EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 2, No. 3, 2009, (302-324)
ISSN 1307-5543 – www.ejpam.com

A Unified Class of Analytic Functions with Varying Argument of Coefficients

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Abstract. The object of the present paper is to investigate several classes of analytic functions with varying argument of coefficients, which are defined here by means of the principle of subordination between analytic functions. Such properties as the coefficient estimates, distortion theorems, subordination theorems, convolution properties, integral means inequalities, and radii of conexity and starlikenes are investigated. Some consequences of our main results for new or known classes of analytic functions are also pointed out.

2000 Mathematics Subject Classifications: Primary 30C45, 30C50; Secondary 30C55.

Key Words and Phrases: Analytic functions; Convex functions; k-Starlike functions; Uniformly convex functions; Varying arguments of coefficients; Subordination between analytic functions; Hadamard product (or convolution); Generalized hypergeometric function; Dziok-Srivastava operator.

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1. Introduction and Definitions

Let $\mathscr{B}$ denote the class of functions

$$f : \mathbb{U} \longrightarrow \mathbb{C},$$

where

$$\mathbb{U} := \mathbb{U}(1) \qquad \text{and} \qquad \mathbb{U}(r) := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < r\}.$$

We also denote by $\widetilde{\mathscr{A}}$ the class of functions $f \in \mathscr{B}$ which are analytic in $\mathbb{U}$ (see, for details, [32]).

We say that a function $f \in \mathscr{B}$ is *subordinate* to a function $F \in \mathscr{B}$, and we write

$$f(z) \prec F(z) \qquad \text{or, simply,} \qquad f \prec F,$$

if and only if there exists a function $w \in \mathscr{B}$ with

$$|w(z)| \leqq |z| \qquad (z \in \mathbb{U}),$$

such that (see, for details, [15])

$$f(z) = F\big(w(z)\big) \qquad (z \in \mathbb{U}).$$

In particular, if F is *univalent* in $\mathbb{U}$, we have the following equivalence:

$$f(z) \prec F(z) \Longleftrightarrow f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

For functions $f, g \in \widetilde{\mathscr{A}}$ of the forms:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

by $f * g$ we denote the *Hadamard product* (or *convolution*) of f and g, defined by

$$\big(f * g\big)(z) := \sum_{n=0}^{\infty} a_n b_n z^n =: \big(g * f\big)(z) \qquad (z \in \mathbb{U}).$$

We denote by $\mathscr{A}$ the class of functions $f \in \mathscr{B}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

We also denote by $\mathscr{T}_\eta \ \ (\eta \in \mathbb{R})$ the class of functions $f \in \mathscr{A}$ of the form (1.1) for which

$$\arg(a_n) = \pi + (1-n)\eta \qquad (n \in \mathbb{N} \setminus \{1\};\ \mathbb{N} := \{1,2,3,\cdots\}). \tag{1.2}$$

For $\eta = 0$, we obtain the familiar class $\mathscr{T}_0$ of functions with negative coefficients. Moreover, we define

$$\mathscr{T} := \bigcup_{\eta \in \mathbb{R}} \mathscr{T}_\eta. \tag{1.3}$$

The class $\mathscr{T}$ was introduced by Silverman [24] (see also [31]). It is called the class of functions with varying argument of coefficients.

Let

$$\alpha \in [0,1) \quad \text{and} \quad r \in (0,1].$$

A function$f \in \mathscr{A}$ is said to be *convex of order α in* $\mathbb{U}(r)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad \big(z \in \mathbb{U}(r);\ 0 \leqq \alpha < 1;\ 0 < r \leqq 1\big).$$

A function $f \in \mathscr{A}$ is said to be *starlike of order α in* $\mathbb{U}(r)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad \big(z \in \mathbb{U}(r);\ 0 \leqq \alpha < 1;\ 0 < r \leqq 1\big). \tag{1.4}$$

We denote by $\mathscr{S}^c(\alpha)$ the class of all functions $f \in \mathscr{A}$, which are *convex of order α in* $\mathbb{U}$, and by $\mathscr{S}^*(\alpha)$ we denote the class of all functions $f \in \mathscr{A}$, which are *starlike of order α in* $\mathbb{U}$. We also set

$$\mathscr{S}^c = \mathscr{S}^c(0) \qquad \text{and} \qquad \mathscr{S}^* = \mathscr{S}^*(0).$$

It is easy to show that, for a function $f \in \mathscr{T}_\eta$, the condition (1.4) is equivalent to the following inequality:

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha \qquad \left(z \in \mathbb{U}(r);\ 0 \leqq \alpha < 1;\ 0 < r \leqq 1\right). \tag{1.5}$$

Let $\mathscr{B}$ be a subclass of the class $\mathscr{A}$. We define the *radius of starlikeness of order* α and the *radius of convexity of order* α for the class $\mathscr{B}$ by

$$R_\alpha^*(\mathscr{B}) = \inf_{f \in \mathscr{B}} \left(\sup \left\{ r \in (0,1] : f \quad \text{is starlike of order } \alpha \ \text{ in } \ \mathbb{U}(r) \right\} \right)$$

and

$$R_\alpha^c(\mathscr{B}) = \inf_{f \in \mathscr{B}} \left(\sup \left\{ r \in (0,1] : f \ \text{is convex of order } \alpha \text{ in } \mathbb{U}(r) \right\} \right),$$

respectively.

Let k, A and B be real parameters such that

$$k \geqq 0, \ \ 0 \leqq B \leqq 1 \quad \text{and} \quad -1 \leqq A < B.$$

Also let $\varphi, \phi \in \mathscr{A}$. By $\mathscr{W}\left(\phi, \varphi; A, B; k\right)$ we denote the class of functions $f \in \mathscr{A}$ such that

$$\left(\varphi * f\right)(z) \neq 0 \qquad (z \in \mathbb{U} \setminus \{0\})$$

and

$$\frac{\left(\phi * f\right)(z)}{\left(\varphi * f\right)(z)} - k \left| \frac{\left(\phi * f\right)(z)}{\left(\varphi * f\right)(z)} - 1 \right| \prec \frac{1 + Az}{1 + Bz}. \tag{1.6}$$

If $0 \leqq B < 1$, then the condition (1.6) is equivalent to the following inequality:

$$\left| \frac{\left(\phi * f\right)(z)}{\left(\varphi * f\right)(z)} - k \left| \frac{\left(\phi * f\right)(z)}{\left(\varphi * f\right)(z)} - 1 \right| - \frac{1 - AB}{1 - B^2} \right| < \frac{B - A}{1 - B^2} \qquad (z \in \mathbb{U}). \tag{1.7}$$

On the other hand, if $B = 1$, then we have

$$\Re \left(\frac{\left(\phi * f\right)(z)}{\left(\varphi * f\right)(z)} \right) - k \left| \frac{\left(\phi * f\right)(z)}{\left(\varphi * f\right)(z)} - 1 \right| > \frac{1 + A}{2} \qquad (z \in \mathbb{U}). \tag{1.8}$$

Related to the function classes $\mathcal{T}$ and $\mathcal{T}_\eta$, we define the following two classes:

$$\mathcal{T}\mathcal{W}\left(\phi,\varphi;A,B;k\right):=\mathcal{T}\cap\mathcal{W}\left(\phi,\varphi;A,B;k\right)$$

and

$$\mathcal{T}\mathcal{W}_\eta\left(\phi,\varphi;A,B;k\right):=\mathcal{T}_\eta\cap\mathcal{W}\left(\phi,\varphi;A,B;k\right).$$

For our present investigation, we assume that φ and ϕ are the functions of the following forms:

$$\varphi(z)=z+\sum_{n=2}^{\infty}\alpha_n z^n \qquad \text{and} \qquad \phi(z)=z+\sum_{n=2}^{\infty}\beta_n z^n \quad (z\in\mathbb{U}), \tag{1.9}$$

where the *real* sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are constrained *further* by

$$0\leqq\alpha_n<\beta_n \qquad (n\in\mathbb{N}\setminus\{1\}).$$

Moreover, let us put

$$d_n:=(k+1)(1+B)\beta_n-(kB+A+k+1)\alpha_n \qquad (n\in\mathbb{N}\setminus\{1\}). \tag{1.10}$$

The function classes $\mathcal{W}\left(\phi,\varphi;A,B;k\right)$ and $\mathcal{W}_\eta\left(\phi,\varphi;A,B;k\right)$ unify and extend various known classes of analytic functions. We choose to list a few of these associated analytic function classes in the last section (Section 8).

The object of the present paper is to investigate coefficient estimates, distortion theorems, subordination theorems, convolution properties, integral means inequalities, and radii of conexity and starlikenes of functions in the general classes

$$\mathcal{T}\mathcal{W}_\eta\left(\phi,\varphi;A,B;k\right) \qquad \text{and} \qquad \mathcal{T}\mathcal{W}\left(\phi,\varphi;A,B;k\right),$$

which we have introduced here.

2. Coefficient Inequalities and Coefficient Estimates

In this section, we first derive a sufficient condition for a function f to belong to the class $\mathscr{W}(\phi, \varphi; A, B; k)$.

Theorem 2.1. *Let $\{d_n\}$ be defined by (1.10), $0 \leqq B \leqq 1$, and $-1 \leqq A < B$. If a function f of the form (1.1) with*

$$(\varphi * f)(z) \neq 0 \qquad (z \in \mathbb{U} \setminus \{0\}),$$

satisfies the following condition:

$$\sum_{n=2}^{\infty} d_n |a_n| \leqq B - A, \tag{2.1}$$

then the function f belongs to the class $\mathscr{W}(\phi, \varphi; A, B; k)$.

Proof. Let $0 \leqq B < 1$. Then, for a function f of the form (1.1), we have

$$\begin{aligned}
&\left| \frac{(\phi * f)(z)}{(\varphi * f)(z)} - k \left| \frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1 \right| - \frac{1 - AB}{1 - B^2} \right| \\
&\qquad \leqq (k+1) \left| \frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1 \right| + \frac{B(B-A)}{1-B^2} \\
&\qquad \leqq (k+1) \frac{\sum_{n=2}^{\infty} (\beta_n - \alpha_n) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \alpha_n |a_n| |z|^{n-1}} + \frac{B(B-A)}{1-B^2}.
\end{aligned}$$

Thus, by (2.1), we obtain (1.7). Consequently, $f \in \mathscr{W}(\phi, \varphi; A, B; k)$. We now suppose that $B = 1$. Then simple calculations give

$$\begin{aligned}
&k \left| \frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1 \right| - \Re \left(\frac{(\phi * f)(z)}{(\varphi * f)(z)} - \frac{1+A}{2} \right) \\
&\qquad \leqq (k+1) \left| \frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1 \right|
\end{aligned}$$

$$\leqq (k+1)\frac{\sum\limits_{n=2}^{\infty}\left(\beta_n-\alpha_n\right)|a_n||z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}\alpha_n|a_n||z|^{n-1}},$$

which, by means of (2.1), leads us to (1.8). Therefore, $f \in \mathscr{W}\left(\phi,\varphi;A,B;k\right)$, and the proof of Theorem 2.1 is completed.

Our next theorem shows that the condition (2.1) is necessary *as well* for functions of the form (1.1) satisfying the argument property (1.2) to belong to the class $\mathscr{T}\mathscr{W}_{\eta}\left(\phi,\varphi;A,B;k\right)$.

Theorem 2.2. *Let f be a function of the form* (1.1) *satisfying the argument property* (1.2). *Then f belongs to the class $\mathscr{T}\mathscr{W}_{\eta}\left(\phi,\varphi;A,B;k\right)$ if and only if the condition* (2.1) *holds true.*

Proof. In view of Theorem 2.1, we need only to show that each function f from the class $\mathscr{T}\mathscr{W}_{\eta}\left(\phi,\varphi;A,B;k\right)$ satisfies the coefficient inequality (2.1). Let f be a function of the form (1.1) and satisfying the argument property (1.2) belong to the class $\mathscr{T}\mathscr{W}_{\eta}\left(\phi,\varphi;A,B;k\right)$. Then, putting $z = r^{i\eta}$ in the conditions (1.7) and (1.8), we obtain

$$(k+1)\frac{\sum\limits_{n=2}^{\infty}\left(\beta_n-\alpha_n\right)|a_n|r^{n-1}}{1-\sum\limits_{n=2}^{\infty}\alpha_n|a_n|r^{n-1}} < \frac{B-A}{1+B}.$$

We thus find that

$$\sum_{n=2}^{\infty}\left[(k+1)(1+B)\beta_n-(k(1+B)+1+A)\alpha_n\right]|a_n|r^{n-1} < B-A,$$

which, upon letting $r \to 1-$, readily yields the assertion (2.1).

Since the condition (2.1) is independent of η, Theorem 2.2 yields the following result.

Theorem 2.3. *Let f be a function of the form* (1.1) *satisfying the agument property* (1.2). *Then $f \in \mathscr{TW}(\phi, \varphi; A, B; k)$ if and only if the condition* (2.1) *holds true.*

From Theorems 2.2 and 2.3 we can obtain the following coefficient estimates for functions in the classes

$$\mathscr{TW}_\eta(\phi, \varphi; A, B; k) \qquad \text{and} \qquad \mathscr{TW}(\phi, \varphi; A, B; k),$$

respectively.

Corollary 2.1. *If a function f of the form* (1.1) *belongs to the class $\mathscr{TW}_\eta(\phi, \varphi; A, B; k)$, then*

$$|a_n| \leqq \frac{B-A}{d_n} \qquad (n \in \mathbb{N} \setminus \{1\}), \tag{2.2}$$

where d_n is defined by (1.10). *The result is sharp and the functions $f_{n,\eta}$ given by*

$$f_{n,\eta}(z) = z - \frac{B-A}{d_n} e^{i(1-n)\eta} z^n \qquad (z \in \mathbb{U};\ n \in \mathbb{N}) \tag{2.3}$$

are the extremal functions.

Corollary 2.2. *If a function f of the form* (1.1) *belongs to the class $\mathscr{TW}(\phi, \varphi; A, B; k)$, then the coefficient estimates given by* (2.2) *hold true. The result is sharp and the functions $f_{n,\eta}$ $(\eta \in \mathbb{R})$ given by* (2.3) *are the extremal functions.*

3. Distortion Theorems

By applying Theorem 2.2, we can deduce the following lemma.

Lemma 3.1. *Let a function f of the form* (1.1) *belong to the class $\mathscr{TW}_\eta(\phi, \varphi; A, B; k)$. If the sequence $\{d_n\}$ defined by* (1.10) *satisfies the following inequality*:

$$d_2 \leqq d_n \qquad (n \in \mathbb{N} \setminus \{1\}), \tag{3.1}$$

then

$$\sum_{n=2}^{\infty} a_n \leqq \frac{B-A}{d_2}.$$

Moreover, if

$$nd_2 \leqq 2d_n \quad (n \in \mathbb{N} \setminus \{1\}), \tag{3.2}$$

then

$$\sum_{n=2}^{\infty} na_n \leqq \frac{2(B-A)}{d_2}.$$

Theorem 3.1. *Let a function f belong to the class $\mathcal{TW}_\eta(\phi, \varphi; A, B; k)$. If the sequence $\{d_n\}$ defined by* (1.10) *satisfies* (3.1), *then*

$$r - \frac{B-A}{d_2} r^2 \leqq |f(z)| \leqq r + \frac{B-A}{d_2} r^2 \qquad (|z| = r < 1). \tag{3.3}$$

Moreover, if (3.2) *holds true, then*

$$1 - \frac{2(B-A)}{d_2} r \leqq |f'(z)| \leqq 1 + \frac{2(B-A)}{d_2} r \qquad (|z| = r < 1). \tag{3.4}$$

The result is sharp and the extremal function $f_{2,\eta}$ is given by (2.3).

Proof. Let a function f of the form (1.1) belong to the class $\mathcal{TW}_\eta(\phi, \varphi; A, B; k)$. Then, for $|z| = r < 1$,

$$\begin{aligned}|f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \leqq r + \sum_{n=2}^{\infty} |a_n| r^n \\ &= r + r^2 \sum_{n=2}^{\infty} |a_n| r^{n-2} \leqq r + r^2 \sum_{n=2}^{\infty} |a_n|\end{aligned}$$

and

$$\begin{aligned}|f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \geqq r - \sum_{n=2}^{\infty} |a_n| r^n \\ &= r - r^2 \sum_{n=2}^{\infty} |a_n| r^{n-2} \geqq r - r^2 \sum_{n=2}^{\infty} |a_n|,\end{aligned}$$

which, in light of Lemma 3.1, yields (3.3). Analogously, we can prove (3.4).

Theorem 3.1 implies the following corollary:

Corollary 3.1. *Let a function f belong to the class $\mathcal{TW}\left(\phi,\varphi;A,B;k\right)$. If the sequence $\{d_n\}$ defined by* (1.10) *satisfies* (3.1), *then the assertion* (3.3) *holds true. Moreover, if we assume that* (3.2) *is satisfied, then the assertion* (3.4) *holds true. The result is sharp and the extremal functions $f_{2,\eta}$ $(\eta \in \mathbb{R})$ are given by* (2.3).

4. Results Involving Subordination Between Analytic Functions

Before stating and proving our subordination theorems for the function classes

$$\mathcal{TW}_{\eta}\left(\phi,\varphi;A,B;k\right) \qquad \text{and} \qquad \mathcal{TW}\left(\phi,\varphi;A,B;k\right),$$

we need the following definition as well as Lemma 4.1.

Definition 4.1. *A sequence $\{b_n\}$ of complex numbers is said to be a subordinating factor sequence if, for each function f of the form (1.1) from the class $\mathcal{S}^c$, we have*

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \qquad \left(a_1 = 1\right). \tag{4.1}$$

Lemma 4.1. (see *[36]*) *The sequence $\{b_n\}$ is a subordinating factor sequence if and only if*

$$\Re\left(1+2\sum_{n=1}^{\infty} b_n z^n\right) > 0 \qquad (z \in \mathbb{U}). \tag{4.2}$$

Theorem 4.1. *Let the sequence $\{d_n\}$, defined by* (1.10), *satisfy the inequality* (3.1). *If $g \in \mathcal{S}^c$ and $f \in \mathcal{TW}_{\eta}\left(\phi,\varphi;A,B;k\right)$, then*

$$\varepsilon(f * g)(z) \prec g(z) \tag{4.3}$$

and

$$\Re\left(f(z)\right) > -\frac{1}{2\varepsilon} \qquad (z \in \mathbb{U}), \tag{4.4}$$

where

$$\varepsilon = \frac{d_2}{2\left(B-A+d_2\right)}. \tag{4.5}$$

The constant factor ε cannot be replaced by a larger number.

Proof. Let a function f of the form (1.1) belong to the class $\mathscr{T}\mathscr{W}_\eta\left(\phi,\varphi;A,B;k\right)$ and suppose that

$$g(z)=z+\sum_{n=2}^{\infty}c_nz^n \qquad (z\in\mathbb{U})$$

belongs to the class $\mathscr{S}^c$. Then

$$\varepsilon(f*g)(z)=\varepsilon z+\sum_{n=2}^{\infty}\left(\varepsilon a_n\right)c_nz^n.$$

Thus, by the above Definition, the subordination result (4.3) holds true if

$$\{\varepsilon a_n\}_{n=1}^{\infty} \qquad (a_1=1)$$

is a subordinating factor sequence. In view of Lemma 4.1, this is equivalent to the following inequality:

$$\Re\left(1+2\sum_{n=1}^{\infty}\varepsilon a_nz^n\right)>0 \qquad (z\in\mathbb{U}). \tag{4.6}$$

By (3.1) for $|z|=r<1$, we have

$$\begin{aligned}\Re\left(1+2\sum_{n=1}^{\infty}\varepsilon a_nz^n\right)&=\Re\left(1+2\varepsilon z+\sum_{n=2}^{\infty}\frac{d_2}{B-A+d_2}a_nz^n\right)\\&\geqq 1-2\varepsilon r-\frac{r}{B-A+d_2}\sum_{n=2}^{\infty}d_n\left|a_n\right|r^{n-1}.\end{aligned}$$

Consequently, by using Theorem 2.2, we obtain

$$\Re\left(1+2\sum_{n=1}^{\infty}\varepsilon a_nz^n\right)\geqq 1-\frac{d_2}{B-A+d_2}r-\frac{B-A}{B-A+d_2}r>0.$$

This evidently proves the inequality (4.6) and hence the subordination result (4.3).

The inequality (4.4) follows from (4.3) by taking

$$g(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n \qquad (z\in\mathbb{U}).$$

We next observe that the function $f_{2,\eta}$ of the form (2.3) belongs to the class $\mathscr{T}\mathscr{W}_{\eta}\left(\phi,\varphi;A,B;k\right)$. It is easily verified that

$$\min\left\{\Re\left(\varepsilon f_{2,\eta}(z)\right)\right\} = -\frac{1}{2} \qquad (z \in \mathbb{U}).$$

This shows that the constant (4.5) cannot be replaced by any larger number.

Directly from Theorem 4.1, we can obtain Theorem 4.2 below.

Theorem 4.2. *Let the sequence* $\{d_n\}$, *defined by* (1.10), *satisfy the inequality* (3.1). *If* $g \in \mathscr{S}^c$ *and* $f \in \mathscr{T}\mathscr{W}\left(\phi,\varphi;A,B;k\right)$, *then the conditions* (4.3) *and* (4.4) *hold true. The constant factor* ε *in* (4.3) *cannot be replaced by a larger number.*

5. Integral Means Inequalities

Following the work of Littlewood [11], we obtain here some integral means inequalities for functions belonging to the class $\mathscr{T}\mathscr{W}_{\eta}\left(\phi,\varphi;A,B;k\right)$.

Lemma 5.1. (see *[11]*) *Let* $f,g \in \widetilde{\mathscr{A}}$. *If* $f \prec g$, *then*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta \leqq \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta \qquad \left(0<r<1;\ \eta>0\right). \tag{5.1}$$

Silverman [23] found that the following function:

$$g(z) = z - \frac{z^2}{2} \qquad (z \in \mathbb{U}),$$

is often extremal over the family of functions with negative coefficients. He applied this function to resolve a certain integral means inequality, which was conjectured in [25] and settled in [26], that (5.1) holds true for all functions f with negative coefficients. Silverman [26] also proved his conjecture for some subclasses of $\mathscr{T}$.

Applying Lemma 5.1 and Theorem 2.2, we now prove the following result.

Theorem 5.1. *Let the sequence* $\{d_n\}$ *defined by* (1.10) *satisfy the inequality* (3.1). *If* $f \in \mathscr{TW}_\eta(\phi, \varphi; A, B; k)$, *then*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\lambda d\theta \leqq \int_0^{2\pi} \left|f_{2,\eta}(re^{i\theta})\right|^\lambda d\theta \qquad (0 < r < 1;\ \lambda > 0), \tag{5.2}$$

where $f_{2,\eta}(z)$ *is defined by* (2.3).

Proof. For a function f of the form (1.1), the inequality (5.2) is equivalent to the following inequality:

$$\int_0^{2\pi} \left|1 + \sum_{n=2}^{\infty} a_n z^{n-1}\right|^\lambda d\theta \leqq \int_0^{2\pi} \left|1 - \frac{B-A}{d_2}\, e^{-i\eta}\, z\right|^\lambda d\theta \qquad \left(z = re^{i\theta}\right).$$

Thus, by Lemma 5.1, it suffices to show that

$$\sum_{n=2}^{\infty} a_n z^{n-1} \prec -\frac{B-A}{d_2} e^{-i\eta} z. \tag{5.3}$$

Upon setting

$$w(z) = \sum_{n=2}^{\infty} \frac{d_2 e^{i\eta}}{A-B}\, a_n\, z^{n-1} \quad (z \in \mathbb{U}),$$

and using (3.1) and Theorem 2.2, we obtain

$$|w(z)| = \left|\sum_{n=2}^{\infty} \frac{d_2}{A-B} a_n\, z^{n-1}\right| \leqq |z| \sum_{n=2}^{\infty} \frac{d_n}{B-A} \left|a_n\right| \leqq |z| \qquad (z \in \mathbb{U}).$$

Since

$$\sum_{n=2}^{\infty} a_n z^{n-1} = -\frac{B-A}{d_2} e^{-i\eta} w(z) \qquad (z \in \mathbb{U}),$$

by the definition of subordination, we have (5.3). This completes the proof of Theorem 5.1.

We can restate Theorem 5.1 as Theorem 5.2 below.

Theorem 5.2. *Let the sequence* $\{d_n\}$ *defined by* (1.10) *satisfy the inequality* (3.1). *If a function* f *of the form* (1.1) *satisfying the argument property* (1.2) *belongs to the class* $\mathscr{TW}(\phi, \varphi; A, B; k)$, *then the integral means inequality* (5.2) *holds true.*

6. The Radii of Convexity and Starlikeness

We begin this section by proving the following result.

Theorem 6.1. *The radius of starlikeness of order* α *for the class* $\mathcal{TW}_\eta(\phi,\varphi;A,B;k)$ *is given by*

$$R_\alpha^*(\mathcal{TW}_\eta(\phi,\varphi;A,B;k)) = \inf_{n\in\mathbb{N}\setminus\{1\}} \left(\frac{(1-\alpha)\,d_n}{(n-\alpha)(B-A)}\right)^{\frac{1}{n-1}}, \tag{6.1}$$

where d_n *is defined by* (1.10).

Proof. A function $f \in \mathcal{T}_\eta$ of the form (1.1) is starlike of order α in the disk $\mathbb{U}(r)$ $(0 < r \leqq 1)$ if and only if it satisfies the condition (1.5). Since

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\sum_{n=2}^{\infty}(n-1)a_n z^n}{z + \sum_{n=2}^{\infty} a_n z^n}\right| \leqq \frac{\sum_{n=2}^{\infty}(n-1)\left|a_n\right| |z|^{n-1}}{1 - \sum_{n=2}^{\infty}\left|a_n\right| |z|^{n-1}},$$

by putting $|z| = r$, the condition (1.5) is true if

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}\left|a_n\right| r^{n-1} \leqq 1. \tag{6.2}$$

By Theorem 2.2, we have

$$\sum_{n=2}^{\infty} \frac{d_n}{B-A}\left|a_n\right| \leqq 1,$$

so that the condition (6.2) is true if

$$\frac{n-\alpha}{1-\alpha} r^{n-1} \leqq \frac{d_n}{B-A} \qquad (n \in \mathbb{N}\setminus\{1\}),$$

that is, if

$$r \leqq \left(\frac{(1-\alpha)\,d_n}{(n-\alpha)(B-A)}\right)^{\frac{1}{n-1}} \qquad (n \in \mathbb{N}\setminus\{1\}).$$

It follows that each function $f \in \mathcal{TW}_\eta(\phi,\varphi;A,B;k)$ is starlike of order α in the disk $\mathbb{U}(r)$, where

$$r = R_\alpha^*(\mathcal{TW}_\eta(\phi,\varphi;A,B;k))$$

is defined by (6.1).

Theorem 6.2. *The radius of convexity of order α for the class $\mathcal{TW}_\eta(\phi,\varphi;A,B;k)$ is given by*

$$R^c_\alpha\left(\mathcal{TW}_\eta(\phi,\varphi;A,B;k)\right)=\inf_{n\in\mathbb{N}\setminus\{1\}}\left(\frac{(1-\alpha)\,d_n}{n(n-\alpha)(B-A)}\right)^{\frac{1}{n-1}},$$

where d_n is defined by (1.10).

Proof. The proof of Theorem 6.2 is analogous to that of Theorem 6.1, and we omit the details involved.

By applying Theorems 6.1 and 6.2, we obtain the following two corollaries.

Corollary 6.1. *The radius of starlikeness of order α for the class $\mathcal{TW}(\phi,\varphi;A,B;k)$ is given by*

$$R^*_\alpha\left(\mathcal{TW}(\phi,\varphi;A,B;k)\right)=\inf_{n\in\mathbb{N}\setminus\{1\}}\left(\frac{(1-\alpha)\,d_n}{(n-\alpha)(B-A)}\right)^{\frac{1}{n-1}},$$

where d_n is defined by (1.10).

Corollary 6.2. *The radius of convexity of order α for the class $\mathcal{TW}(\phi,\varphi;A,B;k)$ is given by*

$$R^c_\alpha\left(\mathcal{TW}(\phi,\varphi;A,B;k)\right)=\inf_{n\in\mathbb{N}\setminus\{1\}}\left(\frac{(1-\alpha)\,d_n}{n(n-\alpha)(B-A)}\right)^{\frac{1}{n-1}},$$

where d_n is defined by (1.10).

7. Convolution Properties

Theorem 7.1. *Let the sequence $\{d_n\}$ defined by* (1.10) *satisfy the inequality* (3.1) *with*

$$d_2 \geqq B-A.$$

If

$$f\in\mathcal{TW}_\eta(\phi,\varphi;A,B;k) \qquad \text{and} \qquad g\in\mathcal{TW}_\mu(\phi,\varphi;A,B;k),$$

then

$$f*g\in\mathcal{TW}_{\eta+\mu}(\phi,\varphi;A,B;k).$$

Proof. Let the functions f and g of the forms:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \qquad \text{and} \qquad g(z)=z+\sum_{n=2}^{\infty}b_n z^n \qquad (z\in\mathbb{U}). \tag{7.1}$$

belong to the classes

$$\mathscr{T}\mathscr{W}_{\eta}\left(\phi,\varphi;A,B;k\right) \qquad \text{and} \qquad \mathscr{T}\mathscr{W}_{\mu}\left(\phi,\varphi;A,B;k\right),$$

respectively. Then, by appealing to Theorem 2.2, we have

$$\sum_{n=2}^{\infty}\frac{d_n}{B-A}\left|a_n\right| \leqq 1 \qquad \text{and} \qquad \sum_{n=2}^{\infty}\frac{d_n}{B-A}\left|b_n\right| \leqq 1.$$

Thus, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=2}^{\infty}\frac{d_n}{B-A}\sqrt{\left|a_n b_n\right|} \leqq 1. \tag{7.2}$$

In order to prove that

$$\sum_{k=2}^{\infty}\frac{d_n}{B-A}\left|a_n b_n\right| \leqq 1,$$

by virtue of (7.2), it is sufficient to show that

$$\left|a_n b_n\right| \leqq \sqrt{\left|a_n b_n\right|} \qquad (n\in\mathbb{N}\setminus\{1\})$$

or, equivalently, that

$$\sqrt{\left|a_n b_n\right|} \leqq 1 \qquad (n\in\mathbb{N}\setminus\{1\}).$$

We note from (7.2) that

$$\sqrt{\left|a_n b_n\right|} \leqq \frac{B-A}{d_n} \qquad (n\in\mathbb{N}\setminus\{1\}).$$

Consequently, we need only to prove that

$$\frac{B-A}{d_n} \leqq 1 \qquad (n\in\mathbb{N}\setminus\{1\}).$$

Since $d_2 \geqq B-A$, by hypothesis, so the last inequality follows from (3.1).

By applying Theorem 7.1, we obtain the following corollary.

Corollary 7.1. *Let the sequence* $\{d_n\}$ *defined by* (1.10) *satisfy the inequality* (3.1) *with*

$$d_2 \geqq B - A.$$

If

$$f, g \in \mathcal{TW}\left(\phi, \varphi; A, B; k\right),$$

then

$$f * g \in \mathcal{TW}\left(\phi, \varphi; A, B; k\right).$$

Theorem 7.2. *Let the sequence* $\{d_n\}$ *defined by* (1.10) *satisfy the inequality* (3.1) *with*

$$d_2 \geqq 2(B - A).$$

If the functions f *and* g *of the form* (7.1) *belong to the class* $\mathcal{TW}_\eta\left(\phi, \varphi; A, B; k\right)$, *then the function* $h(z)$ *given by*

$$h(z) = z - \sum_{n=2}^{\infty}(|a_n|^2 + |b_n|^2) z^n \tag{7.3}$$

belongs to the class $\mathcal{TW}_0\left(\phi, \varphi; A, B; k\right)$.

Proof. Suppose that each of the functions f and g of the form (7.1) belongs to the class $\mathcal{TW}_\eta\left(\phi, \varphi; A, B; k\right)$. Then, by Theorem 2.2, we have

$$\sum_{n=2}^{\infty}\left(\frac{d_n}{B-A}|a_n|\right)^2 \leqq 1 \qquad \text{and} \qquad \sum_{n=2}^{\infty}\left(\frac{d_n}{B-A}|b_n|\right)^2 \leqq 1.$$

We thus obtain

$$\sum_{n=2}^{\infty}\frac{1}{2}\left(\frac{d_n}{B-A}\right)^2\left(|a_n|^2 + |b_n|^2\right) \leqq 1. \tag{7.4}$$

In order to prove that

$$\sum_{k=2}^{\infty}\frac{d_n}{B-A}\left(|a_n|^2 + |b_n|^2\right) \leqq 1,$$

by means of (7.4), it is sufficient to show that

$$\frac{d_n}{B-A} \geqq 2 \qquad (n \in \mathbb{N} \setminus \{1\}).$$

Since $d_2 \geqq 2(B - A)$, by hypothesis, so the last inequality follows from (3.1).

8. Concluding Remarks and Observations

We conclude this paper by observing that, in view of the subordination relation (1.6), by suitably choosing the functions ϕ and φ, we can consider various new or known classes of functions. Let

$$\mathscr{W}_n\left(\varphi;A,B;k\right) := \mathscr{W}\left(z\varphi'(z), \sum_{k=0}^{n-1}\varphi\left(x^k z\right);A,B;k\right) \qquad \left(n \in \mathbb{N};\ x = e^{\frac{2\pi i}{n}}\right).$$

In particular, the class

$$\mathscr{W}_n\left(\varphi;A,B\right) := \mathscr{W}_n\left(\varphi;A,B;0\right)$$

consists of functions $f \in \mathscr{A}$, which satisfy the following subordination condition:

$$\frac{z\left(\varphi * f\right)'(z)}{\sum\limits_{k=0}^{n-1}\left(\varphi * f\right)\left(x^k z\right)} \prec \frac{1+Az}{1+Bz}.$$

It is related to the class of starlike functions with respect to n-symmetric points. Moreover, by putting $n = 1$, we obtain the class

$$\mathscr{W}\left(\varphi;A,B\right) := \mathscr{W}_1\left(\varphi;A,B\right)$$

defined by the following subordination condition:

$$\frac{z\left(\varphi * f\right)'(z)}{\left(\varphi * f\right)(z)} \prec \frac{1+Az}{1+Bz}.$$

This class is related to the familiar class of starlike functions. Analogously, the class defined by

$$\mathscr{H}_n\left(\varphi;\gamma,k\right) := \mathscr{W}_n\left(\varphi;2\gamma-1,1;k\right) \qquad \left(0 \leqq \gamma < 1\right)$$

consists of functions $f \in \mathscr{A}$, which satisfy the following condition:

$$\mathfrak{R}\left(\frac{z\left(\varphi * f\right)'(z)}{\sum\limits_{k=0}^{n-1}\left(\varphi * f\right)\left(x^k z\right)} - \gamma\right) > k\left|\frac{z\left(\varphi * f\right)'(z)}{\sum\limits_{k=0}^{n-1}\left(\varphi * f\right)\left(x^k z\right)} - 1\right| \qquad \left(z \in \mathbb{U}\right).$$

It is related to the class of k-uniformly convex functions of order γ with respect to n-symmetric points. Moreover, by setting $n = 1$, we obtain the function class

$$\mathscr{H}\left(\varphi;\gamma,k\right) := \mathscr{H}_1\left(\varphi;\gamma,k\right),$$

which is defined by the following condition:

$$\Re\left(\frac{z\left(\varphi * f\right)'(z)}{\left(\varphi * f\right)(z)} - \gamma\right) > k\left|\frac{z\left(\varphi * f\right)'(z)}{\left(\varphi * f\right)(z)} - 1\right| \qquad (z \in \mathbb{U}).$$

This class is related to the class of k-uniformly convex functions of order γ. The function classes

$$UST\left(\gamma,k\right) := \mathscr{H}\left(\frac{z}{1-z};\gamma,k\right)$$

and

$$UCV\left(\gamma,k\right) := \mathscr{H}\left(\frac{z}{(1-z)^2};\gamma,k\right)$$

are the well-known classes of of k-starlike functions of order γ and k-uniformly convex functions of order γ, respectively. In particular, the function classes

$$UCV := UCV\left(1,0\right) \qquad \text{and} \qquad k - UCV := UCV(k,0)$$

were introduced by Goodman [8] (see also [14] and [22]) and Kanas *et al.* ([10] and [9]), respectively (see also [7], [20], [21], [33], [34] and [35]).

We note that the following function class:

$$\mathscr{H}_{\mathscr{T}}\left(\varphi;\gamma,k\right) := \mathscr{T}_0 \cap \mathscr{H}\left(\varphi;\gamma,k\right)$$

was investigated recently by Raina and Bansal [19].

If we set

$$h(\alpha_1,z) := z\ {}_qF_s(\alpha_1,\cdots,\alpha_q;\beta_1,\cdots,\beta_s;z),$$

where ${}_qF_s$ is the generalized hypergeometric function (see, for details, [29]), then we obtain the following function class:

$$\mathscr{U}\mathscr{H}\left(q,s,\lambda,\gamma,k\right) := \mathscr{H}_{\mathscr{T}}\left(\lambda h(\alpha_1+1,z) + (1-\lambda)h(\alpha_1,z);\gamma,k\right) \qquad (0 \leqq \lambda \leqq 1)$$

which was introduced and studied by Ramachandran *et al.* [20] (see also several recent works including, for example, [4], [5], [6], [12], [13], [18], [27], [34] and [35], which investigate various properties and applications of the *Dziok-Srivastava operator* defined by means of the Hadamard product involving the generalized hypergeometric function ${}_qF_s$).

Let λ be a convex parameter. A function $f \in \mathscr{A}$ is said to belong to the class

$$\mathscr{V}_\lambda\left(\varphi;A,B\right) := \mathscr{W}\left(\lambda\frac{\varphi\left(z\right)}{z}+\left(1-\lambda\right)\varphi'\left(z\right),z;A,B;0\right)$$

if it satisfies the following condition:

$$\lambda\frac{\left(\varphi*f\right)\left(z\right)}{z}+\left(1-\lambda\right)\left(\varphi*f\right)'\left(z\right)\prec\frac{1+Az}{1+Bz}.$$

Moreover, a function $f \in \mathscr{A}$ is said to belong to the class

$$\mathscr{U}_\lambda\left(\varphi;A,B\right) := \mathscr{W}\left(\lambda\frac{\varphi\left(z\right)}{z}+\left(1-\lambda\right)\varphi'\left(z\right);A,B;0\right)$$

if it satisfies the following condition:

$$\frac{z\left(\varphi*f\right)'\left(z\right)+\left(1-\lambda\right)z^2\left(\varphi*f\right)''\left(z\right)}{\lambda\left(\varphi*f\right)\left(z\right)+\left(1-\lambda\right)z\left(\varphi*f\right)'\left(z\right)}\prec\frac{1+Az}{1+Bz}. \tag{8.1}$$

The above-defined function classes

$$\mathscr{W}_n\left(\varphi;A,B\right),\ \ \mathscr{H}_n\left(\varphi;\gamma,k\right),\ \ \mathscr{U}_\lambda\left(\varphi;A,B\right)\quad\text{and}\quad\mathscr{V}_\lambda\left(\varphi;A,B\right)$$

generalize several important classes, many of which were investigated systematically in earlier works (see, for example, [1], [2], [3], [16], [17], [28] and [30]).

If we apply the results presented in this paper to the classes discussed above, we can easily be led to a remarkably large number of additional new or known results.

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