EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 3, 2022, 856-863
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Double Integral involving the Product of the Bessel Function of the First Kind and modified Bessel Function of the Second Kind: Derivation and Evaluation

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Abstract. A double integral whose kernel involves the Bessel functions $K_v(x\beta)$ and $J_v(y\alpha)$ is derived. This integral is expressed in terms of the Hurwitz-Lerch zeta function and evaluated for various values of the parameters involved. Some examples are evaluated and expressed in terms of fundamental constants. All the results in this work are new.

2020 Mathematics Subject Classifications: 30E20, 33-01, 33-03, 33-04, 33-33B

Key Words and Phrases: Bessel functions, double integral, Cauchy integral

1. Introduction

Integrals involving Bessel functions have been studied in the works by Glasser [3], where the study of wave propagation along a coaxial cable was investigated, Temme [7], where the mathematical discussion of the exchange processes, of heat or of matter (as in ion exchange or adsorption), that arise when a fluid flows through the pores or voids along a column containing matter in the solid state, was studied. Throughout these works the authors derive definite integrals involving the Bessel function or the product of Bessel functions for specific orders. in our present paper we will be expanding on the previous formulae by deriving a double integral of the product of Bessel functions over a general order.

In this paper we derive the double definite integral given by

$$\int_0^\infty \int_0^\infty x^{m-1} y^{1-m} K_v(x\beta) J_v(y\alpha) \log^k\left(\frac{ax}{y}\right) dxdy \tag{1}$$

where the parameters $k, a, \alpha, \beta, v, m$ are general complex numbers and $Re(\alpha, \beta, v, m) > 0, Re(v) < Re(m) < 3/2$. This definite integral will be used to derive special cases in

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DOI: https://doi.org/10.29020/nybg.ejpam.v15i3.4239

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terms of special functions and fundamental constants. The derivations follow the method used by us in [6]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \tag{2}$$

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of x and y, then take a definite double integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (2) by another function of x and y and take the infinite sums of both sides such that the contour integral of both equations are the same.

2. Definite Integral of the Contour Integral

We use the method in [6]. The variable of integration in the contour integral is $t = w + m$. The cut and contour are in the first quadrant of the complex t-plane. The cut approaches the origin from the interior of the first quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form the double integral by replacing y by $\log\left(\frac{ax}{y}\right)$ and multiplying by $x^{m-1}y^{1-m}K_v(x\beta)J_v(y\alpha)$ then taking the definite integral with respect to $x \in [0,\infty)$ and $y \in [0,\infty)$ to obtain

$$\begin{aligned}
&\frac{1}{\Gamma(k+1)}\int_0^\infty\int_0^\infty x^{m-1}y^{1-m}K_v(x\beta)J_v(y\alpha)\log^k\left(\frac{ax}{y}\right)dxdy \\
&= \frac{1}{2\pi i}\int_0^\infty\int_0^\infty\int_C a^w w^{-k-1}x^{m+w-1}y^{-m-w+1}K_v(x\beta)J_v(y\alpha)dwdxdy \\
&= \frac{1}{2\pi i}\int_C\int_0^\infty\int_0^\infty a^w w^{-k-1}x^{m+w-1}y^{-m-w+1}K_v(x\beta)J_v(y\alpha)dxdydw \\
&= \frac{1}{2\pi i}\int_C \frac{1}{2}\pi a^w w^{-k-1}\alpha^{m+w-2}\beta^{-m-w}\csc\left(\frac{1}{2}\pi(m-v+w)\right)dw
\end{aligned} \tag{3}$$

from equations (3.10.1.2) and (3.14.3) in [1] where $Re(\alpha) > 0, |Re(v)| < Re(w+m-v) < 3/2$ and using the reflection formula (8.334.3) in [4] for the Gamma function. We are able to switch the order of integration over x and y using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C} \times [0,\infty) \times [0,\infty)$

3. The Hurwitz-Lerch zeta Function and Infinite Sum of the Contour Integral

In this section we use Equation (2) to derive the contour integral representations for the Hurwitz-Lerch zeta function.

3.1. The Hurwitz-Lerch zeta Function

The Hurwitz-Lerch zeta function (25.14) in [2] has a series representation given by

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^{n} \tag{4}$$

where $|z|< 1, v\neq 0,-1,..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{e^{t}-z}dt \tag{5}$$

where $Re(v)>0$, and either $|z|\leq 1, z\neq 1, Re(s)>0$, or $z=1, Re(s)>1$.

3.2. Infinite sum of the Contour Integral

Using equation (2) and replacing y by $\log(a)+\log(\alpha)-\log(\beta)+\frac{1}{2}i\pi(2y+1)$ then multiplying both sides by $-i\pi\alpha^{m-2}\beta^{-m}e^{\frac{1}{2}i\pi(2y+1)(m-v)}$ taking the infinite sum over $y\in[0,\infty)$ and simplifying in terms of the Hurwitz-Lerch zeta function we obtain

$$\begin{aligned}
&-\frac{1}{\Gamma(k+1)}i\pi^{k+1}\alpha^{m-2}\beta^{-m}e^{\frac{1}{2}i\pi(k+m-v)}\\
&\quad\Phi\left(e^{i\pi(m-v)},-k,\frac{-2i\log(a)-2i\log(\alpha)+2i\log(\beta)+\pi}{2\pi}\right)\\
&\quad=-\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C i\pi w^{-k-1}\alpha^{m-2}\beta^{-m}\exp\Big(w(\log(a)+\log(\alpha)-\log(\beta))\\
&\qquad\qquad+\frac{1}{2}i\pi(2y+1)(m-v+w)\Big)dw \qquad (6)\\
&\quad=-\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}i\pi w^{-k-1}\alpha^{m-2}\beta^{-m}\exp\Big(w(\log(a)+\log(\alpha)-\log(\beta))\\
&\qquad\qquad+\frac{1}{2}i\pi(2y+1)(m-v+w)\Big)dw\\
&\quad=\frac{1}{2\pi i}\int_C\frac{1}{2}\pi a^{w}w^{-k-1}\alpha^{m+w-2}\beta^{-m-w}\csc\left(\frac{1}{2}\pi(m-v+w)\right)dw
\end{aligned}$$

from equation (1.232.2) in [4] where $Im\left(\frac{1}{2}\pi(m-v+w)\right)>0$ in order for the sum to converge.

4. Definite Integral in terms of the Hurwitz-Lerch zeta Function

Theorem 1. *For all* $k, a \in \mathbb{C}, Re(\alpha, \beta, v, m) > 0, Re(v) < Re(m) < 3/2,$

$$\int_0^\infty \int_0^\infty x^{m-1} y^{1-m} K_v(x\beta) J_v(y\alpha) \log^k\left(\frac{ax}{y}\right) dxdy$$
$$= -i\pi^{k+1}\alpha^{m-2}\beta^{-m} e^{\frac{1}{2}i\pi(k+m-v)} \tag{7}$$
$$\Phi\left(e^{i\pi(m-v)}, -k, \frac{-2i\log(a) - 2i\log(\alpha) + 2i\log(\beta) + \pi}{2\pi}\right)$$

Proof. The right-hand sides of relations (3) and (6) are identical; hence, the left-hand sides of the same are identical too. Simplifying with the Gamma function yields the desired conclusion.

Example 1. *The degenerate case.*

$$\int_0^\infty \int_0^\infty x^{m-1} y^{1-m} K_v(x\beta) J_v(y\alpha) dxdy = \frac{1}{2}\pi\alpha^{m-2}\beta^{-m}\csc\left(\frac{1}{2}\pi(m-v)\right) \tag{8}$$

Proof. Use equation (7) and set $k = 0$ and simplify using entry (2) in Table below (64:12:7) in [5].

Example 2. *The Hurwitz zeta function* $\zeta(s, v)$

$$\int_0^\infty \int_0^\infty \frac{\sqrt{x}e^{\beta(-x)}\sin(\alpha y)\log^k\left(\frac{ax}{y}\right)}{\sqrt{y}\sqrt{\beta x}\sqrt{\alpha y}} dxdy$$
$$= -\frac{1}{\sqrt{\alpha}\beta^{3/2}} i e^{\frac{1}{2}i\pi(k+1)}\pi^{k+1}\left(2^k\zeta\left(-k, \frac{-2i\log(a) - 2i\log(\alpha) + 2i\log(\beta) + \pi}{4\pi}\right)\right. \tag{9}$$
$$\left. - 2^k\zeta\left(-k, \frac{1}{2}\left(\frac{-2i\log(a) - 2i\log(\alpha) + 2i\log(\beta) + \pi}{2\pi} + 1\right)\right)\right)$$

Proof. Use equation (7) and set $m = 3/2, v = 1/2$ and simplify using entry (4) in Table below (64:12:7) in [5].

Example 3.

$$\int_0^\infty \int_0^\infty \frac{e^{-x}\sin(y)\log\left(\frac{x}{y}\right)}{y\left(\log^2\left(\frac{x}{y}\right) + \pi^2\right)} dxdy = 0 \tag{10}$$

and

$$\int_0^\infty \int_0^\infty \frac{e^{-x}\sin(y)}{y\left(\log^2\left(\frac{x}{y}\right) + \pi^2\right)} dxdy = \frac{2}{\pi} - \frac{1}{2} \tag{11}$$

Proof. Use equation (9) apply l'Hopital's rule as $k \to -1$ and set $a = -1, \alpha = \beta = 1$ rationalize the denominator and simplify using entry (2) in Table below (64:7) in [5].

Example 4.

$$\int_0^\infty \int_0^\infty \frac{\sqrt{x} K_{\frac{1}{4}}(x) J_{\frac{1}{4}}(y) \log\left(\frac{x}{y}\right)}{\sqrt{y}\left(\log^2\left(\frac{x}{y}\right) + \pi^2\right)} dxdy = \frac{1}{4}\left(\sqrt{2}\pi - 8\sin\left(\frac{\pi}{8}\right) - 2\sqrt{2}\tanh^{-1}\left(\sin\left(\frac{\pi}{8}\right)\right)\right) \tag{12}$$

and

$$\int_0^\infty \int_0^\infty \frac{\sqrt{x} K_{\frac{1}{4}}(x) J_{\frac{1}{4}}(y)}{\sqrt{y}\left(\log^2\left(\frac{x}{y}\right) + \pi^2\right)} dxdy = \frac{8\cos\left(\frac{\pi}{8}\right) - \sqrt{2}\left(\pi + 2\tanh^{-1}\left(\sin\left(\frac{\pi}{8}\right)\right)\right)}{4\pi} \tag{13}$$

Proof. Use equation (7) and set $k = -1, m = 3/2, v = 1/4, \alpha = \beta = 1, a = -1$ and simplify using entry (3) in Table below (64:12:7) in [5].

Example 5.

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{x^{m-1} y^{1-m} K_v(x\beta) J_v(y\alpha)}{\log\left(-\frac{\beta x}{\alpha y}\right)} dxdy \\ &= 2i\alpha^{m-2}\beta^{-m} e^{-\frac{1}{2}i\pi(2m-2v+1)}\left(e^{\frac{1}{2}i\pi(m-v)} - \tanh^{-1}\left(e^{\frac{1}{2}i\pi(m-v)}\right)\right) \end{aligned} \tag{14}$$

Proof. Use equation (7) and set $k = -1, m = 3/2, v = 1/4, a = \beta/\alpha$ and simplify using entry (3) in Table below (64:12:7) in [5].

Example 6. *The Polylogarithm function $Li_k(z)$,*

$$\int_0^\infty \int_0^\infty x^{m-1} y^{1-m} K_v(x) J_v(y) \log^k\left(\frac{ix}{y}\right) dxdy = -i\pi^{k+1} e^{\frac{1}{2}i\pi(k-m+v)} Li_{-k}\left(e^{i\pi(m-v)}\right) \tag{15}$$

Proof. Use equation (7) and set $a = i, \beta = \alpha = 1$ and simplify using equation (64:12:2) in [5].

Example 7. *The Polylogarithm function $Li_2(z)$,*

$$\int_0^\infty \int_0^\infty \frac{x^{m-1} y^{1-m} K_v(x) J_v(y)}{\log^2\left(\frac{ix}{y}\right)} dxdy = -\frac{ie^{-\frac{1}{2}i\pi(m-v+2)} Li_2\left(e^{i\pi(m-v)}\right)}{\pi} \tag{16}$$

Proof. Use equation (7) and set $k = -2, a = i, \beta = \alpha = 1$ and simplify using equation (64:12:2) in [5].

Example 8. *Catalan's constant G,*

$$\int_0^\infty \int_0^\infty \frac{e^{-x}\sqrt{x}\sin(y)\left(\pi^2 - 4\log^2\left(\frac{x}{y}\right)\right)}{y^{3/2}\left(4\log^2\left(\frac{x}{y}\right)+\pi^2\right)^2} dxdy = \frac{48G+\pi^2}{192\sqrt{2}\pi} \tag{17}$$

and

$$\int_0^\infty \int_0^\infty \frac{e^{-x}\sqrt{x}\sin(y)\log\left(\frac{x}{y}\right)}{y^{3/2}\left(4\log^2\left(\frac{x}{y}\right)+\pi^2\right)^2} dxdy = -\frac{\pi^2-48G}{768\sqrt{2}\pi^2} \tag{18}$$

Proof. Use equation (16) and set $m = 2, v = 1/2$ and simplify.

Example 9.

$$\int_0^\infty \int_0^\infty \frac{y^{-m-p+1}K_v(x)J_v(y)\left(y^m x^p - x^m y^p\right)}{x\log\left(\frac{x}{y}\right)} dxdy$$
$$= 2\left(\tanh^{-1}\left(e^{\frac{1}{2}i\pi(m-v)}\right) - \tanh^{-1}\left(e^{\frac{1}{2}i\pi(p-v)}\right)\right) \tag{19}$$

Proof. Use equation (7) and form a second equation by replacing $m \to p$ and taking their difference. Next set $k = -1, a = 1, \alpha = \beta = 1$ and simplify using entry (3) in Table below (64:12:7) in [5].

Example 10.

$$\int_0^\infty \int_0^\infty \frac{\left(x-\sqrt{x}\sqrt{y}\right)K_{\frac{1}{3}}(x)J_{\frac{1}{3}}(y)}{y\log\left(\frac{x}{y}\right)} dxdy = 2\tanh^{-1}\left(1-\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{2}}\right) \tag{20}$$

Proof. Use equation (19) set $v = 1/3, m = 3/2, p = 2$ and simplify.

Example 11.

$$\int_0^\infty \int_0^\infty \frac{x^{2/5}\left(\sqrt[10]{y}-\sqrt[10]{x}\right)K_0(x)J_0(y)}{\sqrt{y}\log\left(\frac{x}{y}\right)} dxdy =$$
$$-\tanh^{-1}\left(\frac{1}{29}\sqrt{2\left(254-31\sqrt{5}-2\sqrt{4505+1109\sqrt{5}}\right)}\right) \tag{21}$$

Proof. Use equation (19) set $v = 0, m = 3/2, p = 7/5$ and simplify.

Example 12.

$$\int_0^\infty \int_0^\infty \frac{\left(\sqrt{x}-x^{2/5}\sqrt[10]{y}\right)K_{\frac{1}{5}}(x)J_{\frac{1}{5}}(y)}{\sqrt{y}\log\left(\frac{x}{y}\right)}dxdy = \frac{1}{2}\left(\log\left(5-2\sqrt{5}\right)+4\tanh^{-1}\left(\tan\left(\frac{3\pi}{40}\right)\right)\right) \tag{22}$$

Proof. Use equation (19) set $v = 1/5, m = 3/2, p = 7/5$ and simplify.

Example 13.

$$\int_0^\infty \int_0^\infty \frac{\sqrt[5]{x}\left(y^{3/10}-x^{3/10}\right)K_{\frac{2}{9}}(x)J_{\frac{2}{9}}(y)}{\sqrt{y}\log\left(\frac{x}{y}\right)}dxdy = -\tanh^{-1}\left(\sin\left(\frac{\pi}{90}\right)\right) - 2\tanh^{-1}\left(\tan\left(\frac{5\pi}{72}\right)\right) \tag{23}$$

Proof. Use equation (19) set $v = 2/9, m = 3/2, p = 6/5$ and simplify.

Example 14.

$$\begin{aligned}\int_0^\infty \int_0^\infty & \frac{\left(x-\sqrt{x}\sqrt{y}\right)K_{\frac{1}{3}}\left(4\sqrt{3}x\right)J_{\frac{1}{3}}\left(2\sqrt{2}y\right)}{y\log\left(\frac{x}{y}\right)}dxdy \\ &= \frac{1}{48}\left((-1)^{7/12}\sqrt[4]{6}\Phi\left(-\sqrt[6]{-1},1,\frac{\pi+i\log(6)}{2\pi}\right)\right. \\ &\left. -(-1)^{5/6}\Phi\left(-(-1)^{2/3},1,\frac{\pi+i\log(6)}{2\pi}\right)\right)\end{aligned} \tag{24}$$

Proof. Use equation (7) and form a second equation by replacing $m \to p$ and taking their difference. Next set $k = -1, a = 1, \alpha = \sqrt{2}, \beta = \sqrt{3}, m = 3/2, p = 2, v = 1/7$ and simplify.

Example 15.

$$\int_0^\infty \int_0^\infty \frac{\sqrt{x}K_{\frac{1}{3}}(3x)J_{\frac{1}{3}}(2y)}{\sqrt{y}\log\left(-\frac{2x}{y}\right)}dxdy = -\frac{(-1)^{7/12}\Phi\left(-\sqrt[6]{-1},1,\frac{3\pi-4i\log(2)+2i\log(3)}{2\pi}\right)}{3\sqrt{6}} \tag{25}$$

Proof. Use equation (7) set $k = -1, a = -2, m = 3/2, v = 1/3, , \alpha = 2, \beta = 3$ and simplify.

Example 16.

$$\int_0^\infty \int_0^\infty \frac{\sqrt{x} K_{\frac{1}{\sqrt{5}}}\left(\frac{5x}{\sqrt{11}}\right) J_{\frac{1}{\sqrt{5}}}\left(\frac{y}{\sqrt{7}}\right)}{\sqrt{y}\sqrt{\log\left(-\frac{3x}{y}\right)}} dxdy$$
$$= \frac{1}{5}\sqrt[4]{7}11^{3/4} e^{-\frac{i\pi}{2\sqrt{5}}}\sqrt{\frac{\pi}{5}}\Phi\left(-ie^{-\frac{i\pi}{\sqrt{5}}}, \frac{1}{2}, \frac{3}{2} + \frac{i\log\left(\frac{175}{99}\right)}{2\pi}\right) \tag{26}$$

Proof. Use equation (7) set $k = -1/2, a = -3, m = 3/2, v = 1/\sqrt{5},, \alpha = 1/\sqrt{7}, \beta = 5/\sqrt{11}$ and simplify.

5. Discussion

In this paper, we have presented a novel method for deriving a new double integral involving the product of Bessel functions along with some interesting definite integrals using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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