangle^k = \triangle^k \boxdot^k$, where the operator $\triangle^k$ is Laplace operator iterated k-times, which is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k \tag{2}$$

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and the operator $\boxdot^k$ is the ultra-hyperbolic operator iterated k-times, which is defined by

$$\boxdot^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k. \tag{3}$$

By putting $p = 1$ and $x_1 = t$ (time) in (3), then we obtain the wave operator

$$\boxdot = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}. \tag{4}$$

In 1997, Kananthai [5] showed that the convolution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is the fundamental solution of the operator $\diamondsuit^k$, that is

$$\diamondsuit^k((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) = \delta, \tag{5}$$

where the function $R^H_{2k}(x)$ is defined by (20) and $R^e_{2k}(x)$ is defined by (19). The fundamental solution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is called the *diamond kernel of Marcel Riesz*. Satsanit [20] showed that

$$\begin{aligned}
\odot^k &= \left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k \\
&= \left(\left(\frac{\triangle + \boxdot}{2} \right)^2 + \left(\frac{\triangle - \boxdot}{2} \right)^2 \right)^k \\
&= \left(\frac{\triangle^2 + \boxdot^2}{2} \right)^k.
\end{aligned} \tag{6}$$

Moreover, Kananthai, Suantai and Longani [7] studied the fundamental solution of the operator $\oplus^k$ and the weak solution of the equation $\oplus^k u(x) = f(x)$, where the operator $\oplus^k$ is defined by

$$\begin{aligned}
\oplus^k &= \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k \\
&= \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \\
&= \diamond^k L_1^k L_2^k \\
&= \diamond^k L^k
\end{aligned} \tag{7}$$

where $p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, k is a non-negative integer, and $f(x)$ is a generalized function.

Next, Kananthai, Suantai and Longani [6] studied the relationship between the operator $\oplus^k$ and the wave operator, and the relationship between the operator $\oplus^k$ and the Laplace operator. Moreover, they studied equation $\oplus^k K(x) = \delta$ and they showed that

$$K(x) = [R^H_{2k}(x) * (-1)^k R^e_{2k}(x)] * S_{2k}(x) * T_{2k}(x)$$

is the fundamental solution of the operator $\oplus^k$. Later, Kananthai [3] studied the inversion of the kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator $\oplus^k$.

In 1988, Trione [22] studied the fundamental solution of the ultra-hyperbolic Klein–Gordon operator iterated k-times, which is defined by

$$(\boxdot + m^2)^k = \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2\right]^k . \quad (8)$$

Later, Lunnaree and Nonlaopon [11] introduced the operator $(\diamondsuit + m^2)^k$, that is named as the diamond Klein-Gordon operator iterated k-times, which is defined by

$$(\diamondsuit + m^2)^k = \left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right)^k , \quad (9)$$

where $p+q = n$ is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, m is a non-negative real number and k is a non-negative integer, see [9, 10, 17, 18] for more details. V.N. Mishra, K. Khatri and L.N. Mishra [15] studied the linear operators to approximate signals of Lip $(\alpha, p), (p \geq 1)$-class, see [2, 12–14, 16] for more details.

Moreover, Kananthai [4] studied the fundamental solution for the $(\diamondsuit + m^4)^k$, which related to the Klein-Gordon operator. From (7) the operator

$$\left[\left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 + \frac{m^2}{2}\right)^2 - \left(\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 - \frac{m^2}{2}\right)^2\right]^k$$

can be expressed in the form

$$\begin{aligned}
&\left[\left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 + \frac{m^2}{2}\right)^2 - \left(\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 - \frac{m^2}{2}\right)^2\right]^k \\
&= \left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right)^k \left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right)^k \\
&= (\diamondsuit + m^2)^k \left(\frac{\triangle^2 + \boxdot^2}{2}\right)^k
\end{aligned}$$

$$= (\Diamond + m^2)^k \odot^k . \tag{10}$$

From (10) with $q = m = 0$ and $k = 1$, we obtain Laplace operator $\triangle_p^4$ of p-dimension, where

$$\triangle_p = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}. \tag{11}$$

In this article, we study the fundamental solution of the equation of the form

$$\left(\left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^2 + \frac{m^2}{2} \right)^2 - \left(\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 - \frac{m^2}{2} \right)^2 \right)^k K(x,m) = \delta,$$

or

$$\left((\Diamond + m^2) \left(\frac{\triangle^2 + \boxdot^2}{2} \right) \right)^k K(x,m) = \delta,$$

where $K(x,m)$ is the fundamental solution, δ is the Dirac delta function, k is a non-negative integer, and m is a non-negative real number. Moreover, we study the Fourier transform of the operator $\left((\Diamond + m^2) \left(\frac{\triangle^2 + \boxdot^2}{2} \right) \right)^k \delta$.

Preliminary Notes

Definition 1. *Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional space $\mathbb{R}^n$,*

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{12}$$

where $p + q = n$.

Define $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$, which designates the interior of the forward cone and $\overline{\Gamma}_+$ designates its closure and the following functions introduce by Nozaki [19, Page 72], that

$$R_\alpha^H(x) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+; \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \tag{13}$$

is called the ultra-hyperbolic kernel of Marcel Riesz. Here, α is a complex parameter and n the dimension of the space. The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)} \tag{14}$$

and p is the number of positive terms of

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad p + q = n$$

and let supp $R_\alpha^H(x) \subset \overline{\Gamma}_+$. *Now,* $R_\alpha^H(x)$ *is an ordinary function if Re* $\alpha \geq n$ *and is a distribution of* α *if Re* $\alpha < n$. *Now, if* $p = 1$ *then* (13) *reduces to the function* $M_\alpha(u)$, *and is defined by*

$$M_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{if } x \in \Gamma_+; \\ 0, & \text{if } x \notin \Gamma_+, \end{cases} \tag{15}$$

where $u = x_1^2 - x_2^2 - \cdots - x_n^2$ *and* $H_n(\alpha) = \pi^{\frac{(n-1)}{2}} 2^{\alpha-1} \Gamma(\frac{\alpha-n+2}{2})$. *The function* $M_\alpha(u)$ *is called the hyperbolic kernel of Marcel Riesz.*

Definition 2. *Let* $f(x) \in L_1(\mathbb{R}^n)$ *(the space of integrable function in* $\mathbb{R}^n$*). The Fourier transform of* $f(x)$ *is defined as*

$$\widehat{f(\xi)} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx, \tag{16}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n), x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \xi \cdot x = (\xi_1 x_1, \xi_2 x_2, \ldots, \xi_n x_n)$ *is the usual inner product in* $\mathbb{R}^n$ *and* $dx = dx_1 dx_2 \ldots dx_n$. *The inverse of the Fourier transform is defined by*

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \widehat{f(\xi)} d\xi. \tag{17}$$

If f *is a distribution with compact supports, by [24, Theorem 7.4-3], Equation* (17) *can be written as*

$$\widehat{f}(\xi) = \mathcal{F} f(x) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i\xi \cdot x} \right\rangle. \tag{18}$$

Lemma 1. [5] *Given the equation* $\triangle^k u(x) = \delta$ *for* $x \in \mathbb{R}^n$, *where* $\triangle^k$ *is the Laplace operator iterated k-times, which is defined by* (2). *Then* $u(x) = (-1)^k R_{2k}^e(x)$ *is the fundamental solution of the operator* $\triangle^k$, *where*

$$R_{2k}^e(x) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k} \pi^{\frac{n}{2}} \Gamma(k)} |x|^{2k-n}. \tag{19}$$

Lemma 2. [22] *If* $\boxdot^k u(x) = \delta$ *for* $x \in \Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$, *where* $\boxdot^k$ *is the ultra-hyperbolic operator iterated k-times, which is defined by* (3). *Then* $u(x) = R_{2k}^H(x)$ *is the unique fundamental solution of the operator* $\boxdot^k$, *where*

$$R_{2k}^H(x) = \frac{u^{\left(\frac{2k-n}{2}\right)}}{K_n(2k)} = \frac{(x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\left(\frac{2k-n}{2}\right)}}{K_n(2k)} \tag{20}$$

and

$$K_n(2k) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+2k-n}{2}\right) \Gamma\left(\frac{1-2k}{2}\right) \Gamma(2k)}{\Gamma\left(\frac{2+2k-p}{2}\right) \Gamma\left(\frac{p-2k}{2}\right)}. \tag{21}$$

Lemma 3. [5] *Given the equation* $\diamondsuit^k u(x) = \delta$ *for* $x \in \mathbb{R}^n$, *then* $u(x) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ *is the unique fundamental solution of the operator* $\diamondsuit^k$, *where* $\diamondsuit^k$ *is the diamond operator iterated k-times, which is defined by* (1), $R^e_{2k}(x)$ *and* $R^H_{2k}(x)$ *are defined by* (19) *and* (20), *respectively. Moreover,* $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ *is a tempered distribution.*

It is not difficult to show that $R^e_{-2k}(x) * R^H_{-2k}(x) = (-1)^k \diamondsuit^k \delta$, for k is a non-negative integer.

Definition 3. *Let* $x = (x_1, x_2, \ldots, x_n)$ *be a point of* $\mathbb{R}^n$, *the function* $P_\alpha(x, m)$ *is defined by*

$$P_\alpha(x,m) = \sum_{r=0}^{\infty} \binom{-\alpha/2}{r} (m^2)^r (-1)^{\alpha/2+r} R^e_{\alpha+2r}(x) * R^H_{\alpha+2r}(x), \tag{22}$$

where α *is a complex parameter,* m *is a non-negative real number,* $R^H_{\alpha+2r}(x)$ *and* $R^e_{\alpha+2r}(x)$ *are defined by* (20) *and* (19), *respectively.*

From the definition of $P_\alpha(x, m)$ and by putting $\alpha = -2k$, we have

$$P_{-2k}(x,m) = \sum_{r=0}^{\infty} \binom{k}{r} (m^2)^r (-1)^{-k+r} R^e_{2(-k+r)}(x) * R^H_{2(-k+r)}(x).$$

Since the operator $(\diamondsuit + m^2)^k$ defined in equation (9) is a linearly continuous and has $1-1$ mapping, then it has inverse. From Lemma 3, we obtain

$$\begin{aligned} P_{-2k}(x,m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \diamondsuit^{-k-r} \delta \\ &= (\diamondsuit + m^2)^k \delta. \end{aligned} \tag{23}$$

By putting $k = 0$ in (23), we have $P_0(x, m) = \delta$. By putting $\alpha = 2k$ into (22), we have

$$\begin{aligned} P_{2k}(x,m) = &\binom{-k}{0} (m^2)^0 (-1)^{k+0} R^e_{2k+0}(x) * R^H_{2k+0}(x) \\ &+ \sum_{r=1}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2k+2r}(x) * R^H_{2k+2r}(x). \end{aligned} \tag{24}$$

The second summand of the right-hand member of (24) vanishes for $m = 0$ and then, we have

$$P_{2k}(x, m = 0) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x) \tag{25}$$

is the fundamental solution of the diamond operator $\diamondsuit^k$.

Lemma 4. *The function* $R^H_{-2k}(x)$ *and* $(-1)^k R^e_{-2k}(x)$ *are the inverse in the convolution algebra of* $R^H_{2k}(x)$ *and* $(-1)^k R^e_{2k}(x)$, *respectively. That is,*

$$R^H_{-2k}(x) * R^H_{2k}(x) = R^H_{-2k+2k}(x) = R^H_0(x) = \delta$$

and

$$(-1)^k R^e_{-2k}(x) * (-1)^k R^e_{2k}(x) = (-1)^{2k} R^e_{-2k+2k}(x) = R^e_0(x) = \delta.$$

For the proof of the this Lemma is given in [1, 21, 23].

Lemma 5. [20]*(Convolution of $R^e_\alpha(x)$ and $R^H_\alpha(x)$). If $R^e_\alpha(x)$ and $R^H_\alpha(x)$ are defined by* (19) *and* (20), *respectively, then*

(i) $R^e_\alpha(x) * R^e_\beta(x) = R^e_{\alpha+\beta}(x)$, *where α and β are complex parameters;*

(ii) $R^H_\alpha(x) * R^H_\beta(x) = R^H_{\alpha+\beta}(x)$, *where α and β are both integers and except only the case both α and β are both integers.*

Lemma 6. [11] *Given the equation $(\diamondsuit + m^2)^k u(x) = \delta$, where $(\diamondsuit + m^2)^k$ is the diamond Klein-Gordon operator, which is defined by*

$$(\diamondsuit + m^2)^k = \left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right)^k, \tag{26}$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, k is a non-negative integer, m is a non-negative real number and δ is the Dirac delta function. Then, we obtain

$$P_{2k}(x,m) = \sum_{r=0}^{\infty} \binom{-k}{r} m^{2r} (-1)^{k+r} R^e_{2k+2r}(x) * R^H_{2k+2r}(x) \tag{27}$$

is the fundamental solution of the operator $(\diamondsuit + m^2)^k$, defined by (9), *where $R^H_{2k}(x)$ and $R^e_{2k}(x)$ are defined by* (20) *and* (19), *respectively. Moreover, $u(x) = P_{2k}(x,m)$ is tempered distribution.*

Lemma 7. [20] *Given the equation*

$$\odot^k G(x) = \delta, \tag{28}$$

where $\odot^k$ is the operator iterated k-times is defined by (6). *Then, we obtain $G(x)$ is the fundamental solution of the equation* (28), *where*

$$G(x) = (R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x)) * (H^{*k}(x))^{*-1} \tag{29}$$

and

$$H(x) = \frac{1}{2} R^H_4(x) + \frac{1}{2}(-1)^2 R^e_4(x). \tag{30}$$

Here, $H^{*k}(x)$ denotes the convolution of $H(x)$ itself k-times, $(H^{*k}(x))^{*-1}$ denotes the inverse of $H^{*k}(x)$ in the convolution algebra. Moreover, $G(x)$ is a tempered distribution.

Lemma 8. *(The Fourier transform of $\left((\diamondsuit + m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k \delta$.)*

Let

$$||\xi|| = \left(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2\right)^{1/2}$$

for $\xi \in \mathbb{R}^n$. *Then*

$$\left|\mathcal{F}\left((\Diamond + m^2)\left(\frac{\triangle^2 + \boxdot^2}{2}\right)\right)^k \delta\right| \leq \frac{1}{(2\pi)^{n/2}}(||\xi||^4 + m^2)^k ||\xi||^{4k}.$$

That is, $\mathcal{F}\left((\Diamond + m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k \delta$ *is bounded and continuous on the space* $\mathcal{S}'$ *of the tempered distribution. Moreover, by the inverse Fourier transformation*

$$\left((\Diamond + m^2)\left(\frac{\triangle^2 + \boxdot^2}{2}\right)\right)^k \delta = \mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}\left[\left((\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2)^2 + \frac{m^2}{2}\right)^2 - \left((\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2)^2 - \frac{m^2}{2}\right)^2\right]^k.$$

Proof. From the Fourier transform (16), we have

$$\begin{aligned}
&\mathcal{F}\left((\Diamond + m^2)\left(\frac{\triangle^2 + \boxdot^2}{2}\right)\right)^k \delta \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle \delta, (\Diamond + m^2)^k \left(\frac{\triangle^2 + \boxdot^2}{2}\right)^k e^{-i\xi\cdot x}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle \delta, (\Diamond + m^2)^k \frac{(-1)^{2k}}{2}\left((\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^2 + (\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 \right.\right. \\
&\quad \left.\left. -\xi_{p+1}^2 - \xi_{p+2}^2 - \cdots - \xi_n^2)^2\right)^k e^{-i\xi\cdot x}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle \delta, \frac{(-1)^{2k}}{2}\left((\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^2 + (\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 \right.\right. \\
&\quad \left.\left. -\xi_{p+1}^2 - \xi_{p+2}^2 - \cdots - \xi_n^2)^2\right)^k (\Diamond + m^2)^k e^{-i\xi\cdot x}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle \delta, \left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]^k \left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2 + m^2\right]^k e^{-i\xi\cdot x}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle \delta, \left[\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 + \frac{m^2}{2}\right)^2 - \left(\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2 - \frac{m^2}{2}\right)^2\right]^k e^{-i\xi\cdot x}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left[\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 + \frac{m^2}{2}\right)^2 - \left(\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2 - \frac{m^2}{2}\right)^2\right]^k \\
&= \frac{1}{(2\pi)^{n/2}}\left[\left((\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2)^2 + \frac{m^2}{2}\right)^2 - \left((\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2)^2 - \frac{m^2}{2}\right)^2\right]^k.
\end{aligned}$$

Next, we consider the boundedness of $\mathcal{F}\left((\diamondsuit+m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k\delta$. Since

$$\begin{aligned}
&\left((\diamondsuit+m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k\\
&=\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+\frac{m^2}{2}\right)^2-\left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)^2-\frac{m^2}{2}\right)^2\right]^k\\
&=\Big[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2-(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_n^2)^2+m^2\right)^k\\
&\quad\times\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_n^2)^2\right)^k\Big]\\
&=\Big[\left((\xi_1^2+\xi_2^2+\cdots+\xi_n^2)(\xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2-\cdots-\xi_n^2)+m^2\right)^k\\
&\quad\times\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+(\xi_{p+1}^2+\cdots+\xi_n^2)^2\right)^k\Big].
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathcal{F}\left((\diamondsuit+m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k\delta\\
&=\frac{1}{(2\pi)^{n/2}}\Big[\left((\xi_1^2+\xi_2^2+\cdots+\xi_n^2)(\xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2-\cdots-\xi_n^2)+m^2\right)^k\\
&\quad\times\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+(\xi_{p+1}^2+\cdots+\xi_n^2)^2\right)^k\Big],\\
&\left|\mathcal{F}\left((\diamondsuit+m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k\delta\right|\\
&=\frac{1}{(2\pi)^{n/2}}\left(\left|\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right|\left|\xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2-\cdots-\xi_n^2\right|+m^2\right)^k\\
&\quad\times\left|\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+(\xi_{p+1}^2+\cdots+\xi_n^2)^2\right)\right|^k\\
&\leq\frac{1}{(2\pi)^{n/2}}\left(\left|\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right|^2+m^2\right)^k\left|\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right|^{2k}\\
&=\frac{1}{(2\pi)^{n/2}}(||\xi||^4+m^2)^k||\xi||^{4k},
\end{aligned}$$

where $||\xi||=\left(\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right)^{1/2}$, $\xi_i(i=1,2,\ldots,n)\in\mathbb{R}$. Hence, we obtain

$$\mathcal{F}\left((\diamondsuit+m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k\delta$$

is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution.

Since $\mathcal{F}$ is $1-1$ transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, then by (17), we have

$$\left((\diamondsuit+m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k\delta$$

$$= \frac{1}{(2\pi)^{n/2}}\mathcal{F}^{-1}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+\frac{m^2}{2}\right)^2-\left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)^2-\frac{m^2}{2}\right)^2\right]^k.$$

Main Results

Theorem 1. $\left(\textbf{The fundamental solution of } \left((\diamondsuit+m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k\right)$.

Given the equation

$$\left((\diamondsuit+m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k K(x,m)=\delta, \tag{31}$$

where $\left((\diamondsuit+m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k$ *is the operator iterated k-times, which is defined by* (10), δ *is the Dirac-delta function,* $x\in\mathbb{R}^n$, m *is a non-negative real number and* k *is a non-negative integer. Then, we obtain*

$$K(x,m)=\left(R_{4k}^H(x)*(-1)^{2k}R_{4k}^e(x)*(H^{*k}(x))^{*-1}\right)*P_{2k}(x,m) \tag{32}$$

is the fundamental solution for the operator iterated k-times, which is defined by (10). *In particular, for* $m=0$ *then* (31) *becomes*

$$\oplus^k K(x,0)=\delta, \tag{33}$$

and we obtain

$$K(x,0)=\left(R_{6k}^H(x)*(-1)^{3k}R_{6k}^e(x)\right)*\left((H^{*k}(x))^{*-1}\right) \tag{34}$$

is the fundamental solution of the o-plus operator $\oplus^k$, *for* $q=m=0$ *then* (31) *becomes*

$$\triangle_p^{4k}K(x,0)=\delta, \tag{35}$$

and we obtain

$$K(x,0)=R_{8k}^e(x) \tag{36}$$

is the fundamental solution of (35), *where* $\triangle_p^{4k}$ *is the Laplace operator of p-dimension, iterated 4k-times which is defined by* (11).

Moreover, from (34), *we obtain*

$$\left(R_{-4k}^H(x)*(-1)^{3k}R_{-6k}^e(x)\right)*\left(H^{*k}(x)\right)*K(x,0)=R_{2k}^H(x) \tag{37}$$

is the fundamental solution of the ultra-hyperbolic operator $\boxdot^k$ *iterated k-times, which defined by* (3), *where* $R_{-6k}^e(x)$ *and* $R_{-4k}^H(x)$ *are inverse of* $R_{6k}^e(x)$ *and* $R_{4k}^H(x)$, *respectively.*

From (34) *and* (37) *with* $p = 1, q = n-1, k = 1, m = 0$ *and* $x_1 = t$ *(time), we obtain*

$$\left(\frac{(-1)^3}{2}R^e_{-6}(x) + M^H_{-4}(u) * \frac{(-1)^5}{2}R^e_{-2}(x)\right) * K(x,0) = M^H_2(u) \tag{38}$$

or

$$\left(\left(-\frac{1}{2}R^e_{-6}(x)\right) + M^H_{-4}(u) * \left(-\frac{1}{2}R^e_{-2}(x)\right)\right) * K(x,0) = M^H_2(u) \tag{39}$$

is the fundamental solution of the wave operator is defined by (4)*, where* $M_2(u)$ *is defined by* (15) *with* $\alpha = 2$.

Proof. From (10) and (31), we have

$$\left((\diamondsuit + m^2)\left(\frac{\triangle^2 + \boxdot^2}{2}\right)\right)^k K(x,m) = (\diamondsuit + m^2)^k \left(\frac{\triangle^2 + \boxdot^2}{2}\right)^k K(x,m) = \delta. \tag{40}$$

Convolving both sides of (40) by $\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right) * P_{2k}(x,m)$, we obtain

$$\left[\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right) * P_{2k}(x,m)\right] * (\diamondsuit + m^2)^k \left(\frac{\triangle^2 + \boxdot^2}{2}\right)^k K(x,m)$$
$$= \left[\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right) * P_{2k}(x,m)\right] * \delta.$$

By properties of the convolution, we have

$$(\diamondsuit + m^2)^k (P_{2k}(x,m)) * \left(\frac{\triangle^2 + \boxdot^2}{2}\right)^k \left(\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right)\right) * K(x,m)$$
$$= \left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right) * P_{2k}(x,m).$$

By Lemma 6 and Lemma 7, we obtain,

$$\delta * \delta * K(x,m) = K(x,m) = \left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right) * P_{2k}(x,m) \tag{41}$$

is the fundamental solution of $\left((\diamondsuit + m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k$ operator.

In particular, for $m = 0$ then (31) becomes

$$\oplus^k K(x,0) = \delta. \tag{42}$$

From Lemma 5, (22) and (41), we obtain

$$K(x,0) = \left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right) * P_{2k}(x,0)$$
$$= \left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right) * ((-1)^k R^e_{2k}(x) * R^H_{2k}(x))$$

$$= \left(R^H_{6k}(x) * (-1)^{3k} R^e_{6k}(x)\right) * \left((H^{*k}(x))^{*-1}\right) \tag{43}$$

is the fundamental solution of the o-plus operator $\oplus^k$.

Putting $q = m = 0$, then (31) becomes

$$\triangle^{4k}_p K(x, 0) = \delta, \tag{44}$$

where $\triangle^{4k}_p$ is Laplace operator of p-dimension iterated $4k$-times. By Lemma 1, we have

$$K(x, 0) = (-1)^{4k} R^e_{8k}(x) = R^e_{8k}(x)$$

is the fundamental solution of (44).

On the other hand, we can also find $K(x, m)$ from (41). Since $q = 0$, we have $R^H_{2k}(x)$ reduces to $(-1)^k R^e_{2k}(x)$. Thus, by (41) for $q = m = 0$, we obtain

$$\begin{aligned} K(x,0) &= \left((-1)^{2k} R^e_{4k}(x) * (-1)^{2k} R^e_{4k}(x)\right) * \left((-1)^{2k} R^e_{4k}(x)\right)^{*-1} * P_{2k}(x,0) \\ &= (-1)^{4k} R^e_{4k+4k}(x) * \left((-1)^{2k} R^e_{4k}(x)\right)^{*-1} * ((-1)^k R^e_{2k}(x) * (-1)^k R^e_{2k}(x)) \\ &= (-1)^{8k} R^e_{8k}(x) = R^e_{8k}(x), \end{aligned}$$

where $(R^e_{4k}(x))^{*-1}$ is the inverse of $R^e_{4k}(x)$ in the convolution algebra.

From (41), we have

$$K(x, 0) = \left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x)\right) * \left(H^{*k}(x)\right)^{*-1} * P_{2k}(x, 0).$$

Convolving the above equation by $\left(R^H_{-4k}(x) * (-1)^{3k} R^e_{-6k}(x)\right) * \left(H^{*k}(x)\right)$. By Lemma 4, Lemma 5, and (25), we obtain

$$\begin{aligned} &\left(R^H_{-4k}(x) * (-1)^{3k} R^e_{-6k}(x)\right) * \left(H^{*k}(x)\right) * K(x, 0) \\ &= \left(R^H_{4k}(x) * R^H_{-4k}(x)) * ((-1)^{2k} R^e_{4k}(x) * (-1)^{3k} R^e_{-6k}(x)\right)) \\ &* (\left(H^{*k}(x)\right) * \left(H^{*k}(x)\right)^{*-1}) * P_{2k}(x, 0) \end{aligned}$$

or

$$\begin{aligned} &\left(R^H_{-4k}(x) * (-1)^{3k} R^e_{-6k}(x)\right) * \left(H^{*k}(x)\right) * K(x, 0) \\ &= \delta(x) * (-1)^{5k} R^e_{-2k}(x) * \delta(x) * P_{2k}(x, 0) \\ &= \delta(x) * (-1)^{5k} R^e_{-2k}(x) * \delta(x) * ((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) \\ &= \delta(x) * \delta(x) * \delta(x) * R^H_{2k}(x) = R^H_{2k}(x). \end{aligned}$$

It follows that

$$\left(R^H_{-4k}(x) * (-1)^{3k} R^e_{-6k}(x)\right) * \left(H^{*k}(x)\right) * K(x, 0) = R^H_{2k}(x) \tag{45}$$

as the fundamental solution of the ultra-hyperbolic operator iterated k-times defined by (3).

In particular, if we put $p=1, q=n-1, k=1, m=0$ and $x_1=t$ (time) in (41) then $R^H_{-4}(x)$ reduces to $M^H_{-4}(u)$ and $R^H_2(x)$ reduce to $M^H_2(u)$, where $M^H_4(u)$ and $M^H_2(u)$ are defined by (15) with $\alpha=-4, \alpha=2$, respectively. Thus, (45) becomes

$$\left(M^H_{-4}(u) * (-1)^3 R^e_{-6}(x)\right) * \left(\frac{1}{2} M^H_4(x) + \frac{(-1)^2}{2} R^e_4(x)\right) * K(x,0) = M^H_2(u). \tag{46}$$

By Lemma 7, we obtain

$$\left(\frac{(-1)^3}{2} R^e_{-6}(x) + M^H_{-4}(u) * \frac{(-1)^5}{2} R^e_{-2}(x)\right) * K(x,0) = M^H_2(u) \tag{47}$$

or

$$\left(\left(-\frac{1}{2} R^e_{-6}(x)\right) + M^H_{-4}(u) * \left(-\frac{1}{2} R^e_{-2}(x)\right)\right) * K(x,0) = M^H_2(u) \tag{48}$$

as the fundamental solution of the wave operator defined by

$$\boxdot = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}, \tag{49}$$

where $R^e_{-6}(x)$ defined by (19). This completes the proof.

Theorem 2.

$$\begin{aligned} &\mathcal{F}\left[\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right) * P_{2k}(x,m)\right] \\ &= \frac{1}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+\frac{m^2}{2}\right)^2 - \left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2-\frac{m^2}{2}\right)^2\right]^k} \\ &= \left|\mathcal{F}\left[\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right) * P_{2k}(x,m)\right]\right| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} M \end{aligned} \tag{50}$$

for a large $\xi_i \in \mathbb{R}$, where m is a non-negative real number and M is a constant. That is, $\mathcal{F}$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distributions.

Proof. By Theorem 1, we obtain

$$\left((\diamondsuit + m^2)\left(\frac{\triangle^2 + \boxdot^2}{2}\right)\right)^k \left(\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right) * P_{2k}(x,m)\right) = \delta$$

or

$$\left(\left((\diamondsuit + m^2)\left(\frac{\triangle^2 + \boxdot^2}{2}\right)\right)^k \delta\right) * \left(\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right) * P_{2k}(x,m)\right) = \delta.$$

Taking the Fourier transform on both sides of the above equation, we obtain

$$\mathcal{F}\left(\left(\left((\diamondsuit + m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k \delta\right) * \left(\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right)\right.\right.$$
$$\left.\left. * P_{2k}(x,m)\right)\right) = \mathcal{F}\delta = \frac{1}{(2\pi)^{n/2}}.$$

By (18), we have

$$\frac{1}{(2\pi)^{n/2}}\left\langle \left(\left((\diamondsuit + m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k \delta\right) * \left(\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right)\right.\right.$$
$$\left.\left. * P_{2k}(x,m)\right), e^{-i(\xi\cdot x)}\right\rangle = \frac{1}{(2\pi)^{n/2}}.$$

By the definition of convolution

$$\frac{1}{(2\pi)^{n/2}}\left\langle \left(\left((\diamondsuit + m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k \delta\right) * \left(\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\right)\right.\right.$$
$$\left.\left. * P_{2k}(x,m)\right), e^{-i\xi\cdot(x+r)}\right\rangle = \frac{1}{(2\pi)^{n/2}},$$
$$\frac{1}{(2\pi)^{n/2}}\left\langle (R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}) * P_{2k}(x,m), e^{-i(\xi\cdot r)}\right\rangle$$
$$\times \left\langle \left((\diamondsuit + m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k \delta, e^{-i(\xi\cdot x)}\right\rangle = \frac{1}{(2\pi)^{n/2}},$$
$$\mathcal{F}((R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}) * P_{2k}(x,m))(2\pi)^{\frac{n}{2}}\mathcal{F}\left(\left((\diamondsuit + m^2)\left(\frac{\triangle^2+\boxdot^2}{2}\right)\right)^k \delta\right)$$
$$= \frac{1}{(2\pi)^{n/2}}.$$

By Lemma 8, we obtain

$$\mathcal{F}((R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}) * P_{2k}(x,m))$$
$$\times \left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2 + \frac{m^2}{2}\right)^2 - \left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2 - \frac{m^2}{2}\right)^2\right]^k$$
$$= \frac{1}{(2\pi)^{n/2}}.$$

It follows that

$$\mathcal{F}((R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}) * P_{2k}(x,m))$$

$$= \frac{1}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+\frac{m^2}{2}\right)^2-\left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)^2-\frac{m^2}{2}\right)^2\right]^k}.$$

Since

$$\frac{1}{\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+\frac{m^2}{2}\right)^2-\left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)^2-\frac{m^2}{2}\right)^2\right]}$$
$$= \frac{1}{\left[(\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)^2\right]}$$
$$\times \frac{1}{\left[(\xi_1^2+\xi_2^2+\cdots+\xi_n^2)(\xi_1^2+\xi_2^2+\cdots+\xi_p^2-\xi_{p+1}^2-\cdots-\xi_{p+q}^2)+m^2\right]}. \quad (51)$$

Let $\xi=(\xi_1,\xi_2,\ldots,\xi_n)\in\Gamma_+$ with Γ_+ defined by Definition 1. Then $(\xi_1^2+\xi_2^2+\cdots+\xi_p^2+\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)>0$ and for a large k, the right-hand side of (51) tend to zero. It follows that it is bounded by a positive constant M say, that is we obtain (50) as required and also by (50) $\mathcal{F}$ is continuous on the space $\mathcal{S}'$ of the tempered distribution.

Theorem 3.

$$\begin{aligned}
&\mathcal{F}\left(\left[\left(R_{4k}^H(x)*(-1)^{2k}R_{4k}^e(x)*(H^{*k}(x))^{*-1}\right)*P_{2k}(x,m)\right]\right.\\
&\quad\left.*\left[\left(R_{4l}^H(x)*(-1)^{2l}R_{4l}^e(x)*(H^{*l}(x))^{*-1}\right)*P_{2l}(x,m)\right]\right)\\
&=(2\pi)^{n/2}\mathcal{F}\left[\left(R_{4k}^H(x)*(-1)^{2k}R_{4k}^e(x)*(H^{*k}(x))^{*-1}\right)*P_{2k}(x,m)\right]\\
&\quad\times\mathcal{F}\left[\left(R_{4l}^H(x)*(-1)^{2l}R_{4l}^e(x)*(H^{*l}(x))^{*-1}\right)*P_{2l}(x,m)\right]\\
&=\frac{1}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+\frac{m^2}{2}\right)^2-\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2-\frac{m^2}{2}\right)^2\right]^{k+l}},
\end{aligned}$$

where k and l are non-negative integers and $\mathcal{F}$ is bounded and continuous on the space $\mathcal{S}'$ of tempered distribution.

Proof. Since $R_{4k}^H(x)$, $R_{4k}^e(x)$ and $P_{2k}(x,m)$ are tempered distribution with compact support,

$$\begin{aligned}
&\left(\left[\left(R_{4k}^H(x)*(-1)^{2k}R_{4k}^e(x)*(H^{*k}(x))^{*-1}\right)*P_{2k}(x,m)\right]\right.\\
&\quad\left.*\left[\left(R_{4l}^H(x)*(-1)^{2l}R_{4l}^e(x)*(H^{*l}(x))^{*-1}\right)*P_{2l}(x,m)\right]\right)\\
&=\left[R_{4k}^H(x)*R_{4l}^H(x)\right]*\left[(-1)^{2k+2l}R_{4k}^e(x)*R_{4l}^e(x)\right]*\left[(H^{*k}(x))^{*-1}(H^{*l}(x))^{*-1}\right]\\
&\quad*\left[P_{2k}(x,m)*P_{2l}(x,m)\right]
\end{aligned}$$

$$= \left[R^H_{4(k+l)}(x)\right] * \left[(-1)^{2(k+l)} R^e_{4(k+l)}(x)\right] * \left[(H^{*(k+l)}(x))^{*-1}\right] * \left[P_{2(k+l)}(x,m)\right]$$

by [8, Pages 156–159] and [21, Lemma 2.45]. Taking the Fourier transform on both sides and using Theorem 2, we obtain

$$\begin{aligned}
&\mathcal{F}\Big(\Big[\Big(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\Big) * P_{2k}(x,m)\Big] \\
&\quad * \Big[\Big(R^H_{4l}(x) * (-1)^{2l} R^e_{4l}(x) * (H^{*l}(x))^{*-1}\Big) * P_{2l}(x,m)\Big]\Big) \\
&= \frac{1}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+\frac{m^2}{2}\right)^2-\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2-\frac{m^2}{2}\right)^2\right]^{k+l}} \\
&= \frac{1}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+\frac{m^2}{2}\right)^2-\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2-\frac{m^2}{2}\right)^2\right]^{k}} \\
&\quad \times \frac{(2\pi)^{n/2}}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2+\frac{m^2}{2}\right)^2-\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2-\frac{m^2}{2}\right)^2\right]^{l}} \\
&= (2\pi)^{n/2}\mathcal{F}\Big[\Big(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x) * (H^{*k}(x))^{*-1}\Big) * P_{2k}(x,m)\Big] \\
&\quad \times \mathcal{F}\Big[\Big(R^H_{4l}(x) * (-1)^{2l} R^e_{4l}(x) * (H^{*l}(x))^{*-1}\Big) * P_{2l}(x,m)\Big].
\end{aligned}$$

Since $\left(R^H_{4(k+l)}(x) * (-1)^{2(k+l)} R^e_{4(k+l)}(x) * (H^{*(k+l)}(x))^{*-1}\right) * \left(P_{2(k+l)}(x,m)\right) \in \mathcal{S}'$, the space of tempered distribution and by Theorem 2, we obtain that $\mathcal{F}$ is bounded and continuous on $\mathcal{S}'$.

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