On \( k \)-Cost Effective Domination Number in the Join of Graphs

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Abstract. In this paper, we characterized the \( k \)-cost effective domination in the join of graphs. Further, we investigate the \( k \)-cost effective domination, cost effective domination index, maximal cost effective domination in the join of graphs.

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1. Introduction

Let \( G = (V(G), E(G)) \) be a connected simple graph and \( v \in V(G) \). The neighborhood of \( v \) in the set \( N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\} \). The degree of a vertex \( v \) in a graph \( G \), denoted by \( \text{deg}_G(v) \), is \( |N(v)| \). A subset \( S \) of \( V(G) \) is a dominating set of \( G \) if for every \( v \in V(G) \setminus S \), there exists \( u \in S \) such that \( uv \in E(G) \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of a dominating set of \( G \). A subset \( S \) of \( V(G) \) is an independent set of \( G \) if \( \text{deg}(uv) = 0 \) for distinct pairs of vertices \( u \) and \( v \) in \( S \). An independent dominating set in \( G \) is an independent set in \( G \) which is dominating in \( G \). The minimum cardinality \( \gamma_i(G) \) of an independent dominating set in \( G \) is called independence domination number.

Let \( k \geq 0 \) be an integer. Consider a vertex \( v \), its neighborhood set, \( N(v) \) and the vertex-set of \( G \), \( V(G) \). A vertex \( v \in S \subseteq V(G) \) is said to be \( k \)-cost effective if \( |N(v) \cap (V(G) \setminus S)| \geq |N(v) \cap S| + k \). A dominating set \( S \) is \( k \)-cost effective, if every vertex in \( S \) is \( k \)-cost effective. The minimum cardinality of a \( k \)-cost effective dominating set in \( G \) is called the \( k \)-cost effective domination number, denoted by \( \gamma_k(G) \).

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set of $G$ is the $k$-cost effective domination number $\gamma_{ce}^k(G)$ of $G$. In cases where there is no $k$-cost effective dominating set for $G$, the $k$-cost effective domination number of $G$ is infinity. The \emph{k-cost effective domination index} of $G$, denoted by $\eta(G)$, is the maximum value of $k$ such that $k$-cost effective domination number is finite. That is,

$$\eta(G) = \max \{ k : \gamma_{ce}^k(G) \text{ is finite} \}.$$

The \emph{maximal cost effective domination number} of $G$ is equal to $\gamma_{ce}^{\eta(G)}(G)$.

2. Results

\textbf{Theorem 1.} Let $G$ and $H$ be connected graphs, $k \geq \max \{|V(G)|, |V(H)|\}$, and $S \subseteq V(G + H)$. Then $S$ is a $k$-cost effective dominating set in $G + H$ if and only if one of the following holds:

(i) $S$ is $(k - |V(H)|)$-cost effective dominating set in $G$;

(ii) $S$ is $(k - |V(G)|)$-cost effective dominating set in $H$;

(iii) $V(G) \cap S$ is $(k - k_1)$-cost effective dominating set in $G$, where $k_1 = |V(H)| - 2|V(H) \cap S|$ and $V(H) \cap S$ is $(k - k_2)$-cost effective dominating set in $H$, where $k_2 = |V(G)| - 2|V(G) \cap S|$.

\textbf{Proof:} Let $k \geq \max \{|V(G)|, |V(H)|\}$, and $S \subseteq V(G + H)$. Suppose $S$ is a $k$-cost effective dominating set in $G + H$ and let $x \in S$. Then

$$|N_{G + H}(x) \setminus S| - |N_{G + H}(x) \cap S| \geq k.$$ 

Suppose $S \subseteq V(G)$. Then $S$ is a dominating set in $G$. Now,

$$|N_{G + H}(x) \setminus S| - |N_{G + H}(x) \cap S| = |V(H)| + |N_G(x) \setminus S| - |N_G(x) \cap S| \geq k.$$

This implies that,

$$|N_G(x) \setminus S| - |N_G(x) \cap S| \geq k - |V(H)|.$$

Hence, $S$ is $(k - |V(H)|)$-cost effective dominating set in $G$. Similarly, if $S \subseteq V(H)$, then $S$ is $(k - |V(G)|)$-cost effective dominating set in $H$.

Suppose that $S_1 = V(G) \cap S \neq \emptyset$ and $S_2 = V(H) \cap S \neq \emptyset$. Since $S$ is a $k$-cost effective dominating set in $G + H$,

$$|N_{G + H}(x) \setminus S| - |N_{G + H}(x) \cap S| \geq k.$$

Let $x \in S_1 \subseteq S$. Then

$$|N_{G + H}(x) \setminus S| - |N_{G + H}(x) \cap S| = |N_G(x) \setminus S_1| + |V(H) \setminus S_2| - |N_G(x) \cap S_1| - |S_2|$$
Similarly, for each $x^k$, let

$$\text{Corollary 1.}$$

This implies that,

$$|N_G(x) \setminus S_1| - |N_G(x) \cap S_1| \geq k - |V(H)| + 2|V(H) \cap S|$$

$$= k - (|V(H)| - 2|V(H) \cap S|)$$

$$= k - k_1,$$

where $k_1 = |V(H)| - 2|V(H) \cap S|$. Thus, $S_1 = V(G) \cap S$ is $(k - k_1)$-cost effective dominating set in $G$. Similarly, $S_2 = V(H) \cap S$ is $(k - k_2)$-cost effective dominating set in $H$.

Conversely, suppose that $S$ satisfies Property (i). Then $S$ is a dominating set in $G + H$ and

$$|N_G(x) \setminus S| - |N_G(x) \cap S| \geq k - |V(H)|, \forall x \in S.$$ 

Now,

$$|N_G(H) \setminus S| - |N_G(H) \cap S| = |V(H)| + |N_G(x) \setminus S| - |N_G(x) \cap S|$$

$$\geq |V(H)| + k - |V(H)|$$

$$= k,$$

for all $x \in S$. Since $x$ is arbitrary, $S$ is a $k$-cost effective dominating set in $G + H$. Similarly, if $S$ satisfies Property (ii), then $S$ is a $k$-cost effective dominating set in $G + H$. Suppose $S$ satisfies Property (iii) and $x \in V(G) \cap S$. Then

$$|N_G(x) \setminus S| - |N_G(x) \cap S| \geq k - k_1,$$

where $k_1 = |V(H)| - 2|V(H) \cap S|$. Now,

$$|N_G(H) \setminus S| - |N_G(H) \cap S| = |N_G(x) \setminus S| + |V(H) \setminus S| - |N_G(x) \cap S| + |V(H) \cap S|$$

$$= |N_G(x) \setminus S| - |N_G(x) \cap S| + |V(H) \setminus S| - |V(H) \cap S|$$

$$= |N_G(x) \setminus S| - |N_G(x) \cap S| + |V(H)| - 2|V(H) \cap S|$$

$$\geq k - k_1 + k_1$$

$$= k.$$

Similarly, for each $x \in V(H) \cap S$, $|N_G(H) \setminus S| - |N_G(H) \cap S| \geq k$. Therefore, $S$ is a $k$-cost effective dominating set in $G + H$.

**Corollary 1.** Let $G$ and $H$ be connected graphs, $k \geq \max\{|V(G)|, |V(H)|\}$. If $S$ is a $k$-cost effective dominating set in $G + H$, then one of the following holds:

(i) $S \subseteq V(G)$ and $k \leq \eta(G) + |V(H)|$

(ii) $S \subseteq V(H)$ and $k \leq \eta(H) + |V(G)|$.
Thus, $\gamma$ and $\eta$

Let Corollary 3.

Theorem 2. Let $G$ and $H$ be connected graphs such that $\gamma(G) = 1$ or $\gamma(H) = 1$ and $0 \leq k \leq |V(H)| + |V(G)| - 1$. Then $S \subseteq V(G + H)$ is a $\gamma^k_{ce}$-set in $G + H$ if and only if $S$ is a $\gamma$-set in $G$ or $S$ is a $\gamma$-set in $H$.

Corollary 2. Let $G$ and $H$ be connected graphs such that $\gamma(G) = 1$ or $\gamma(H) = 1$. Then

$$\gamma^k_{ce}(G + H) = \begin{cases} 1, & \text{if } 0 \leq k \leq |V(H)| + |V(G)| - 1 \\ \infty, & \text{if } k > |V(H)| + |V(G)| - 1. \end{cases}$$

Corollary 3. Let $G$ and $H$ be connected graphs such that $\gamma(G) = 1$ or $\gamma(H) = 1$. Then $\eta(G + H) = |V(H)| + |V(G)| - 1$ and $\eta^k_{ce}(G + H) = 1$.

In the succeeding theorems, $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$ and assume that $\Delta(G) + |V(H)| \leq \Delta(H) + |V(G)|$.

Theorem 3. Let $G$ and $H$ be connected graphs such that $\min\{\gamma(G), \gamma(H)\} \geq 2$ and $0 \leq k \leq \Delta(G) + |V(H)| - 2$. Then $S$ is a $\gamma^k_{ce}$-set in $G + H$ if and only if $|S| = 2$ and one of the following holds:

(i) $|V(G) \cap S| = 1$ and $|V(H) \cap S| = 1$;

(ii) $S$ is a $\gamma$-set in $G$ such that $k - |V(H)| + 2 \leq \delta(S : G)$;

(iii) $S$ is a $\gamma$-set in $H$ such that $k - |V(G)| + 2 \leq \delta(S : H)$.

Proof: Suppose that $A = \{a, b\}$ such that $\text{deg}_G(a) = \Delta(G)$ and $\text{deg}_H(b) = \Delta(H)$. Clearly, $A$ is a dominating set in $G + H$. Moreover,

$$|N_{G+H}(a) \setminus A| - |N_{G+H}(a) \cap A| = \text{deg}_G(a) + |V(H)|$$
$$= \Delta(G) + |V(H)|$$
$$\geq \Delta(G) + |V(H)| - 2$$
$$\geq k.$$

and

$$|N_{G+H}(b) \setminus A| - |N_{G+H}(b) \cap A| = \text{deg}_H(b) + |V(G)|$$
$$= \Delta(H) + |V(G)|$$
$$\geq \Delta(G) + |V(H)|$$
$$> \Delta(G) + |V(H)| - 2$$
$$\geq k.$$

Thus, $A$ is a $k$-cost effective dominating set in $G + H$. Accordingly, $\gamma^k_{ce}(G + H) = |S| \leq 2$. Suppose that $|S| = 1$. Then $\gamma(G) = 1$ or $\gamma(H) = 1$, which is
a contradiction to that fact that \( \min \{ \gamma(G), \gamma(H) \} \geq 2 \). Therefore, \( \gamma^*_c(G + H) = 2 \). Since \( S \) is a \( \gamma^*_c \)-set in \( G + H \), \( |S| = 2 \).

Clearly, \( |V(G) \cap S| = 1 \) and \( |V(H) \cap S| = 1 \). Thus, Property (i) holds.

Suppose that \( S \subseteq V(G) \). Since \( S \) is a dominating set in \( G + H \), \( S \) is a dominating set in \( G \). Now, \( \gamma(G) \geq 2 \), so \( S \) is a minimum dominating set in \( G \), that is, \( S \) is a \( \gamma \)-set in \( G \).

Let \( S = \{ a_1, a_2 \} \subseteq V(G) \). Suppose \( a_1 \) and \( a_2 \) are adjacent in \( S \). Then

\[
|N_{G+H}(a_i) \setminus S| - |N_{G+H}(a_i) \cap S| = (|V(H)| + \deg_G(a_i) - 1) - 1
= |V(H)| + \deg_G(a_i) - 2
\geq |V(H)| + \delta(S : G) - 2
\geq k, \ i = 1, 2.
\]

Thus, \( k - |V(H)| + 2 \leq \delta(S : G) \). Suppose \( a_1 \) and \( a_2 \) are not adjacent in \( S \). Then

\[
|N_{G+H}(a_i) \setminus S| - |N_{G+H}(a_i) \cap S| = |V(H)| + \deg_G(a_i)
\geq |V(H)| + \delta(S : G) - 2
= k, \ i = 1, 2.
\]

Thus, \( k - |V(H)| + 2 \leq \delta(S : G) \). Similarly, \( k - |V(G)| + 2 \leq \delta(S : H) \).

Conversely, suppose that \( S \) satisfies Property (i). Then \( S \) is a \( \gamma \)-set in \( G + H \). Moreover,

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = |V(H)| - 1 + \deg_G(a) - 1
= |V(H)| + \Delta(G) - 2
\geq k.
\]

and

\[
|N_{G+H}(b) \setminus S| - |N_{G+H}(b) \cap S| = |V(G)| - 1 + \deg_H(b) - 1
= |V(G)| + \Delta(H) - 2
= |V(H)| + \Delta(G) - 2
\geq k.
\]

Thus, \( S \) is a \( k \)-cost effective dominating set in \( G + H \). Hence, \( S \) is a \( \gamma^*_c \)-set in \( G + H \).

Suppose that \( S \) satisfies Property (ii). Then \( S \) is a \( \gamma \)-set in \( G + H \). Suppose \( a_1 \) and \( a_2 \) are adjacent in \( S \). Then

\[
|N_{G+H}(a_i) \setminus S| - |N_{G+H}(a_i) \cap S| = (|V(H)| + \deg_G(a_i) - 1) - 1
= |V(H)| + \deg_G(a_i) - 2
\geq |V(H)| + \delta(S : G) - 2
\geq k.
\]
Suppose \(a_1\) and \(a_2\) are not adjacent in \(S\). Then
\[
|N_{G+H}(a_i) \setminus S| - |N_{G+H}(a_i) \cap S| = |V(H)| + \deg_G(a_i)
> |V(H)| + \deg_G(a_i) - 2
\geq |V(H)| + \delta(S : G) - 2
= k.
\]

Thus, \(S\) is a \(k\)-cost effective dominating set in \(G + H\). Suppose that a singleton set is a dominating set in \(G + H\). Then \(\gamma(G) = 1\) or \(\gamma(H) = 1\), which a contradiction to the fact that \(\min\{\gamma(G), \gamma(H)\} \geq 2\). Hence, \(S\) is a \(\gamma^k_{ce}\)-set in \(G + H\). Similarly, if \(S\) satisfies Property (iii), then \(S\) is a \(\gamma^k_{ce}\)-set in \(G + H\).

Therefore, \(S\) is a \(\gamma^k_{ce}\)-set in \(G + H\).

\[\Box\]

**Theorem 4.** Let \(G\) and \(H\) be connected graphs such that \(\min\{\gamma(G), \gamma(H)\} \geq 2\) and \(k = \Delta(G) + |V(H)| - 1\). Then \(S\) is a \(k\)-cost effective dominating set in \(G + H\) if and only if one of the following holds:

(i) \(S\) is an independent dominating set in \(G\) such that \(\delta(S : G) \geq \Delta(G) - 1\);

(ii) \(S\) is a dominating set in \(H\) such that \(0 \leq r_H(a) + 2|N_H(a) \cap S| - t \leq 1\), where
\[
r_H(a) = \Delta(H) - \deg_H(a) \quad \text{and} \quad t = \Delta(H) + |V(G)| - \Delta(G) - |V(H)|,
\]
and
\[
\deg_H(a) + |V(G)| - 2|N_H(a) \cap S| = \Delta(G) + |V(H)| - 1.
\]

**Proof:** Suppose that \(S\) is a \(k\)-cost effective dominating set in \(G + H\). Consider the following cases:

Case 1: \(V(G) \cap S \neq \emptyset\) and \(V(H) \cap S \neq \emptyset\).

Let \(a \in V(G) \cap S\). Then
\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| \leq \Delta(G) - 1 + |V(H)| - 1
< \Delta(G) + |V(H)| - 1
= k,
\]
a contradiction. Thus, this case is not possible.

Case 2: \(S \subseteq V(G)\).

Suppose \(S\) is not an independent dominating set \(G\). Let \(a \in S\). Then there exists \(a' \in S\) such that \(d_G(a, a') = 1\). Now
\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| \leq \Delta(G) - 1 + |V(H)| - 1
< \Delta(G) + |V(H)| - 1
= k,
\]
a contradiction. Thus, in this case \(S\) is an independent dominating set in \(G\). Let \(r_G(a) = \Delta(H) - \deg_G(a)\). Now, \(S\) is a \(k\)-cost effective dominating set in \(G + H\), so
\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = \deg_G(a) + |V(H)|
\]
Hence, \( r_G(a) \leq 1 \) and \( \text{deg}_G(a) \geq \Delta(G) - 1 \) for all \( a \in S \). Hence, \( \delta(S : G) \geq \Delta(G) - 1 \).

Case 3: \( S \subseteq V(H) \).

Since \( S \) is a \( k \)-cost effective dominating set in \( G + H \), \( S \) is a dominating set in \( H \). Let \( a \in S \) and \( r_H(a) = \Delta(H) - \text{deg}_H(a) \), and \( t = \Delta(H) + |V(G)| - \Delta(G) - |V(H)| \). Then

\[
\begin{align*}
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &= \text{deg}_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S| \\
&= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)| \\
&= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S| \\
&= \Delta(G) + |V(H)| - (r_H(a) + 2|N_H(a) \cap S| - t).
\end{align*}
\]

Thus, \( 0 \leq r_H(a) + 2|N_H(a) \cap S| - t \leq 1 \). Hence, \( \text{deg}_H(a) + |V(G)| - 2|N_H(a) \cap S| = \Delta(G) + |V(H)| - 1 \).

Conversely, suppose that \( S \) satisfies Property (i). Then \( S \) is a dominating set in \( G + H \). Let \( a \in S \). Then

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = \text{deg}_G(a) + |V(H)| \\
= \Delta(G) - 1 + |V(H)| \\
= k.
\]

Hence, \( S \) is a \( k \)-cost effective dominating set in \( G + H \).

Suppose that \( S \) satisfies Property (ii). Then \( S \) is a dominating set in \( G + H \). Now,

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = \text{deg}_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S| \\
= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)| \\
= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S| \\
= \Delta(G) + |V(H)| - (r_H(a) + 2|N_H(a) \cap S| - t).
\]

If \( r_H(a) + 2|N_H(a) \cap S| - t = 0 \), then

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = \text{deg}_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S| \\
= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)| \\
= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S| \\
= \Delta(G) + |V(H)| - (r_H(a) + 2|N_H(a) \cap S| - t) \\
= \Delta(G) + |V(H)| - 1 \\
= k.
\]

Hence, \( S \) is a \( k \)-cost effective dominating set in \( G + H \). If

\[
r_H(a) + 2|N_H(a) \cap S| - t = 1
\]
Hence, $S$ is a $k$-cost effective dominating set in $G + H$.

Therefore, $S$ is a $k$-cost effective dominating set in $G + H$. 

**Theorem 5.** Let $G$ and $H$ be connected graphs such that

$$\min\{\gamma(G), \gamma(H)\} \geq 2$$

and $k = \Delta(G) + |V(H)|$. Then $S$ is a $k$-cost effective dominating set in $G + H$ if and only if one of the following holds:

(i) $S$ is an independent dominating set in $G$ such that $\delta(S : G) = \Delta(G)$;

(ii) $S$ is a dominating set in $H$ such that $\deg_H(a) + |V(G)| = 2|N_H(a) \cap S| + \Delta(G) + |V(H)|$

and $r_H(a) + 2|N_H(a) \cap S| - t = 0$, where $r_H(a) = \Delta(H) - \deg_H(a)$,

$t = \Delta(H) + |V(G)| - \Delta(G) - |V(H)|$.

**Proof:** Suppose that $S$ is a $k$-cost effective dominating set in $G + H$. Consider the following cases:

Case 1: $V(G) \cap S \neq \emptyset$ and $V(H) \cap S \neq \emptyset$.

Let $a \in V(G) \cap S$. Then

$$|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = \deg_G(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S|$$

$$= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)|$$

$$= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S|$$

$$= \Delta(G) + |V(H)| - (r_H(a) + 2|N_H(a) \cap S| - t)$$

$$= \Delta(G) + |V(H)| - 1$$

$$= k.$$

a contradiction. Thus, this case is not possible.

Case 2: $S \subseteq V(G)$.

Suppose $S$ is not an independent dominating set $G$. Let $a \in S$. Then there exists $a' \in S$ such that $d_G(a, a') = 1$. Now

$$|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| \leq \Delta(G) - 1 + |V(H)| - 1$$

$$< \Delta(G) + |V(H)|$$

$$= k,$$

a contradiction. Thus, in this case $S$ is an independent dominating set in $G$. Now,

$$|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = \deg_G(a) + |V(H)|$$

$$= \Delta(G) + |V(H)|$$
Thus, $\deg_G(a) = \Delta(G) \forall a \in S$. Hence, $\delta(S : G) = \Delta(G)$.

Case 3: $S \subseteq V(H)$.

Since $S$ is a $k$-cost effective dominating set in $G + H$, $S$ is a dominating set in $H$. Let $a \in S$ and $r_H(a) = \Delta(H) - \deg_H(a)$, and $t = \Delta(H) + |V(G)| - \Delta(G) - |V(H)|$. Then

$$|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = \deg_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S|$$

$$= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)|$$

$$= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S|$$

$$= \Delta(G) + |V(H)| - (r_H(a) + 2|N_H(a) \cap S| - t).$$

Thus, $r_H(a) + 2|N_H(a) \cap S| - t = 0$. Hence, $\deg_H(a) + |V(G)| = 2|N_H(a) \cap S| + \Delta(G) + |V(H)|$.

Conversely, suppose that $S$ satisfies Property (i). Then $S$ is a dominating set in $G + H$. Let $a \in S$. Then

$$|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = \deg_G(a) + |V(H)|$$

$$= \delta(S : G) + |V(H)|$$

$$= \Delta(G) + |V(H)|$$

$$= k.$$ 

Hence, $S$ is a $k$-cost effective dominating set in $G + H$.

Suppose that $S$ satisfies Property (ii). Then $S$ is a dominating set in $G + H$. Now,

$$|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = \deg_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S|$$

$$= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)|$$

$$= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S|$$

$$= \Delta(G) + |V(H)| + \Delta(H) + |V(G)| - \Delta(G)$$

$$- |V(H)| - \Delta(H) + \deg_H(a) - 2|N_H(a) \cap S|$$

$$= |V(G)| + \deg_H(a) - 2|N_H(a) \cap S|$$

$$= \Delta(G) + |V(H)|$$

$$= k.$$ 

Thus, $S$ is a $k$-cost effective dominating set in $G + H$.

Therefore, $S$ is a $k$-cost effective dominating set in $G + H$. \qed

**Theorem 6.** Let $G$ and $H$ be connected graphs such that $\min\{\gamma(G), \gamma(H)\} \geq 2$ and $\Delta(G) + |V(H)| + 1 \leq k \leq \Delta(H) + |V(G)|$. Then $S$ is a $k$-cost effective dominating set in $G + H$ if and only if $S$ is a dominating set in $H$ such that $t - r_H(a) - 2|N_H(a) \cap S| \geq p$, where $1 \leq p \leq t$ and $t = \Delta(H) + |V(G)| - \Delta(G) - |V(H)|$, and $r_H(a) = \Delta(H) - \deg_H(a)$ and $\deg_H(a) + |V(G)| \geq p + 2|N_H(a) \cap S| + \Delta(G) + |V(H)|$. 

Proof: Suppose that $S$ is a $k$-cost effective dominating set in $G + H$. Consider the following cases:

Case 1: $V(G) \cap S \neq \emptyset$ and $V(H) \cap S \neq \emptyset$.

Let $a \in V(G) \cap S$. Then

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| \leq \Delta(G) - 1 + |V(H)| - 1
\]

\[
< \Delta(G) + |V(H)| + 1
\]

\[
\leq k,
\]

a contradiction. Thus, in this case is not possible.

Case 2: $S \subseteq V(G)$.

Suppose $S$ is not an independent dominating set G. Let $a \in S$. Then there exists $a' \in S$ such that $d_G(a, a') = 1$. Now

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| \leq \Delta(G) - 1 + |V(H)| - 1
\]

\[
< \Delta(G) + |V(H)| + 1
\]

\[
\leq k,
\]

a contradiction. Thus, in this case $S$ is an independent dominating set in $G$. Thus,

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = deg_G(a) + |V(H)|
\]

\[
= \Delta(G) - r_G(a) + |V(H)|
\]

\[
\leq \Delta(G) + |V(H)| + 1
\]

\[
\geq k,
\]

a contradiction. Thus, in this case is not possible.

Case 3: $S \subseteq V(H)$.

Since $S$ is a $k$-cost effective dominating set in $G + H$, $S$ is a dominating set in $H$. Let $a \in S$, $r_H(a) = \Delta(H) - deg_H(a)$ and $1 \leq p \leq t$, where $t = \Delta(H) + |V(G)| - \Delta(G) - |V(H)|$. Then

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = deg_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S|
\]

\[
= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)|
\]

\[
= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S|.
\]

Thus, $t - r_H(a) - 2|N_H(a) \cap S| \geq p$. Hence, $deg_H(a) + |V(G)| \geq p + 2|N_H(a) \cap S| + \Delta(G) + |V(H)|$.

Conversely, suppose that $S$ is a dominating set in $H$ such that $t - r_H(a) - 2|N_H(a) \cap S| \geq p$, where $1 \leq p \leq t$ and $t = \Delta(H) + |V(G)| - \Delta(G) - |V(H)|$, and $r_H(a) = \Delta(H) - deg_H(a)$ and $deg_H(a) + |V(G)| \geq p + 2|N_H(a) \cap S| + \Delta(G) + |V(H)|$. Then

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = deg_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S|
\]
Let \( k \geq \Delta(G) + |V(G)| + 1 \). Then \( \gamma^k_{ce}(G + H) = \infty \).

**Proof:** Let \( k \geq \Delta(G) + |V(G)| + 1 \). Suppose that there exists a \( k \)-cost effective dominating set \( S \) in \( G + H \). Consider the following cases:

---

**Case 1:** \( V(G) \cap S \neq \emptyset \) and \( V(H) \cap S \neq \emptyset \).

Let \( a \in V(H) \cap S \). Then

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| \leq \Delta(H) - 1 + |V(G)| - 1 \\
\leq \Delta(H) + |V(G)| + 1 \\
= k,
\]

a contradiction. Thus, in this case \( S \) is not an independent dominating set in \( G \). Let \( a \in S \). Then there exists \( a' \in S \) such that \( d_G(a, a') = 1 \). Now

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| \leq \Delta(G) - 1 + |V(H)| - 1 \\
\leq \Delta(H) + |V(G)| + 1 \\
= k,
\]

a contradiction. Thus, in this case \( S \) is an independent dominating set in \( G \). Thus,

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = \text{deg}_G(a) + |V(H)| \\
= \Delta(G) - r_G(a) + |V(H)| \\
= \Delta(H) - r_G(a) + |V(G)| \\
< \Delta(H) + |V(G)| + 1 \\
= k,
\]

a contradiction.

**Case 3:** \( S \subseteq V(H) \). Let \( a \in S \). Then

\[
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = \text{deg}_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S| \\
= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)| \\
= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S| \\
= \Delta(G) + |V(H)| - (r_H(a) + 2|N_H(a) \cap S| - t)
\]
\[
\begin{align*}
\Delta(H) + |V(G)| - (r_H(a) + 2|N_H(a) \cap S| - t) \\
< \Delta(H) + |V(G)| + 1 \\
= k,
\end{align*}
\]
a contradiction. Hence, \(\gamma^k_{\text{ce}}(G + H) = \infty.\)

The next result follows from Theorem 3, Theorem 4, Theorem 5, Theorem 6 and Theorem 7.

**Corollary 4.** Let \(G\) and \(H\) be connected graphs such that \(\gamma(G) \geq 2\), \(\gamma(H) \geq 2\) and \(|V(H)| + \Delta(G) \leq |V(G)| + \Delta(H)\). Then

\[
\gamma^k_{\text{ce}}(G + H) = \begin{cases} 
2, & \text{if } 0 \leq k \leq |V(G)| + \Delta(H) - 2 \\
\min\{\gamma^*(G), \gamma^*(H)\}, & \text{if } |V(G)| + \Delta(H) - 1 \leq k \leq \Delta(G) + |V(H)| \\
\gamma(H), & \text{if } |V(H)| + \Delta(G) + 1 \leq k \leq |V(G)| + \Delta(H), \\
\infty, & \text{if } k \geq |V(G)| + \Delta(H) + 1
\end{cases}
\]

where

\[
\gamma^*_i(G) = \min\{|S| : S \text{ is a } \gamma_i\text{-set in } G \text{ and } \delta(S : G) \geq \Delta(G) - 1\},
\]

\[
\gamma^*(H) = \min\{|S| : S \text{ is a } \gamma\text{-set in } G \text{ and } 0 \leq \Delta(G) + |V(H)| - |V(G)| - \text{deg}_H(a) + 2|N_H(a) \cap S| \leq 1\}, \text{ and}
\]

\[
\gamma(H) = \min\{|S| : S \text{ is a } \gamma\text{-set in } G \text{ and } \text{deg}_H(a) + |V(G)| - |V(H)| - 2|N_H(a) \cap S| \geq p\}.
\]

**Corollary 5.** Let \(G\) and \(H\) be connected graphs such that \(\gamma(G) \geq 2\), \(\gamma(H) \geq 2\) and \(|V(H)| + \Delta(G) \leq |V(G)| + \Delta(H)\). Then \(\eta(G + H) = |V(G)| + \Delta(H)\) and \(\gamma^\eta_{\text{ce}}(G + H) = \gamma(H)\).

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References


