On $(\Lambda, p)$-closed sets and the related notions in topological spaces

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Abstract. This article deals with the concepts of $\Lambda_p$-sets and $(\Lambda, p)$-closed sets which are defined by utilizing the notions of preopen sets and preclosed sets. We also introduce and characterize some new low separation axioms. Characterizations of $\Lambda_p$-$R_0$ spaces are given. Moreover, we introduce the concept of weakly $(\Lambda, p)$-continuous functions. In particular, several characterizations of weakly $(\Lambda, p)$-continuous functions are established.

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1. Introduction

In 1982, Mashhour et al. [16] introduced the notion of preopen sets which is also known under the name of locally dense sets [7] in the literature. Since then, this notion received wide usage in general topology. Kar and Bhattachryya [12] introduced new separation axioms pre-$T_0$, pre-$T_1$ and pre-$T_2$ by using preopen sets due to Mashhour et al. [16]. Caldas [3] and Jafari [11] introduced independently the notions of $p$-$D$-sets and a separation axiom $p$-$D_1$ which is strictly between pre-$T_0$ and pre-$T_1$. Caldas et al. [4] introduced two new classes of topological spaces called pre-$R_0$ and pre-$R_1$ spaces in terms of concept of preopen sets and investigated some of their fundamental properties. Mashhour et al. [15] introduced and studied the concept of supra topological spaces by dropping a finite intersection condition of topological spaces. El-Shafei et al. [9] defined some concepts on supra topological spaces using supra preopen sets and investigated main properties. Al-shami et al. [2] introduced and investigated new separation axioms, namely supra semi $T_i$-spaces ($i = 0, 1, 2, 3, 4$). In [1], the present author introduce the version of complete Hausdorffness and complete regularity on supra topological spaces and discussed their

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fundamental properties. Cammaroto and Noiri [6] defined $\Lambda_m$-sets and generalized $\Lambda_m$-sets in an $m$-spaces $(X, m)$ which is equivalent to a generalized topological spaces [14] and investigated properties of several low separation axioms of topologies constructed by the families of these sets. Ganster et al. [10] introduced the notions of a pre-$\Lambda$-set and a pre-$V$-set in a topological space and studied the fundamental properties of pre-$\Lambda$-sets and pre-$V$-sets. Caldas et al. [5] introduced and studied two new weak separation axioms called $\Lambda_\theta R_0$ and $\Lambda_\theta R_1$ spaces by using the notions of $(\Lambda, \theta)$-open sets and $(\Lambda, \theta)$-closure operators. The concept of weak continuity due to Levine [13] is one of the most important weak forms of continuity in topological spaces. Rose [18] introduced the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. Popa and Noiri [17] introduced the concept of weakly $(\tau, m)$-continuous functions as functions from a topological space into a set satisfying some minimal conditions and investigated several characterizations of such functions. The paper is organized as follows. In section 3, we obtain fundamental properties of $\Lambda_p$-sets and investigate low separation axioms of an Alexandorff spaces $(X, \Lambda_p)$. In section 4, we introduce the concept of $(\Lambda, p)$-closed sets and investigate properties of several low separation axioms of topologies constructed by the families of these sets. In section 5, we investigate some characterizations of $\Lambda_p R_0$ spaces. In the last section, we introduce the concept of weakly $(\Lambda, p)$-continuous functions and investigate several characterizations of such functions.

2. Preliminaries

Throughout the present paper, spaces $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of a topological space $(X, \tau)$, $\text{Cl}(A)$ and $\text{Int}(A)$ represent the closure and the interior of $A$, respectively. A subset $A$ of a topological space $(X, \tau)$ is said to be preopen [16] (resp. preclosed [16]) if $A \subseteq \text{Int}\{\text{Cl}(A)\}$ (resp. $\text{Cl}\{\text{Int}(A)\} \subseteq A$). By $PO(X, \tau)$ and $PC(X, \tau)$, we denote the collection of all preopen sets and the collection of all preclosed sets of a topological space $(X, \tau)$, respectively. The intersection of all preclosed sets containig $A$ is called the preclosure [8] of $A$ and is denoted by $p\text{Cl}(A)$.

**Definition 1.** A topological space $(X, \tau)$ is said to be:

1. pre-$T_0$ [12] if, for each pair of distinct points of $X$, there exists a preopen set containing one of the points but not the other;
2. pre-$T_1$ [12] if, for each pair of distinct points $x$ and $y$ of $X$, there exists a pair of preopen sets one containing $x$ but not $y$ and the other containing $y$ but not $x$;
3. pre-$R_0$ [4] if every preopen set contains the preclosure of each of its singletons.

**Definition 2.** Let $A$ be a subset of a topological space $(X, \tau)$. A subset $\Lambda_p(A)$ [10] is defined as follows: $\Lambda_p(A) = \cap\{O \in PO(X, \tau) | A \subseteq O\}$. 
Lemma 1. [10] For subsets $A$, $B$ and $A_i (i \in I)$ of a topological space $(X, \tau)$, the following properties hold:

1. $A \subseteq \Lambda_p(A)$.
2. If $A \subseteq B$, then $\Lambda_p(A) \subseteq \Lambda_p(B)$.
3. $\Lambda_p(\Lambda_p(A)) = \Lambda_p(A)$.
4. $\Lambda_p(\bigcap\{A_i | i \in I\}) \subseteq \bigcap\{\Lambda_p(A_i) | i \in I\}$.
5. $\Lambda_p(\bigcup\{A_i | i \in I\}) = \bigcup\{\Lambda_p(A_i) | i \in I\}$.

3. $\Lambda_p$-sets and a topological space $(X, \Lambda_p)$

In this section, we obtain fundamental properties of $\Lambda_p$-sets and investigate low separation axioms of an Alexandorff space $(X, \Lambda_p)$.

Definition 3. A subset $A$ of a topological space $(X, \tau)$ is called a $\Lambda_p$-set (pre-$\Lambda_p$-set [10]) if $A = \Lambda_p(A)$. The family of all $\Lambda_p$-sets of $(X, \tau)$ is denoted by $\Lambda_p(X, \tau)$ (or simply $\Lambda_p$).

Lemma 2. For a subset $A$ of a topological space $(X, \tau)$, the following properties hold:

1. $\Lambda_p(A)$ is a $\Lambda_p$-set.
2. If $A$ is preopen, then $A$ is a $\Lambda_p$-set.

Proof. This follows readily from Lemma 1.

Lemma 3. [10] For subsets $A$ and $A_i (i \in I)$ of a topological space $(X, \tau)$, the following properties hold:

1. $\emptyset$ and $X$ are pre-$\Lambda$-sets.
2. Every union of pre-$\Lambda$-sets is a pre-$\Lambda$-set.
3. Every intersection of pre-$\Lambda$-sets is a pre-$\Lambda$-set.

Theorem 1. For a topological space $(X, \tau)$, the pair $(X, \Lambda_p)$ is an Alexandroff space.

Proof. This is an immediate consequence of Lemma 3.

Theorem 2. Let $(X, \tau)$ be a topological space. Then, $\Lambda_p = \Lambda_{\Lambda_p}$.

Proof. By Lemma 2, $PO(X, \tau) \subseteq \Lambda_p$. Let $A$ be any subset of $X$. Then,

$\Lambda_{\Lambda_p} = \cap\{U | A \subseteq U, U \in \Lambda_p\} \subseteq \{U | A \subseteq U, U \in PO(X, \tau)\} = \Lambda_p(A)$.

Thus, $\Lambda_{\Lambda_p}(A) \subseteq \Lambda_p(A)$. Now, we suppose that $x \notin \Lambda_{\Lambda_p}(A)$. Then, there exists $U \in \Lambda_p$ such that $A \subseteq U$ and $x \notin U$. Since $x \notin U$, there exists $V \in PO(X, \tau)$ such that $U \subseteq V$ and $x \notin V$. Therefore, $x \notin \Lambda_p(A)$. This shows that $\Lambda_{\Lambda_p}(A) \supseteq \Lambda_p(A)$ and hence $\Lambda_p(A) = \Lambda_{\Lambda_p}(A)$. 

Theorem 3. A topological space \((X, \tau)\) is pre-\(R_0\) if and only if the topological space \((X, \Lambda_p)\) is \(R_0\).

Proof. Let \(V \in \Lambda_p\) and let \(x \in V\). Then, \(x \in \Lambda_p(V) = \cap\{U \mid V \subseteq U, U \in PO(X, \tau)\}\) and \(x \in U\) for any \(U \in PO(X, \tau)\) containing \(V\). Since \((X, \tau)\) is pre-\(R_0\), \(pCl(\{x\}) \subseteq U\) for every \(U \in PO(X, \tau)\) containing \(V\). Thus,

\[
pCl(\{x\}) \subseteq \cap\{U \mid V \subseteq U, U \in PO(X, \tau)\} = \Lambda_p(V) = V.
\]

Since \(PO(X, \tau) \subseteq \Lambda_p, \Lambda_p-Cl(\{x\}) \subseteq pCl(\{x\}) \subseteq V\), where \(\Lambda_p-Cl(\{x\})\) denotes the closure of the singleton \(\{x\}\) in the topological space \((X, \Lambda_p)\). This shows that \((X, \Lambda_p)\) is \(R_0\).

Conversely, suppose that \((X, \Lambda_p)\) is \(R_0\). Let \(V \in \Lambda_p\) and \(x \in V\). Since \(PO(X, \tau) \subseteq \Lambda_p\), we have \(\Lambda_p-Cl(\{x\}) \subseteq V\). Since \(X - \Lambda_p-Cl(\{x\}) \in \Lambda_p\),

\[
X - \Lambda_p-Cl(\{x\}) = \cap\{U \mid X - \Lambda_p-Cl(\{x\}) \subseteq U, U \in PO(X, \tau)\}.
\]

Then, there exists \(U \in PO(X, \tau)\) such that \(X - \Lambda_p-Cl(\{x\}) \subseteq U\) and \(x \notin U\). Thus, \(x \in X - U \subseteq \Lambda_p-Cl(\{x\}) \subseteq V\). Since \(X - U\) is preclosed, \(pCl(\{x\}) \subseteq X - U \subseteq V\). Consequently, we obtain \((X, \tau)\) is pre-\(R_0\).

Theorem 4. A topological space \((X, \tau)\) is pre-\(T_0\) if and only if the topological space \((X, \Lambda_p)\) is \(T_0\).

Proof. This is obvious since \(PO(X, \tau) \subseteq \Lambda_p\).

Conversely, let \(x\) and \(y\) be any pair of distinct points of \(X\). Since \((X, \Lambda_p)\) is \(T_0\), there exists \(V \in \Lambda_p\) such that either \(x \in V\) and \(y \notin V\) or \(x \notin V\) and \(y \in V\). In case \(x \in V\) and \(y \notin V\), there exists \(U \in PO(X, \tau)\) such that \(V \subseteq U\) and \(y \notin U\). However, since \(x \in V, x \in U\). In case \(x \notin V\) and \(y \in V\), similarly there exists \(U \in PO(X, \tau)\) such that \(x \notin U\) and \(y \in U\). Thus, \((X, \tau)\) is pre-\(T_0\).

Lemma 4. For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is pre-\(T_1\);
2. For each \(x \in X\), the singleton \(\{x\}\) is preclosed in \((X, \tau)\);
3. For each \(x \in X\), the singleton \(\{x\}\) is a \(\Lambda_p\)-set.

Proof. (1) \(\Rightarrow\) (2): Let \(y\) be any point of \(X\) and let \(x \in X - \{y\}\). Then, there exists a preopen set \(V_x\) such that \(x \in V_x\) and \(y \notin V_x\). Thus, \(X - \{y\} = \cup_{x \in X - \{y\}} V_x\) and hence the singleton \(\{y\}\) is preclosed in \(X\).

(2) \(\Rightarrow\) (3): Let \(x\) be any point of \(X\) and let \(y \in X - \{x\}\). Then,

\[
x \in (X - \{y\}) \in PO(X, \tau)
\]

and \(\Lambda_p(\{x\}) \subseteq X - \{y\}\). Therefore, \(y \notin \Lambda_p(\{x\})\) and hence \(\Lambda_p(\{x\}) \subseteq \{x\}\). Thus, \(\Lambda_p(\{x\}) = \{x\}\). This shows that \(\{x\}\) is a \(\Lambda_p\)-set.
Suppose that the singleton \{x\} is a \(\Lambda_p\)-set for each \(x \in X\). Let \(x\) and \(y\) be any distinct points. Then, \(y \not\in \Lambda_p(\{x\})\) and there exists a preopen set \(U_x\) such that \(x \in U_x\) and \(y \notin U_x\). Similarly, \(x \not\in \Lambda_p(\{y\})\) and there exists a preopen set \(U_y\) such that \(y \in U_y\) and \(x \notin U_y\). This shows that \((X, \tau)\) is pre-\(T_1\).

**Theorem 5.** A topological space \((X, \tau)\) is pre-\(T_1\) if and only if the topological space \((X, \Lambda_p)\) is discrete.

**Proof.** Suppose that \((X, \tau)\) is pre-\(T_1\). Let \(x \in X\). By Lemma 4, \(\{x\}\) is a \(\Lambda_p\)-set and hence \(\{x\}\) is open in \((X, \Lambda_p)\). Thus, every subset of \(X\) is open in \((X, \Lambda_p)\). This shows that \((X, \Lambda_p)\) is discrete.

Conversely, suppose that a topological space \((X, \Lambda_p)\) is discrete. For any point \(x \in X\), \(\{x\}\) is open in \((X, \Lambda_p)\) and hence \(\{x\}\) is a \(\Lambda_p\)-set, by Lemma 4, we have \((X, \tau)\) is pre-\(T_1\).

**Corollary 1.** For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is pre-\(T_1\);
2. \((X, \tau)\) is pre-\(R_0\) and pre-\(T_0\);
3. \((X, \Lambda_p)\) is \(R_0\) and \(T_0\);
4. \((X, \Lambda_p)\) is \(T_1\);
5. \((X, \Lambda_p)\) is discrete.

**Proof.**

1. \(\Rightarrow\) (2): By Lemma 4, every pre-\(T_1\) space is pre-\(R_0\) and pre-\(T_0\).

2. (2) \(\Rightarrow\) (1): Since \((X, \tau)\) is pre-\(T_0\), for any distinct point \(x, y\) of \(X\), there exists a preopen set \(U\) of \(X\) such that \(x \in U\) and \(y \notin U\). Hence, \(p\text{Cl}(\{x\}) \subseteq U\) since \((X, \tau)\) is pre-\(R_0\). Thus, \(x \notin X - p\text{Cl}(\{x\})\) and hence \(y \in X - U \subseteq X - p\text{Cl}(\{x\}) \in \text{PO}(X, \tau)\). This shows that \((X, \tau)\) is pre-\(T_1\).

3. \(\Leftrightarrow\) (3): This is an immediate consequence of Theorem 3 and Theorem 4.

4. \(\Leftrightarrow\) (4): This proof is obvious.

5. \(\Leftrightarrow\) (1): This is an immediate consequence of Theorem 5.

4. \((\Lambda, p)\)-closed sets

In this section, we introduce the notion of \((\Lambda, p)\)-closed sets in topological spaces. Moreover, some properties of \((\Lambda, p)\)-closed sets are discussed.

**Definition 4.** A subset \(A\) of a topological space \((X, \tau)\) is called \((\Lambda, p)\)-closed if \(A = T \cap C\), where \(T\) is a \(\Lambda_p\)-set and \(C\) is a preclosed set. The collection of all \((\Lambda, p)\)-closed sets in a topological space \((X, \tau)\) is denoted by \(\Lambda_p C(X, \tau)\).

**Theorem 6.** For a subset \(A\) of a topological space \((X, \tau)\), the following properties are equivalent:
(1) A is \((\Lambda, p)\)-closed;
(2) \(A = T \cap \text{pCl}(A)\), where \(T\) is a \(\Lambda_p\)-set;
(3) \(A = \Lambda_p(A) \cap \text{pCl}(A)\).

Proof. (1) \(\Rightarrow\) (2): Let \(A = T \cap C\), where \(T\) is a \(\Lambda_p\)-set and \(C\) is a preclosed set. Since \(A \subseteq C\), we have \(\text{pCl}(A) \subseteq C\) and hence \(A = T \cap C \supseteq T \cap \text{pCl}(A) \supseteq A\). Consequently, we obtain \(A = T \cap \text{pCl}(A)\).

(2) \(\Rightarrow\) (3): Let \(A = T \cap \text{pCl}(A)\), where \(T\) is a \(\Lambda_p\)-set. Since \(A \subseteq T\), \(\Lambda_p(A) \subseteq \Lambda_p(T) = T\) and hence \(A \subseteq \Lambda_p(A) \cap \text{pCl}(A) \subseteq T \cap \text{pCl}(A) = A\). Thus, \(A = \Lambda_p(A) \cap \text{pCl}(A)\).

(3) \(\Rightarrow\) (1): Since \(\Lambda_p(A)\) is a \(\Lambda_p\)-set, \(\text{pCl}(A)\) is preclosed and \(A = \Lambda_p(A) \cap \text{pCl}(A)\). This shows that \(A\) is \((\Lambda, p)\)-closed.

Definition 5. A subset \(A\) of a topological space \((X, \tau)\) is said to be \((\Lambda, p)\)-open if the complement of \(A\) is \((\Lambda, p)\)-closed. The collection of all \((\Lambda, p)\)-open sets in a topological space \((X, \tau)\) is denoted by \(\Lambda_pO(X, \tau)\).

Theorem 7. For a subset \(A_{\gamma}(\gamma \in \Gamma)\) of a topological space \((X, \tau)\), the following properties hold:

(1) If \(A_{\gamma}\) is \((\Lambda, p)\)-closed for each \(\gamma \in \Gamma\), then \(\cap \{A_{\gamma} \mid \gamma \in \Gamma\}\) is \((\Lambda, p)\)-closed.
(2) If \(A_{\gamma}\) is \((\Lambda, p)\)-open for each \(\gamma \in \Gamma\), then \(\cup \{A_{\gamma} \mid \gamma \in \Gamma\}\) is \((\Lambda, p)\)-open.

Proof. (1) Suppose that \(A_{\gamma}\) is \((\Lambda, p)\)-closed for each \(\gamma \in \Gamma\). Then, for each \(\gamma\), there exist a \(\Lambda_p\)-set \(T_{\gamma}\) and a preclosed set \(C_{\gamma}\) such that \(A_{\gamma} = T_{\gamma} \cap C_{\gamma}\). Thus,

\[\cap_{\gamma \in \Gamma} A_{\gamma} = \cap_{\gamma \in \Gamma} (T_{\gamma} \cap C_{\gamma}) = (\cap_{\gamma \in \Gamma} T_{\gamma}) \cap (\cap_{\gamma \in \Gamma} C_{\gamma}).\]

Since \(\cap_{\gamma \in \Gamma} C_{\gamma}\) is a preclosed set and by Lemma 3, we have \(\cap_{\gamma \in \Gamma} T_{\gamma}\) is a \(\Lambda_p\)-set. This shows that \(\cap_{\gamma \in \Gamma} A_{\gamma}\) is \((\Lambda, p)\)-closed.

(2) Let \(A_{\gamma}\) be \((\Lambda, p)\)-open for each \(\gamma \in \Gamma\). Then, \(X - A_{\gamma}\) is \((\Lambda, p)\)-closed, by (1), we have \(X - \cup_{\gamma \in \Gamma} A_{\gamma} = \cap_{\gamma \in \Gamma} (X - A_{\gamma})\) is \((\Lambda, p)\)-closed and hence \(\cup_{\gamma \in \Gamma} A_{\gamma}\) is \((\Lambda, p)\)-open.

Theorem 8. Let \((X, \tau)\) be a pre-\(R_0\) space. For each \(x \in X\), \(\{x\}\) is \((\Lambda, p)\)-closed if and only if \(\{x\}\) is preclosed.

Proof. Suppose that \(\{x\}\) is a \((\Lambda, p)\)-closed set. By Theorem 6,

\[\{x\} = \Lambda_p(\{x\}) \cap \text{pCl}(\{x\}).\]

For any preopen set \(U\) containing \(x\), \(\text{pCl}(\{x\}) \subseteq U\) and hence \(\text{pCl}(\{x\}) \subseteq \Lambda_p(\{x\})\). Thus, \(\{x\} = \Lambda_p(\{x\}) \cap \text{pCl}(\{x\}) \supseteq \text{pCl}(\{x\})\). This shows that \(\{x\}\) is preclosed.

Conversely, suppose that \(\{x\}\) is a preclosed set. Since \(\{x\} \subseteq \Lambda_p(\{x\})\), we have \(\Lambda_p(\{x\}) \cap \text{pCl}(\{x\}) = \Lambda_p(\{x\}) \cap \{x\} = \{x\}\), by Theorem 6, \(\{x\}\) is \((\Lambda, p)\)-closed.
Theorem 9. A topological space \((X, \tau)\) is pre-\(T_0\) if and only if for each \(x \in X\), the singleton \(\{x\}\) is \((\Lambda, p)\)-closed.

Proof. Suppose that \((X, \tau)\) is pre-\(T_0\). For each \(x \in X\), it is obvious that

\[
\{x\} \subseteq \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\}).
\]

If \(y \neq x\), (i) there exists a preopen set \(V_x\) such that \(y \notin V_x\) and \(x \in V_x\) or (ii) there exists a preopen set \(V_y\) such that \(x \notin V_y\) and \(y \in V_y\). In case of (i), \(y \notin \Lambda_p(\{x\})\) and \(y \notin \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})\). Thus, \(\{x\} \supseteq \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})\). In case (ii), \(y \notin p\text{Cl}(\{x\})\) and \(y \notin \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})\). This shows that \(\{x\} \supseteq \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})\). Consequently, we obtain \(\{x\} = \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})\).

Conversely, suppose that \((X, \tau)\) is not pre-\(T_0\). There exist two distinct points \(x, y\) such that (i) \(y \in V_x\) for every preopen set \(V_x\) containing \(x\) and (ii) \(x \in V_y\) for every preopen set \(V_y\) containing \(y\). From (i) and (ii), we obtain \(y \in \Lambda_p(\{x\})\) and \(y \in p\text{Cl}(\{x\})\), respectively. Therefore, we have \(y \in \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})\). By Theorem 6, \(\{x\} = \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})\) since \(\{x\}\) is \((\Lambda, p)\)-closed. This is contrary to \(x \neq y\).

Definition 6. Let \(A\) be a subset of a topological space \((X, \tau)\). A point \(x \in X\) is called a \((\Lambda, p)\)-cluster point of \(A\) if \(A \cap U \neq \emptyset\) for every \((\Lambda, p)\)-open set \(U\) of \(X\) containing \(x\). The set of all \((\Lambda, p)\)-cluster points of \(A\) is called the \((\Lambda, p)\)-closure of \(A\) and is denoted by \(A^{(\Lambda, p)}\).

Lemma 5. For subsets \(A, B\) of a topological space \((X, \tau)\), the following properties hold:

1. \(A \subseteq A^{(\Lambda, p)}\) and \(A^{(\Lambda, p)}(A^{(\Lambda, p)}) = A^{(\Lambda, p)}\).
2. If \(A \subseteq B\), then \(A^{(\Lambda, p)} \subseteq B^{(\Lambda, p)}\).
3. \(A^{(\Lambda, p)} = \cap \{F | A \subseteq F\text{ and } F\text{ is } (\Lambda, p)\text{-closed}\}\).
4. \(A^{(\Lambda, p)}\) is \((\Lambda, p)\)-closed.
5. \(A\) is \((\Lambda, p)\)-closed if and only if \(A = A^{(\Lambda, p)}\).

Remark 1. Every \(\Lambda_p\)-set is \((\Lambda, p)\)-closed.

The converse of Remark 1 is not true in general as shown by the following example.

Example 1. Let \(X = \{-2, -1\}\) with a topology \(\tau = \{\emptyset, \{-2\}, X\}\). Then, \(-1\) is a \((\Lambda, p)\)-closed set, but \(-1\) is not a \(\Lambda_p\)-set.

Lemma 6. For a subset \(A\) of a topological space \((X, \tau)\), the following properties hold:

1. If \(A\) is preclosed, then \(A\) is \((\Lambda, p)\)-closed.
2. \(A\) is \((\Lambda, p)\)-closed if and only if \(A = \Lambda_p(A) \cap A^{(\Lambda, p)}\).
Theorem 10. For a topological space \( (X, \tau) \), the following properties hold:

1. \( \Lambda_{(\Lambda, p)}(A) = \{ x \in X \mid A \cap \{x\}^{(\Lambda, p)} \neq \emptyset \} \) for each subset \( A \) of \( X \).
2. For each \( x \in X \), \( \Lambda_{(\Lambda, p)}(\langle x \rangle_p) = \Lambda_{(\Lambda, p)}(\{x\}) \).

Proof. (1) It is sufficient to observe that \( A = X \cap A \), where the whole set \( X \) is a \( \Lambda_{(\Lambda, p)} \)-set.

(2) Let \( A \) be a \( (\Lambda, p) \)-closed set. Then, there exist a \( \Lambda_{(\Lambda, p)} \)-set \( T \) and a preclosed set \( C \) such that \( A = T \cap C \). Since \( A \subseteq T \), we have \( A \subseteq \Lambda_{(\Lambda, p)}(A) \subseteq \Lambda_{(\Lambda, p)}(T) = T \). Since \( C \) is preclosed, by (1), \( C \) is \( (\Lambda, p) \)-closed. Since \( A \subseteq C \), \( A \subseteq A^{(\Lambda, p)} \subseteq C^{(\Lambda, p)} = C \) and hence \( A \subseteq \Lambda_{(\Lambda, p)}(A) \cap A^{(\Lambda, p)} \subseteq T \cap C = A \).

Thus, \( A = \Lambda_{(\Lambda, p)}(A) \cap A^{(\Lambda, p)} \).

Conversely, let \( A = \Lambda_{(\Lambda, p)}(A) \cap A^{(\Lambda, p)} \). Since \( \Lambda_{(\Lambda, p)}(A) \) is a \( \Lambda_{(\Lambda, p)} \)-set, by Remark 1, \( \Lambda_{(\Lambda, p)}(A) \) is \( (\Lambda, p) \)-closed. Since \( A^{(\Lambda, p)} \) is \( (\Lambda, p) \)-closed, by Theorem 7(1), \( \Lambda_{(\Lambda, p)}(A) \cap A^{(\Lambda, p)} \) is \( (\Lambda, p) \)-closed and hence \( A \) is \( (\Lambda, p) \)-closed.

The following example shows that the converse of Lemma 6(1) is not true in general.

Example 2. Let \( X = \{-2, -1, 0, 1, 2\} \) with a topology \( \tau = \{\emptyset, \{-2\}, \{2\}, \{-2, 2\}, X\} \). Then, \( \{-2, 2\} \) is \( (\Lambda, p) \)-closed, but \( \{-2, 2\} \) is not preclosed.

Definition 7. Let \( A \) be a subset of a topological space \( (X, \tau) \). A subset \( \Lambda_{(\Lambda, p)}(A) \) is defined as follows: \( \Lambda_{(\Lambda, p)}(A) = \cap\{U \in \Lambda_{(\Lambda, p)}(X, \tau) \mid A \subseteq U\} \).

Lemma 7. For subsets \( A, B \) of a topological space \( (X, \tau) \), the following properties hold:

1. \( A \subseteq \Lambda_{(\Lambda, p)}(A) \).
2. If \( A \subseteq B \), then \( \Lambda_{(\Lambda, p)}(A) \subseteq \Lambda_{(\Lambda, p)}(B) \).
3. \( \Lambda_{(\Lambda, p)}[\Lambda_{(\Lambda, p)}(A)] = \Lambda_{(\Lambda, p)}(A) \).
4. If \( A \) is \( (\Lambda, p) \)-open, then \( \Lambda_{(\Lambda, p)}(A) = A \).

Lemma 8. Let \( (X, \tau) \) be a topological space and let \( x, y \in X \). Then, \( y \in \Lambda_{(\Lambda, p)}(\{x\}) \) if and only if \( x \in \{y\}^{(\Lambda, p)} \).

Proof. Let \( y \notin \Lambda_{(\Lambda, p)}(\{x\}) \). Then, there exists a \( (\Lambda, p) \)-open set \( V \) containing \( x \) such that \( y \notin V \). Hence, \( x \notin \{y\}^{(\Lambda, p)} \). The converse is similarly shown.

A subset \( N_x \) of a topological space \( (X, \tau) \) is said to be \( (\Lambda, p) \)-neighbourhood of a point \( x \in X \) if there exists a \( (\Lambda, p) \)-open set \( U \) such that \( x \in U \subseteq N_x \).

Lemma 9. A subset \( A \) of a topological space \( (X, \tau) \) is \( (\Lambda, p) \)-open if and only if \( A \) is \( (\Lambda, p) \)-neighbourhood of each \( x \in A \).

Definition 8. Let \( A \) be a subset of a topological space \( (X, \tau) \). A subset \( \langle x \rangle_p \) is defined as follows: \( \langle x \rangle_p = \Lambda_{(\Lambda, p)}(\{x\}) \cap \{x\}^{(\Lambda, p)} \).

Theorem 10. For a topological space \( (X, \tau) \), the following properties hold:

1. \( \Lambda_{(\Lambda, p)}(A) = \{ x \in X \mid A \cap \{x\}^{(\Lambda, p)} \neq \emptyset \} \) for each subset \( A \) of \( X \).
2. For each \( x \in X \), \( \Lambda_{(\Lambda, p)}(\langle x \rangle_p) = \Lambda_{(\Lambda, p)}(\{x\}) \).
(3) For each \( x \in X \), \( (x)_p^{(\Lambda,p)} = \{x\}^{(\Lambda,p)} \).

(4) If \( U \) is \((\Lambda,p)\)-open and \( x \in U \), then \( (x)_p \subseteq U \).

(5) If \( F \) is \((\Lambda,p)\)-closed and \( x \in F \), then \( (x)_p \subseteq F \).

Proof. (1) Suppose that \( A \cap \{x\}^{(\Lambda,p)} = \emptyset \). Then, we have \( x \notin X - \{x\}^{(\Lambda,p)} \) which is a \((\Lambda,p)\)-open set containing \( A \). Therefore, \( x \notin A_{(\Lambda,p)}(A) \). Consequently, we have \( A_{(\Lambda,p)}(A) \subseteq \{ x \in X \mid A \cap \{x\}^{(\Lambda,p)} \neq \emptyset \} \). Next, let \( x \in X \) such that \( A \cap \{x\}^{(\Lambda,p)} \neq \emptyset \) and suppose that \( x \notin A_{(\Lambda,p)}(A) \). Then, there exists a \((\Lambda,p)\)-open set \( U \) containing \( A \) and \( x \notin U \). Let \( y \in A \cap \{x\}^{(\Lambda,p)} \). Hence, \( U \) is a \((\Lambda,p)\)-neighbourhood of \( y \) which does not contain \( x \). By this contradiction \( x \in A_{(\Lambda,p)}(A) \).

(2) Let \( x \in X \). Then, we have \( \{x\} \subseteq \{x\}^{(\Lambda,p)} \cap A_{(\Lambda,p)}(\{x\}) = (x)_p \). By Lemma 7, \( A_{(\Lambda,p)}(\{x\}) \subseteq A_{(\Lambda,p)}((x)_p) \). Next, we show the opposite implication. Suppose that \( y \notin A_{(\Lambda,p)}((x)_p) \). Then, there exists a \((\Lambda,p)\)-open set \( V \) such that \( x \in V \) and \( y \notin V \). Since \( (x)_p \subseteq A_{(\Lambda,p)}(\{x\}) \subseteq A_{(\Lambda,p)}(V) = V \), we have \( A_{(\Lambda,p)}((x)_p) \subseteq V \). Since \( y \notin V \), we have \( y \notin A_{(\Lambda,p)}((x)_p) \). Thus, \( A_{(\Lambda,p)}((x)_p) \subseteq A_{(\Lambda,p)}(\{x\}) \) and hence \( A_{(\Lambda,p)}((x)_p) = A_{(\Lambda,p)}(\{x\}) \).

(3) By the definition of \( (x)_p \), we have \( \{x\} \subseteq (x)_p \) and \( \{x\}^{(\Lambda,p)} \subseteq (x)_p^{(\Lambda,p)} \) by Lemma 5. On the other hand, we have \( (x)_p \subseteq \{x\}^{(\Lambda,p)} \) and \( (x)_p^{(\Lambda,p)} \subseteq \{x\}^{(\Lambda,p)} \). Thus, \( (x)_p^{(\Lambda,p)} = \{x\}^{(\Lambda,p)} \).

(4) Let \( U \) be a \((\Lambda,p)\)-open set and let \( x \in U \). By Lemma 7, \( A_{(\Lambda,p)}(\{x\}) \subseteq U \) and hence \( (x)_p \subseteq U \).

(5) Let \( F \) be a \((\Lambda,p)\)-closed set and let \( x \in F \). By Lemma 5, we have \( (x)_p = \{x\}^{(\Lambda,p)} \cap A_{(\Lambda,p)}(\{x\}) \subseteq \{x\}^{(\Lambda,p)} \subseteq F^{(\Lambda,p)} = F \).

Lemma 10. For any points \( x \) and \( y \) in a topological space \((X, \tau)\), the following properties are equivalent:

(1) \( \lambda_{(\Lambda,p)}(\{x\}) \neq \lambda_{(\Lambda,p)}(\{y\}) \);

(2) \( \{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)} \).

Proof. (1) \( \Rightarrow \) (2): Suppose that \( \lambda_{(\Lambda,p)}(\{x\}) \neq \lambda_{(\Lambda,p)}(\{y\}) \). There exists a point \( z \in X \) such that \( z \in \lambda_{(\Lambda,p)}(\{x\}) \) and \( z \notin \lambda_{(\Lambda,p)}(\{y\}) \) or \( z \notin \lambda_{(\Lambda,p)}(\{x\}) \) and \( z \notin \lambda_{(\Lambda,p)}(\{y\}) \). We prove only the first case being the second analogous. From \( z \in \lambda_{(\Lambda,p)}(\{x\}) \) it follows that \( \{x\} \cap \{z\}^{(\Lambda,p)} \neq \emptyset \) which implies \( x \in \{z\}^{(\Lambda,p)} \). By \( z \notin \lambda_{(\Lambda,p)}(\{y\}) \), \( \{y\} \cap \{z\}^{(\Lambda,p)} = \emptyset \). Since \( x \in \{z\}^{(\Lambda,p)} \), \( \{x\}^{(\Lambda,p)} \subseteq \{z\}^{(\Lambda,p)} \) and \( \{y\} \cap \{x\}^{(\Lambda,p)} = \emptyset \). Therefore, it follows that \( \{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)} \). Thus, \( \lambda_{(\Lambda,p)}(\{x\}) \neq \lambda_{(\Lambda,p)}(\{y\}) \) implies that \( \{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)} \).

(2) \( \Rightarrow \) (1): Suppose that \( \{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)} \). Then, there exists a point \( z \in X \) such that \( z \in \{x\}^{(\Lambda,p)} \) and \( z \notin \{y\}^{(\Lambda,p)} \) or \( z \notin \{x\}^{(\Lambda,p)} \) and \( z \notin \{y\}^{(\Lambda,p)} \). We prove only the first case being the second analogous. It follows that there exists a \((\Lambda,p)\)-open set containing \( z \) and therefore \( x \) but not \( y \), namely, \( y \notin \lambda_{(\Lambda,p)}(\{x\}) \) and hence \( \lambda_{(\Lambda,p)}(\{x\}) \neq \lambda_{(\Lambda,p)}(\{y\}) \).
Lemma 11. For any points $x$ and $y$ in a topological space $(X, \tau)$, the following properties hold:

(1) $y \in \Lambda_{(\Lambda, p)}(\{x\})$ if and only if $x \in \{y\}^{(\Lambda, p)}$;

(2) $\Lambda_{(\Lambda, p)}(\{x\}) = \Lambda_{(\Lambda, p)}(\{y\})$ if and only if $\{x\}^{(\Lambda, p)} = \{y\}^{(\Lambda, p)}$.

Proof. (1) Let $x \notin \{y\}^{(\Lambda, p)}$. Then, there exists a $(\Lambda, p)$-open set $U$ such that $x \in U$ and $y \notin U$. Therefore, $y \notin \Lambda_{(\Lambda, p)}(\{x\})$. The converse is similarly shown.

(2) Suppose that $\Lambda_{(\Lambda, p)}(\{x\}) = \Lambda_{(\Lambda, p)}(\{y\})$ for any points $x$ and $y$ in $X$. Since

$$x \in \Lambda_{(\Lambda, p)}(\{x\}),$$

$x \in \Lambda_{(\Lambda, p)}(\{y\})$ and by (1), $y \in \{x\}^{(\Lambda, p)}$. By Lemma 5, $\{y\}^{(\Lambda, p)} \subseteq \{x\}^{(\Lambda, p)}$. Similarly, we have $\{x\}^{(\Lambda, p)} \subseteq \{y\}^{(\Lambda, p)}$ and hence $\{x\}^{(\Lambda, p)} = \{y\}^{(\Lambda, p)}$.

Conversely, suppose that $\{x\}^{(\Lambda, p)} = \{y\}^{(\Lambda, p)}$. Since $x \in \{x\}^{(\Lambda, p)}$, $x \in \{y\}^{(\Lambda, p)}$ and by (1), $y \in \Lambda_{(\Lambda, p)}(\{x\})$. By Lemma 7, $\Lambda_{(\Lambda, p)}(\{y\}) \subseteq \Lambda_{(\Lambda, p)}(\Lambda_{(\Lambda, p)}(\{x\})) = \Lambda_{(\Lambda, p)}(\{x\})$. Similarly, we have $\Lambda_{(\Lambda, p)}(\{x\}) \subseteq \Lambda_{(\Lambda, p)}(\{y\})$ and hence $\Lambda_{(\Lambda, p)}(\{x\}) = \Lambda_{(\Lambda, p)}(\{y\})$.

5. Characterizations of $\Lambda_p$-$R_0$ spaces

In this section, we introduce the concept of $\Lambda_p$-$R_0$ spaces. Moreover, some characterizations of $\Lambda_p$-$R_0$ spaces are investigated.

Definition 9. A topological space $(X, \tau)$ is called a $\Lambda_p$-$R_0$ space if, for each $(\Lambda, p)$-open set $U$ and each $x \in U$, $\{x\}^{(\Lambda, p)} \subseteq U$.

Theorem 11. For a topological space $(X, \tau)$, the following properties are equivalent:

(1) $(X, \tau)$ is $\Lambda_p$-$R_0$;

(2) for each $(\Lambda, p)$-closed set $F$ and each $x \in X - F$, there exists a $(\Lambda, p)$-open set $U$ such that $F \subseteq U$ and $x \notin U$;

(3) for each $(\Lambda, p)$-closed set $F$ and each $x \in X - F$, $F \cap \{x\}^{(\Lambda, p)} = \emptyset$;

(4) for each $x, y \in X$, $\{x\}^{(\Lambda, p)} = \{y\}^{(\Lambda, p)}$ or $\{x\}^{(\Lambda, p)} \cap \{y\}^{(\Lambda, p)} = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let $F$ be a $(\Lambda, p)$-closed set and let $x \in X - F$. Then, we have $\{x\}^{(\Lambda, p)} \subseteq X - F$. Let $U = X - \{x\}^{(\Lambda, p)}$, then $U$ is a $(\Lambda, p)$-open set such that $F \subseteq U$ and $x \notin U$.

(2) $\Rightarrow$ (3): Let $F$ be a $(\Lambda, p)$-closed set and let $x \in X - F$. There exists a $(\Lambda, p)$-open set $U$ such that $F \subseteq U$ and $x \notin U$. Thus, $U \cap \{x\}^{(\Lambda, p)} = \emptyset$ and hence $F \cap \{x\}^{(\Lambda, p)} = \emptyset$.

(3) $\Rightarrow$ (4): Let $x, y$ be distinct points of $X$. Suppose that $\{x\}^{(\Lambda, p)} \neq \{y\}^{(\Lambda, p)}$. By (3), $x \in \{y\}^{(\Lambda, p)}$ and $y \in \{x\}^{(\Lambda, p)}$. Thus, $\{x\}^{(\Lambda, p)} \subseteq \{y\}^{(\Lambda, p)} \subseteq \{x\}^{(\Lambda, p)}$ and hence $\{x\}^{(\Lambda, p)} = \{y\}^{(\Lambda, p)}$. 


(4) ⇒ (1): Let \( U \) be a \((\Lambda, p)\)-open set and let \( x \in U \). For each \( y \notin U \), we have \( U \cap \{y\}^{(\Lambda, p)} = \emptyset \) and hence \( x \notin \{y\}^{(\Lambda, p)} \). Therefore, \( \{y\}^{(\Lambda, p)} \neq \{x\}^{(\Lambda, p)} \). By (4),
\[
\{x\}^{(\Lambda, p)} \cap \{y\}^{(\Lambda, p)} = \emptyset.
\]
Since \( X - U \) is \((\Lambda, p)\)-closed, \( y \in \{y\}^{(\Lambda, p)} \subseteq X - U \) and \( \cup_{y \in X - U} \{y\}^{(\Lambda, p)} = X - U \). Thus, \( \{x\}^{(\Lambda, p)} \cap (X - U) = \{x\}^{(\Lambda, p)} \cap [\cup_{y \in X - U} \{y\}^{(\Lambda, p)}] = \cup_{y \in X - U} [\{x\}^{(\Lambda, p)} \cap \{y\}^{(\Lambda, p)}] = \emptyset \) and hence \( \{x\}^{(\Lambda, p)} \subseteq U \). This shows that \((X, \tau)\) is \(\Lambda_p\)-R_0.

**Corollary 2.** A topological space \((X, \tau)\) is \((\Lambda_p)\)-R_0 if and only if, for each \( x, y \in X \), \( \{x\}^{(\Lambda, p)} \neq \{y\}^{(\Lambda, p)} \) implies \( \{x\}^{(\Lambda, p)} \cap \{y\}^{(\Lambda, p)} = \emptyset \).

**Proof.** This is obvious by Theorem 11.

Conversely, let \( U \) be a \((\Lambda, p)\)-open set and let \( x \in U \). If \( y \notin U \), then \( U \cap \{y\}^{(\Lambda, p)} = \emptyset \). Thus, \( x \notin \{y\}^{(\Lambda, p)} \) and \( \{x\}^{(\Lambda, p)} \neq \{y\}^{(\Lambda, p)} \). By the hypothesis, \( \{x\}^{(\Lambda, p)} \cap \{y\}^{(\Lambda, p)} = \emptyset \) and hence \( y \notin \{x\}^{(\Lambda, p)} \). Therefore, \( \{x\}^{(\Lambda, p)} \subseteq U \). Thus, \((X, \tau)\) is \(\Lambda_p\)-R_0.

**Theorem 12.** A topological space \((X, \tau)\) is \((\Lambda_p)\)-R_0 if and only if, for each \( x, y \in X \), \( \Lambda_{\{x\}}(\{y\}) \neq \Lambda_{\{y\}}(\{x\}) \) implies \( \Lambda_{\{x\}}(\{y\}) \cap \Lambda_{\{y\}}(\{x\}) = \emptyset \).

**Proof.** Suppose that \( \Lambda_{\{x\}}(\{y\}) \cap \Lambda_{\{y\}}(\{x\}) = \emptyset \). Let \( z \in \Lambda_{\{x\}}(\{y\}) \cap \Lambda_{\{y\}}(\{x\}) \). Then, \( z \in \Lambda_{\{x\}}(\{y\}) \) and by Lemma 11, \( x \in \{z\}^{(\Lambda, p)} \) and \( y \in \{z\}^{(\Lambda, p)} \). Similarly, we have \( \{z\}^{(\Lambda, p)} = \{y\}^{(\Lambda, p)} \) and by Lemma 11, \( \Lambda_{\{y\}}(\{x\}) = \Lambda_{\{x\}}(\{y\}) \).

Conversely, we shows the sufficiency by using Corollary 2. Suppose that
\[
\{x\}^{(\Lambda, p)} \neq \{y\}^{(\Lambda, p)}.
\]
By Lemma 11, \( \Lambda_{\{x\}}(\{y\}) = \Lambda_{\{y\}}(\{x\}) \) and hence \( \Lambda_{\{x\}}(\{y\}) \cap \Lambda_{\{y\}}(\{x\}) = \emptyset \). Therefore, \( \{x\}^{(\Lambda, p)} \cap \{y\}^{(\Lambda, p)} = \emptyset \). In fact, assume \( z \in \{x\}^{(\Lambda, p)} \cap \{y\}^{(\Lambda, p)} \). Then, \( z \in \{x\}^{(\Lambda, p)} \) implies \( x \in \Lambda_{\{x\}}(\{z\}) \) and hence \( x \in \Lambda_{\{y\}}(\{z\}) \). By the hypothesis, \( \Lambda_{\{x\}}(\{z\}) = \Lambda_{\{y\}}(\{z\}) \) and by Lemma 11, \( \{z\}^{(\Lambda, p)} = \{x\}^{(\Lambda, p)} \). Similarly, we have \( \{z\}^{(\Lambda, p)} = \{y\}^{(\Lambda, p)} \) and hence \( \{x\}^{(\Lambda, p)} = \{y\}^{(\Lambda, p)} \). This contradicts that \( \{x\}^{(\Lambda, p)} \neq \{y\}^{(\Lambda, p)} \). Thus, \( \{x\}^{(\Lambda, p)} \cap \{y\}^{(\Lambda, p)} = \emptyset \). This shows that \((X, \tau)\) is \(\Lambda_p\)-R_0.

**Theorem 13.** For a topological space \((X, \tau)\), the following properties are equivalent:

(1) \((X, \tau)\) is \(\Lambda_p\)-R_0;

(2) \( x \in \{y\}^{(\Lambda, p)} \) if and only if \( y \in \{x\}^{(\Lambda, p)} \).

**Proof.** (1) ⇒ (2): Suppose that \((X, \tau)\) is \(\Lambda_p\)-R_0. Let \( x \in \{y\}^{(\Lambda, p)} \). By Lemma 11, \( y \in \Lambda_{\{x\}}(\{x\}) \) and hence \( \Lambda_{\{x\}}(\{x\}) \cap \Lambda_{\{y\}}(\{y\}) \neq \emptyset \). By Theorem 12, we have
\[
\Lambda_{\{x\}}(\{x\}) = \Lambda_{\{y\}}(\{y\})
\]
and hence \( x \in \Lambda_{(\Lambda, p)}(\{y\}) \). Thus, \( y \in \{x\}^{(\Lambda, p)} \) by Lemma 11. The converse is similarly shown.

(2) \( \Rightarrow \) (1): Let \( U \) be a \((\Lambda, p)\)-open set and let \( x \in U \). If \( y \notin U \), then \( \{y\}^{(\Lambda, p)} \cap U = \emptyset \). Thus, \( x \notin \{y\}^{(\Lambda, p)} \) and hence \( y \notin \{x\}^{(\Lambda, p)} \). This implies that \( \{x\}^{(\Lambda, p)} \subseteq U \). Therefore, \((X, \tau)\) is \( \Lambda_{p}-R_0 \).

**Theorem 14.** For a topological space \((X, \tau)\), the following properties are equivalent:

(1) \((X, \tau)\) is \( \Lambda_{p}-R_0 \);

(2) for each nonempty subset \( A \) of \( X \) and each \((\Lambda, p)\)-open set \( U \) such that \( A \cap U \neq \emptyset \), there exists a \((\Lambda, p)\)-closed set \( F \) such that \( A \cap F \neq \emptyset \) and \( F \subseteq U \);

(3) \( F = \Lambda_{(\Lambda, p)}(F) \) for each \((\Lambda, p)\)-closed set \( F \);

(4) \( \{x\}^{(\Lambda, p)} = \Lambda_{(\Lambda, p)}(\{x\}) \) for each \( x \in X \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( A \) be a nonempty subset of \( X \) and let \( U \in \Lambda_{p}O(X, \tau) \) such that \( A \cap U \neq \emptyset \). Then, there exists \( x \in A \cap U \) and hence \( \{x\}^{(\Lambda, p)} \subseteq U \). Put \( F = \{x\}^{(\Lambda, p)} \), then \( F \) is \((\Lambda, p)\)-closed, \( A \cap F \neq \emptyset \) and \( F \subseteq U \).

(2) \( \Rightarrow \) (3): Let \( F \) be a \((\Lambda, p)\)-closed set. By Lemma 7, we have \( F \subseteq \Lambda_{(\Lambda, p)}(F) \). Next, we show \( F \supseteq \Lambda_{(\Lambda, p)}(F) \). Let \( x \notin F \). Then, \( x \in (X - F) \in \Lambda_{p}O(X, \tau) \) and by (2), there exists a \((\Lambda, p)\)-closed set \( K \) such that \( x \in K \) and \( K \subseteq X - F \). Now, put \( U = X - K \). Then, \( F \subseteq U \in \Lambda_{p}O(X, \tau) \) and \( x \notin F \). Thus, \( x \notin \Lambda_{(\Lambda, p)}(F) \). This shows that \( F \supseteq \Lambda_{(\Lambda, p)}(F) \).

(3) \( \Rightarrow \) (4): Let \( x \in X \) and let \( y \notin \Lambda_{(\Lambda, p)}(\{x\}) \). There exists a \((\Lambda, p)\)-open set \( U \) such that \( x \in U \) and \( y \notin U \). Therefore, \( \{y\}^{(\Lambda, p)} \cap U = \emptyset \). By (3), we have

\[
\Lambda_{(\Lambda, p)}(\{y\}^{(\Lambda, p)}) \cap U = \emptyset.
\]

Since \( x \notin \Lambda_{(\Lambda, p)}(\{y\}^{(\Lambda, p)}) \), there exists a \((\Lambda, p)\)-open set \( G \) such that \( \{y\}^{(\Lambda, p)} \subseteq G \) and \( x \notin G \). Hence, \( \{x\}^{(\Lambda, p)} \cap G = \emptyset \). Since \( y \in G \), we have \( y \notin \{x\}^{(\Lambda, p)} \) and hence \( \{x\}^{(\Lambda, p)} \subseteq \Lambda_{(\Lambda, p)}(\{x\}) \). Moreover, \( \{x\}^{(\Lambda, p)} \subseteq \Lambda_{(\Lambda, p)}(\{x\}) \subseteq \Lambda_{(\Lambda, p)}(\{x\}^{(\Lambda, p)}) = \{x\}^{(\Lambda, p)} \). Consequently, we obtain \( \{x\}^{(\Lambda, p)} = \Lambda_{(\Lambda, p)}(\{x\}) \).

(4) \( \Rightarrow \) (5): The proof is obvious.

(5) \( \Rightarrow \) (1): Let \( U \in \Lambda_{p}O(X, \tau) \) and let \( x \in U \). If \( y \notin U \), then \( \{y\}^{(\Lambda, p)} \cap U = \emptyset \) and \( x \notin \{y\}^{(\Lambda, p)} \). By Lemma 11, \( y \notin \Lambda_{(\Lambda, p)}(\{x\}) \) and by (5), \( y \notin \{x\}^{(\Lambda, p)} \). Thus, \( \{x\}^{(\Lambda, p)} \subseteq U \) and hence \((X, \tau)\) is \( \Lambda_{p}-R_0 \).

**Corollary 3.** A topological space \((X, \tau)\) is \( \Lambda_{p}-R_0 \) if and only if \( \{x\}^{(\Lambda, p)} \subseteq \Lambda_{(\Lambda, p)}(\{x\}) \) for each \( x \in X \).

**Proof.** This is obvious by Theorem 14.

Conversely, let \( x \in \{y\}^{(\Lambda, p)} \). By Lemma 11, we have \( y \in \Lambda_{(\Lambda, p)}(\{x\}) \) and hence \( y \in \{x\}^{(\Lambda, p)} \). Similarly, if \( y \in \{x\}^{(\Lambda, p)} \), then \( x \in \{y\}^{(\Lambda, p)} \). It follows from Theorem 13 that \((X, \tau)\) is \( \Lambda_{p}-R_0 \).
Theorem 15. For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is \(\Lambda_p\)-\(R_0\);
2. \(\langle x \rangle_p = \{x\}^{(\Lambda_p)}\) for each \(x \in X\);
3. \(\langle x \rangle_p\) is \((\Lambda, p)\)-closed for each \(x \in X\).

Proof. (1) \(\Rightarrow\) (2): By Theorem 14, \(\{x\}^{(\Lambda_p)} = \Lambda_{(\Lambda, p)}(\{x\})\) for each \(x \in X\) and hence 
\[\{x\}^{(\Lambda_p)} = \{x\}^{(\Lambda, p)} \cap \Lambda_{(\Lambda, p)}(\{x\}) = \langle x \rangle_p.\]
(2) \(\Rightarrow\) (1): Since \(\{x\}^{(\Lambda_p)} = \langle x \rangle_p\) for each \(x \in X\), we have \(\{x\}^{(\Lambda_p)} \subseteq \Lambda_{(\Lambda, p)}(\{x\})\). By Corollary 3, \((X, \tau)\) is \(\Lambda_p\)-\(R_0\).
(2) \(\Leftrightarrow\) (3): This is a consequence of Lemma 7.

6. Characterizations of weakly \((\Lambda, p)\)-continuous functions

In this section, we introduce the notion of weakly \((\Lambda, p)\)-continuous functions and obtain several characterizations of weakly \((\Lambda, p)\)-continuous functions.

Definition 10. Let \(A\) be a subset of a topological space \((X, \tau)\). The union of all \((\Lambda, p)\)-open sets contained in \(A\) is called the \((\Lambda, p)\)-interior of \(A\) and is denoted by \(A^{(\Lambda, p)}\).

Lemma 12. Let \(A\) and \(B\) be subsets of a topological space \((X, \tau)\). For the \((\Lambda, p)\)-interior, the following properties hold:

1. \(A^{(\Lambda, p)} \subseteq A\) and \([A^{(\Lambda, p)}]^{(\Lambda, p)} = A^{(\Lambda, p)}\).
2. If \(A \subseteq B\), then \(A^{(\Lambda, p)} \subseteq B^{(\Lambda, p)}\).
3. \(A^{(\Lambda, p)} = \cup\{G \mid G \subseteq A\text{ and }G\text{ is }\Lambda_p\text{-open}\}\).
4. \(A^{(\Lambda, p)}\) is \((\Lambda, p)\)-open.
5. \(A\) is \((\Lambda, p)\)-open if and only if \(A^{(\Lambda, p)} = A\).
6. \([X - A]^{(\Lambda, p)} = X - A^{(\Lambda, p)}\).

Definition 11. A function \(f : (X, \tau) \to (Y, \sigma)\) is said to be weakly \((\Lambda, p)\)-continuous at a point \(x \in X\) if, for each \((\Lambda, p)\)-open set \(V\) containing \(f(x)\), there exists a \((\Lambda, p)\)-open set \(U\) containing \(x\) such that \(f(U) \subseteq V^{(\Lambda, p)}\). A function \(f : (X, \tau) \to (Y, \sigma)\) is said to be \((\Lambda, p)\)-continuous if \(f\) has this property at each point \(x \in X\).

Theorem 16. A function \(f : (X, \tau) \to (Y, \sigma)\) is weakly \((\Lambda, p)\)-continuous at \(x \in X\) if and only if for each \((\Lambda, p)\)-open set \(V\) containing \(f(x)\), \(x \in [f^{-1}(V^{(\Lambda, p)})]^{(\Lambda, p)}\).
Theorem 17. A function \( f : (X, \tau) \to (Y, \sigma) \) is weakly \((\Lambda, p)\)-continuous if and only if for every \((\Lambda, p)\)-open set \(V\) of \(Y\), \(f^{-1}(V) \subseteq f^{-1}(V^{(\Lambda, p)})\).

Proof. Let \( V \) be any \((\Lambda, p)\)-open set of \(Y\) and let \( x \in f^{-1}(V) \). Then \( f(x) \in V \). Since \( f \) is weakly \((\Lambda, p)\)-continuous at \( x \), by Theorem 16, \( x \in f^{-1}(V^{(\Lambda, p)})\) and hence \( f^{-1}(V) \subseteq f^{-1}(V^{(\Lambda, p)})\).

Conversely, let \( x \in X \) and let \( V \) be any \((\Lambda, p)\)-open set of \(Y\) containing \( f(x) \). Then, we have \( x \in f^{-1}(V) \subseteq f^{-1}(V^{(\Lambda, p)})\) and hence \( x \in f^{-1}(V^{(\Lambda, p)})\). Thus, \( f \) is weakly \((\Lambda, p)\)-continuous by Theorem 16.

Theorem 18. A function \( f : (X, \tau) \to (Y, \sigma) \) is weakly \((\Lambda, p)\)-continuous if and only if for every \((\Lambda, p)\)-open set \(V\) of \(Y\), \(f^{-1}(V) \subseteq f^{-1}(V^{(\Lambda, p)})\).

Proof. Let \( V \) be any \((\Lambda, p)\)-open subset of \(Y\) and let \( x \in f^{-1}(V) \). There exists a \((\Lambda, p)\)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V^{(\Lambda, p)} \). Since \( x \in U \subseteq f^{-1}(V^{(\Lambda, p)}) \), we have \( x \in f^{-1}(V^{(\Lambda, p)})\) and hence \( f^{-1}(V) \subseteq f^{-1}(V^{(\Lambda, p)})\).

Conversely, let \( x \in X \) and let \( V \) be any \((\Lambda, p)\)-open set containing \( f(x) \). Since \( V \cap [Y - V^{(\Lambda, p)}] = \emptyset \), \( f(x) \notin [Y - V^{(\Lambda, p)}]\) and hence \( x \notin f^{-1}([Y - V^{(\Lambda, p)}])\). By the hypothesis, \( x \notin f^{-1}(V^{(\Lambda, p)}) = [X - f^{-1}(V^{(\Lambda, p)})] \) and there exists a \((\Lambda, p)\)-open set \( U \) containing \( x \) such that \( U \cap [X - f^{-1}(V^{(\Lambda, p)})] = \emptyset \). Thus, \( f(U) \subseteq V^{(\Lambda, p)} \). This shows that \( f \) is weakly \((\Lambda, p)\)-continuous.

Theorem 19. For a function \( f : (X, \tau) \to (Y, \sigma) \), the following properties are equivalent:

1. \( f \) is weakly \((\Lambda, p)\)-continuous;
2. \( f^{-1}(U) \subseteq f^{-1}(U^{(\Lambda, p)})\) for every \((\Lambda, p)\)-open subset \( U \) of \( Y \);
3. \( f^{-1}(F^{(\Lambda, p)}) \subseteq f^{-1}(F) \) for every \((\Lambda, p)\)-closed subset \( F \) of \( Y \);
4. \( f^{-1}([A^{(\Lambda, p)}]) \subseteq f^{-1}(A) \) for every subset \( A \) of \( Y \);
5. \( f^{-1}(A_{(\Lambda, p)}) \subseteq f^{-1}(A_{(\Lambda, p)})\) for every subset \( A \) of \( Y \);
Theorem 20. For a function \( f : (X, \tau) \to (Y, \sigma) \), the following properties are equivalent:

1. \( f \) is weakly \((\Lambda, p)\)-continuous;
2. \([f^{-1}(F)](\Lambda, p) \subseteq f^{-1}(F)\) for every \((\Lambda, p)\)-closed subset \( F \) of \( Y \);
3. \([f^{-1}([U^{(\Lambda, p)}](\Lambda, p))](\Lambda, p) \subseteq f^{-1}(U^{(\Lambda, p)})\) for every \((\Lambda, p)\)-open subset \( U \) of \( Y \);

Proof. (1) \( \Rightarrow \) (2): It follows from Theorem 18.
(2) \( \Rightarrow \) (3): Let \( F \) be any \((\Lambda, p)\)-closed subset of \( Y \). Then, \( Y - F \) is \((\Lambda, p)\)-open, by (2), \( f^{-1}(Y - F) \subseteq [f^{-1}([Y - F](\Lambda, p))](\Lambda, p) = [f^{-1}(Y - F(\Lambda, p))](\Lambda, p) = X - [f^{-1}(F(\Lambda, p))](\Lambda, p) \).

Thus, \([f^{-1}(F(\Lambda, p))](\Lambda, p) \subseteq f^{-1}(F)\).
(3) \( \Rightarrow \) (4): Let \( A \) be any subset of \( Y \). Since \( A^{(\Lambda, p)} \) is \((\Lambda, p)\)-closed, by (3),
\[
[f^{-1}([A^{(\Lambda, p)}](\Lambda, p))](\Lambda, p) \subseteq f^{-1}(A^{(\Lambda, p)}).
\]

(4) \( \Rightarrow \) (5): Let \( A \) be any subset of \( Y \). By (4), we have
\[
f^{-1}(A_{(\Lambda, p)}) = X - f^{-1}([Y - A](\Lambda, p))
\subseteq X - [f^{-1}([Y - A](\Lambda, p))](\Lambda, p)
= [f^{-1}([A^{(\Lambda, p)}](\Lambda, p))](\Lambda, p).
\]

Thus, we get the result.

(5) \( \Rightarrow \) (6): Let \( U \) be any \((\Lambda, p)\)-open subset of \( Y \). Suppose that \( x \notin f^{-1}(U^{(\Lambda, p)}) \). Then, \( f(x) \notin U^{(\Lambda, p)} \) and so there exists a \((\Lambda, p)\)-open set \( V \) containing \( x \) such that \( U \cap V = \emptyset \).
Thus, \( U \cap V^{(\Lambda, p)} = \emptyset \). By (5), \( x \in f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda, p)})](\Lambda, p) \). There exists a \((\Lambda, p)\)-open set \( W \) containing \( x \) such that \( x \in W \subseteq f^{-1}(V^{(\Lambda, p)}) \). Since \( U \cap V^{(\Lambda, p)} = \emptyset \) and \( f(W) \subseteq V^{(\Lambda, p)} \), we have \( W \cap f^{-1}(U) = \emptyset \) and hence \( x \notin [f^{-1}(U)](\Lambda, p) \). This shows that \([f^{-1}(U)](\Lambda, p) \subseteq f^{-1}(U^{(\Lambda, p)})\).

(6) \( \Rightarrow \) (1): This is obvious from Theorem 18.

Definition 12. A subset \( A \) of a topological space \((X, \tau)\) is said to be:

(i) \((\Lambda, p)\)-open if \( A \subseteq [A^{(\Lambda, p)}](\Lambda, p) \);
(ii) \((\Lambda, p)\)-open if \( A \subseteq [A(\Lambda, p)](\Lambda, p) \);
(iii) \((\Lambda, p)\)-closed if \( A \subseteq [\Lambda(\Lambda, p)](\Lambda, p) \);
(iv) \((\Lambda, p)\)-closed if \( A = [A^{(\Lambda, p)}](\Lambda, p) \).

The complement of a \((\Lambda, p)\)-open (resp. \((\Lambda, p)\)-open, \((\Lambda, p)\)-closed, \((\Lambda, p)\)-closed, \((\Lambda, p)\)-closed) set is called \((\Lambda, p)\)-closed (resp. \((\Lambda, p)\)-closed, \((\Lambda, p)\)-closed, \((\Lambda, p)\)-closed).
Theorem 22. \[ f^{-1}([U_{\Lambda,p}]_{(\Lambda,p)})^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)}) \text{ for every } s(\Lambda,p)\text{-open subset } U \text{ of } Y. \]

Proof. (1) \( \Rightarrow \) (2): Let \( F \) be any \( r(\Lambda,p)\)-closed subset of \( Y \). Then, \( F_{(\Lambda,p)} \) is \( (\Lambda,p)\)-open, by Theorem 19, \[ f^{-1}(F_{(\Lambda,p)})^{(\Lambda,p)} \subseteq f^{-1}([F_{(\Lambda,p)}]^{(\Lambda,p)}). \] Since \( F \) is \( r(\Lambda,p)\)-closed, we have \[ f^{-1}([F_{(\Lambda,p)}]^{(\Lambda,p)}) = f^{-1}(F). \]

(2) \( \Rightarrow \) (3): Let \( U \) be any \( \beta(\Lambda,p)\)-open set. Then, \( U^{(\Lambda,p)} \subseteq [[U^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)} \subseteq U^{(\Lambda,p)} \) and hence \( U^{(\Lambda,p)} \) is \( r(\Lambda,p)\)-closed. By (2), \[ f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)})^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)}). \]

(3) \( \Rightarrow \) (4): The proof is obvious.

(4) \( \Rightarrow \) (1): Let \( U \) be any \( (\Lambda,p)\)-open subset of \( Y \). Then, we have \( U \) is \( s(\Lambda,p)\)-open and by (4), \[ f^{-1}(U)_{(\Lambda,p)} \subseteq [f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)}). \] Thus, \( f \) is weakly \( (\Lambda,p)\)-continuous by Theorem 19.

Theorem 21. For a function \( f : (X, \tau) \to (Y, \sigma) \), the following properties are equivalent:

(1) \( f \) is weakly \( (\Lambda,p)\)-continuous;

(2) \[ f^{-1}([U_{\Lambda,p}]^{(\Lambda,p)})^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)}) \text{ for every } p(\Lambda,p)\text{-open subset } U \text{ of } Y; \]

(3) \[ f^{-1}(U)^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)}) \text{ for every } p(\Lambda,p)\text{-open subset } U \text{ of } Y; \]

(4) \[ f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]^{(\Lambda,p)} \text{ for every } p(\Lambda,p)\text{-open subset } U \text{ of } Y. \]

Proof. (1) \( \Rightarrow \) (2): Let \( U \) be any \( p(\Lambda,p)\)-open subset of \( Y \). Then, we have \[ U^{(\Lambda,p)} = [[U^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)} \] and hence \( U^{(\Lambda,p)} \) is \( r(\Lambda,p)\)-closed. By Theorem 20, \[ f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)})^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)}). \]

(2) \( \Rightarrow \) (3): Let \( U \) be any \( p(\Lambda,p)\)-open subset of \( Y \). Then, \( U \subseteq [U^{(\Lambda,p)}]_{(\Lambda,p)} \) and by (2), we have \[ f^{-1}(U)^{(\Lambda,p)} \subseteq [f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)}). \]

(3) \( \Rightarrow \) (4): Let \( U \) be any \( p(\Lambda,p)\)-open subset of \( Y \). By (3), we have \[ f^{-1}(U) \subseteq f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)}) \]
\[ = X - f^{-1}([Y - U^{(\Lambda,p)}]^{(\Lambda,p)}) \]
\[ = X - [f^{-1}(Y - U^{(\Lambda,p)})]^{(\Lambda,p)} \]
\[ = [f^{-1}(U^{(\Lambda,p)})]^{(\Lambda,p)}. \]

(4) \( \Rightarrow \) (1): Since every \( (\Lambda,p)\)-open set is \( p(\Lambda,p)\)-open, by (4) and Theorem 19, it follows that \( f \) is weakly \( (\Lambda,p)\)-continuous.

Theorem 22. For a function \( f : (X, \tau) \to (Y, \sigma) \), the following properties are equivalent:

(1) \( f \) is weakly \( (\Lambda,p)\)-continuous;

(2) \[ f^{-1}([A^{(\Lambda,p)}]_{(\Lambda,p)})^{(\Lambda,p)} \subseteq f^{-1}(A^{(\Lambda,p)}) \text{ for every subset } A \text{ of } Y; \]
(3) \([f^{-1}(F_{\Lambda,p})]^{(\Lambda,p)} \subseteq f^{-1}(F)\) for every \(r(\Lambda,p)\)-closed subset \(F\) of \(Y\);

(4) \([f^{-1}(U)]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})\) for every \((\Lambda,p)\)-open subset \(U\) of \(Y\);

(5) \(f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{\Lambda,p}\) for every \((\Lambda,p)\)-open subset \(U\) of \(Y\);

(6) \([f^{-1}(U)]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})\) for every \(p(\Lambda,p)\)-open subset \(U\) of \(Y\);

(7) \(f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{\Lambda,p}\) for every \(p(\Lambda,p)\)-open subset \(U\) of \(Y\).

**Proof.** (1) \(\Rightarrow\) (2): Let \(A\) be any subset of \(Y\) and let \(x \in X - f^{-1}(A^{(\Lambda,p)})\). Then, \(f(x) \in Y - A^{(\Lambda,p)}\) and there exists a \((\Lambda,p)\)-open set \(U\) containing \(f(x)\) such that \(U \cap A = \emptyset\) and hence \(U^{(\Lambda,p)} \cap [A^{(\Lambda,p)}]_{\Lambda,p} = \emptyset\). Since \(f\) is weakly \((\Lambda,p)\)-continuous, there exists a \((\Lambda,p)\)-open set \(W\) containing \(x\) such that \(f(W) \subseteq U^{(\Lambda,p)}\). Then \(W \cap f^{-1}([A^{(\Lambda,p)}]_{\Lambda,p}) = \emptyset\) and hence \(x \in X - f^{-1}([A^{(\Lambda,p)}]_{\Lambda,p})\) which shows that

\[f^{-1}([A^{(\Lambda,p)}]_{\Lambda,p})_{\Lambda,p} \subseteq f^{-1}(A^{(\Lambda,p)}).\]

(2) \(\Rightarrow\) (3): Let \(F\) be any \((\Lambda,p)\)-closed subset of \(Y\). By (2), we have

\[f^{-1}(F_{\Lambda,p})_{\Lambda,p} = f^{-1}([F^{(\Lambda,p)}]_{\Lambda,p})_{\Lambda,p} \subseteq f^{-1}([F^{(\Lambda,p)}]_{\Lambda,p}) = f^{-1}(F).\]

(3) \(\Rightarrow\) (4): Let \(U\) be any \((\Lambda,p)\)-open subset of \(Y\). Since \(U^{(\Lambda,p)}\) is \((\Lambda,p)\)-closed and by (3), \([f^{-1}(U)]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})\).

(4) \(\Rightarrow\) (5): Let \(U\) be any \((\Lambda,p)\)-open subset of \(Y\). Since \(Y - U^{(\Lambda,p)}\) is \((\Lambda,p)\)-open, by (4), \(X - f^{-1}(U^{(\Lambda,p)})_{\Lambda,p} = f^{-1}([Y - U^{(\Lambda,p)}]_{\Lambda,p}) \subseteq f^{-1}([Y - U^{(\Lambda,p)}]_{\Lambda,p}) \subseteq X - f^{-1}(U)\) and hence \(f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{\Lambda,p}\).

(5) \(\Rightarrow\) (1): Let \(x \in X\) and let \(U\) be any \((\Lambda,p)\)-open subset of \(Y\) containing \(f(x)\). By (5), \(x \in f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{\Lambda,p}\). Put \(W = [f^{-1}(U^{(\Lambda,p)})]_{\Lambda,p}\). Thus, \(f(W) \subseteq U^{(\Lambda,p)}\) and hence \(f\) is weakly \((\Lambda,p)\)-continuous at \(x\). This shows that \(f\) is weakly \((\Lambda,p)\)-continuous.

(1) \(\Rightarrow\) (6): Let \(U\) be any \(p(\Lambda,p)\)-open subset of \(Y\) and let \(x \in X - f^{-1}(U^{(\Lambda,p)})\). There exists a \((\Lambda,p)\)-open set \(V\) containing \(f(x)\) such that \(V \cap U = \emptyset\) and hence \([V \cap U]^{(\Lambda,p)} = \emptyset\). Since \(U\) is \((\Lambda,p)\)-open, we have \(U \cap V^{(\Lambda,p)} \subseteq [U \cap V]^{(\Lambda,p)} = \emptyset\). Since \(f\) is weakly \((\Lambda,p)\)-continuous and \(V\) is a \((\Lambda,p)\)-open set containing \(f(x)\), there exists a \((\Lambda,p)\)-open set \(W\) containing \(x\) such that \(f(W) \subseteq V^{(\Lambda,p)}\). Then, \(f(W) \cap U = \emptyset\) and hence \(W \cap f^{-1}(U) = \emptyset\). Thus, \(x \in X - f^{-1}(U)^{(\Lambda,p)}\). This shows that \([f^{-1}(U)]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})\).

(6) \(\Rightarrow\) (7): Let \(U\) be any \(p(\Lambda,p)\)-open subset of \(Y\). Since \(Y - U^{(\Lambda,p)}\) is \((\Lambda,p)\)-open and by (6), we have

\[X - [f^{-1}(U^{(\Lambda,p)})]_{\Lambda,p} = [f^{-1}(Y - U^{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}([Y - U^{(\Lambda,p)}]^{(\Lambda,p)}) \subseteq X - f^{-1}(U)\]

and hence \(f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{\Lambda,p}\).

(7) \(\Rightarrow\) (1): Let \(x \in X\) and let \(V\) be any \((\Lambda,p)\)-open subset of \(Y\) containing \(f(x)\). Then, \(V\) is \(p(\Lambda,p)\)-open, by (7), \(x \in f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda,p)})]_{\Lambda,p}\). Put \(U = [f^{-1}(V^{(\Lambda,p)})]^{(\Lambda,p)}\). Then, \(f(U) \subseteq V^{(\Lambda,p)}\) and hence \(f\) is weakly \((\Lambda,p)\)-continuous at \(x\). Thus, \(f\) is weakly \((\Lambda,p)\)-continuous.
Definition 13. A topological space $(X, \tau)$ is said to be $\Lambda_p$-$T_2$ if, for any disjoint pair of points $x$ and $y$ in $X$, there exist $(\Lambda, p)$-open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 14. A topological space $(X, \tau)$ is said to be $\Lambda_p$-Urysohn if, for each distinct points $x, y \in X$, there exist $(\Lambda, p)$-open sets $U$ and $V$ containing $x$ and $y$, respectively, such that $U^{(\Lambda,p)} \cap V^{(\Lambda,p)} = \emptyset$.

Theorem 23. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a weakly $(\Lambda, p)$-continuous injection and $(Y, \sigma)$ is $\Lambda_p$-Urysohn, then $(X, \tau)$ is $\Lambda_p$-$T_2$.

Proof. Let $x, y$ be distinct points of $X$. Then, $f(x) \neq f(y)$. Since $(Y, \sigma)$ is $\Lambda_p$-Urysohn, there exist $(\Lambda, p)$-open sets $U$ and $V$ containing $f(x)$ and $f(y)$, respectively, such that $U^{(\Lambda,p)} \cap V^{(\Lambda,p)} = \emptyset$. Since $f$ is weakly $(\Lambda, p)$-continuous, there exist $(\Lambda, p)$-open sets $G$ and $W$ containing $x$ and $y$, respectively, such that $f(G) \subseteq U^{(\Lambda,p)}$ and $f(W) \subseteq V^{(\Lambda,p)}$. This shows that $G \cap W = \emptyset$. Thus, $(X, \tau)$ is $\Lambda_p$-$T_2$.

Theorem 24. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is weakly $(\Lambda, p)$-continuous and $(Y, \sigma)$ is $\Lambda_p$-$T_2$, then $f$ has $(\Lambda, p)$-closed point inverses.

Proof. Let $y \in Y$. We show that $f^{-1}(y) = \{ x \in X \mid f(x) = y \}$ is $(\Lambda, p)$-closed, or equivalently $G = \{ x \in X \mid f(x) \neq y \}$ is $(\Lambda, p)$-open. Let $x \in G$. Since $f(x) \neq y$ and $(Y, \sigma)$ is $\Lambda_p$-$T_2$, there exist disjoint $(\Lambda, p)$-open sets $U$ and $V$ such that $f(x) \in U$ and $y \in V$. Since $U \cap V = \emptyset$, $U^{(\Lambda,p)} \cap V = \emptyset$ and hence $y \notin U^{(\Lambda,p)}$. Since $f$ is weakly $(\Lambda, p)$-continuous, there exists a $(\Lambda, p)$-open set $W$ containing $x$ such that $f(W) \subseteq U^{(\Lambda,p)}$. Now, suppose that $W$ is not contained in $G$. Then, there exists a point $z \in W$ such that $f(z) = y$. Since $f(W) \subseteq U^{(\Lambda,p)}$, $y = f(z) \in U^{(\Lambda,p)}$. This is a contradiction. Therefore, $W \subseteq G$ and by Lemma 9, $G$ is $(\Lambda, p)$-open.

Theorem 25. Let $(X, \tau)$ be a topological space. If for each pair of distinct points $x_1$ and $x_2$ in $X$, there exists a function $f : (X, \tau) \rightarrow (Y, \sigma)$ such that

1. $(Y, \sigma)$ is $\Lambda_p$-Urysohn,
2. $f(x_1) \neq f(x_2)$ and
3. $f$ is weakly $(\Lambda, p)$-continuous at $x_1$ and $x_2$, then $(X, \tau)$ is $\Lambda_p$-$T_2$.

Proof. Let $x_1, x_2$ be any distinct points of $X$. By the hypothesis, there exists a function $f : (X, \tau) \rightarrow (Y, \sigma)$ which satisfies the conditions (1), (2) and (3). Let $y_i = f(x_i)$ for $i = 1, 2$. Then, $y_1 \neq y_2$. Since $(Y, \sigma)$ is $\Lambda_p$-Urysohn, there exist $(\Lambda, p)$-open sets $V_i$ in $(Y, \sigma)$ containing $y_i$ such that $V_1^{(\Lambda,p)} \cap V_2^{(\Lambda,p)} = \emptyset$. Since $f$ is weakly $(\Lambda, p)$-continuous at $x_1$ and $x_2$, for $i = 1, 2$, there exist $(\Lambda, p)$-open sets $U_i$ in $(X, \tau)$ containing $x_i$ such that $f(U_i) \subseteq V_i^{(\Lambda,p)}$. Hence, we get $U_1 \cap U_2 = \emptyset$. This shows that $(X, \tau)$ is $\Lambda_p$-$T_2$. 
Corollary 4. If \( f : (X, \tau) \to (Y, \sigma) \) is a weakly \((\Lambda, p)\)-continuous injection and \((Y, \sigma)\) is \(\Lambda_p\)-Urysohn, then \((X, \tau)\) is \(\Lambda_p\)-T2.

Definition 15. Let \( A \) be a subset of a topological space \((X, \tau)\). The \(\theta(\Lambda, p)\)-closure of \( A \), \( A^{\theta(\Lambda, p)} \), is defined as follows:

\[
A^{\theta(\Lambda, p)} = \{ x \in X \mid A \cap U^{(\Lambda, p)} \neq \emptyset \text{ for each } (\Lambda, p)\text{-open set } U \text{ containing } x \}.
\]

A subset \( A \) of a topological space \((X, \tau)\) is called \(\theta(\Lambda, p)\)-closed if \( A = A^{\theta(\Lambda, p)} \). The complement of a \(\theta(\Lambda, p)\)-closed set is said to be \(\theta(\Lambda, p)\)-open.

Lemma 13. Let \( A \) be a subset of a topological space \((X, \tau)\). Then, \( x \in A^{(\Lambda, p)} \) if and only if \( U \cap A \neq \emptyset \) for every \((\Lambda, p)\)-open set \( U \) containing \( x \).

Lemma 14. For a subset \( A \) of a topological space \((X, \tau)\), the following properties hold:

1. If \( A \) is \((\Lambda, p)\)-open in \((X, \tau)\), then \( A^{(\Lambda, p)} = A^{\theta(\Lambda, p)} \).

2. \( A^{\theta(\Lambda, p)} \) is \((\Lambda, p)\)-closed for every subset \( A \) of \( X \).

Proof. (1) In general, we have \( A^{(\Lambda, p)} \subseteq A^{\theta(\Lambda, p)} \). Suppose that \( x \notin A^{(\Lambda, p)} \). By Lemma 13, there exists a \((\Lambda, p)\)-open set \( U \) containing \( x \) such that \( U \cap A = \emptyset \); hence \( A \cap U^{(\Lambda, p)} = \emptyset \) since \( A \) is \((\Lambda, p)\)-open. Thus, \( x \notin A^{\theta(\Lambda, p)} \). Consequently, we obtain \( A^{\theta(\Lambda, p)} = A^{(\Lambda, p)} \).

(2) Let \( x \in X - A^{\theta(\Lambda, p)} \). Then, we have \( x \notin A^{\theta(\Lambda, p)} \). There exists a \((\Lambda, p)\)-open set \( U_x \) containing \( x \) such that \( A \cap U_x^{(\Lambda, p)} = \emptyset \) and hence \( U_x \cap A^{(\Lambda, p)} = \emptyset \). Therefore, \( x \in U_x \subseteq X - A^{\theta(\Lambda, p)} \). Thus, \( X - A^{\theta(\Lambda, p)} = \bigcup_{x \in X - A^{\theta(\Lambda, p)}} U_x \) and hence \( X - A^{\theta(\Lambda, p)} \) is \((\Lambda, p)\)-open. This shows that \( A^{\theta(\Lambda, p)} \) is \((\Lambda, p)\)-closed.

Theorem 26. For a function \( f : (X, \tau) \to (Y, \sigma) \), the following properties are equivalent:

1. \( f \) is weakly \((\Lambda, p)\)-continuous;

2. \( f(A^{(\Lambda, p)}) \subseteq [f(A)]^{\theta(\Lambda, p)} \) for every subset \( A \) of \( X \);

3. \([f^{-1}(B)]^{(\Lambda, p)} \subseteq f^{-1}(B^{\theta(\Lambda, p)}) \) for every subset \( B \) of \( Y \);

4. \([f^{-1}(V)]^{(\Lambda, p)} \subseteq f^{-1}(V^{\theta(\Lambda, p)}) \) for every \((\Lambda, p)\)-open subset \( V \) of \( Y \).

Proof. (1) \(\Rightarrow\) (2): Let \( A \) be any subset of \( X \). Let \( x \in A^{(\Lambda, p)} \) and \( V \) be any \((\Lambda, p)\)-open set containing \( f(x) \). Since \( f \) is weakly \((\Lambda, p)\)-continuous, there exists a \((\Lambda, p)\)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V^{(\Lambda, p)} \). Since \( x \in A^{(\Lambda, p)} \), we have \( U \cap A \neq \emptyset \). It follows that \( \emptyset \neq f(U) \cap f(A) \subseteq V^{(\Lambda, p)} \cap f(A) \) and hence \( V^{(\Lambda, p)} \cap f(A) \neq \emptyset \). Thus, \( f(x) \in [f(A)]^{\theta(\Lambda, p)} \). Consequently, we obtain \( f(A^{(\Lambda, p)}) \subseteq [f(A)]^{\theta(\Lambda, p)} \).

(2) \(\Rightarrow\) (3): Let \( B \) be any subset of \( Y \). By (2), we have

\[
f([f^{-1}(B)]^{(\Lambda, p)}) \subseteq [f(f^{-1}(B))]^{\theta(\Lambda, p)} \subseteq B^{\theta(\Lambda, p)}
\]

and hence \([f^{-1}(B)]^{(\Lambda, p)} \subseteq f^{-1}(B^{\theta(\Lambda, p)}) \).
(3) \( \Rightarrow (4) \): Let \( V \) be any \((\Lambda,p)\)-open subset of \( Y \). By Lemma 14, \( V^{(\Lambda,p)} = V^{\theta(\Lambda,p)} \). Thus, the proof is obvious.

(4) \( \Rightarrow (1) \): Let \( V \) be any \((\Lambda,p)\)-open set containing \( f(x) \). Since \( V \cap [Y - V^{(\Lambda,p)}] = \emptyset \), we have \( f(x) \notin [Y - V^{(\Lambda,p)}](\Lambda,p) \) and hence \( x \notin f^{-1}([Y - V^{(\Lambda,p)}](\Lambda,p)) \). Since \( Y - V^{(\Lambda,p)} \) is \((\Lambda,p)\)-open, by (4), \( x \notin [f^{-1}([Y - V^{(\Lambda,p)}]])^{(\Lambda,p)} \) and there exists a \((\Lambda,p)\)-open set \( U \) containing \( x \) such that \( U \cap f^{-1}(Y - V^{(\Lambda,p)}) = \emptyset \); hence \( f(U) \cap [Y - V^{(\Lambda,p)}] = \emptyset \). This shows that \( f(U) \subseteq V^{(\Lambda,p)} \). Thus, \( f \) is weakly \((\Lambda,p)\)-continuous.

**Definition 16.** A topological space \((X, \tau)\) is said to be \( \Lambda_p \)-regular if, for each \((\Lambda,p)\)-closed set \( F \) and each \( x \notin F \), there exist disjoint \((\Lambda,p)\)-open sets \( U \) and \( V \) such that \( x \in U \) and \( F \subseteq V \).

**Lemma 15.** A topological space \((X, \tau)\) is \( \Lambda_p \)-regular if and only if for each \( x \in X \) and each \((\Lambda,p)\)-open set \( U \) containing \( x \), there exists a \((\Lambda,p)\)-open set \( V \) such that \( x \in V \subseteq V^{(\Lambda,p)} \subseteq U \).

**Proof.** Let \( x \in X \) and let \( U \) be a \((\Lambda,p)\)-open set containing \( x \). Then, \( x \notin X - U \) and \( X - U \) is \((\Lambda,p)\)-closed. There exist disjoint \((\Lambda,p)\)-open sets \( V \) and \( W \) such that \( x \in V \) and \( x \notin W \subseteq U \). Thus, \( V \subseteq X - W \subseteq U \). Since \( X - W \) is \((\Lambda,p)\)-closed, we have \( V^{(\Lambda,p)} \subseteq X - W \subseteq U \) and hence \( x \in V \subseteq V^{(\Lambda,p)} \subseteq U \).

Conversely, let \( F \) be a \((\Lambda,p)\)-closed set and let \( x \notin F \). Then, \( x \in X - F \). Since \( X - F \) is \((\Lambda,p)\)-open, there exists a \((\Lambda,p)\)-open set \( V \) such that \( x \in V \subseteq V^{(\Lambda,p)} \subseteq X - F \) and hence \( F \subseteq X - V^{(\Lambda,p)} \). This shows that \((X, \tau)\) is \( \Lambda_p \)-regular.

**Lemma 16.** Let \((X, \tau)\) be a \( \Lambda_p \)-regular space. Then, the following properties hold:

1. \( A^{(\Lambda,p)} = A^{\theta(\Lambda,p)} \) for every subset \( A \) of \( X \).
2. Every \((\Lambda,p)\)-open set is \( \theta(\Lambda,p) \)-open.

**Proof.** (1) In general, we have \( A^{(\Lambda,p)} \subseteq A^{\theta(\Lambda,p)} \) for every subset \( A \) of \( X \). Next, we show that \( A^{\theta(\Lambda,p)} \subseteq A^{(\Lambda,p)} \). Let \( x \in A^{\theta(\Lambda,p)} \) and \( U \) be any \((\Lambda,p)\)-open set containing \( x \). By Lemma 15, there exists a \((\Lambda,p)\)-open set \( V \) such that \( x \in V \subseteq V^{(\Lambda,p)} \subseteq U \). Since \( x \in A^{\theta(\Lambda,p)} \), it follows that \( A \cap V^{(\Lambda,p)} \neq \emptyset \) and hence \( U \cap A \neq \emptyset \). Thus, \( x \in A^{(\Lambda,p)} \). Consequently, we obtain \( A^{\theta(\Lambda,p)} \subseteq A^{(\Lambda,p)} \).

(2) Let \( V \) be a \((\Lambda,p)\)-open set. By (1), we have \( X - V = [X - V]^{(\Lambda,p)} = [X - V]^{\theta(\Lambda,p)} \) and hence \( X - V \) is \( \theta(\Lambda,p) \)-closed. Thus, \( V \) is \( \theta(\Lambda,p) \)-open.

**Theorem 27.** Let \((Y, \sigma)\) be a \( \Lambda_p \)-regular space. For a function \( f : (X, \tau) \rightarrow (Y, \sigma) \), the following properties are equivalent:

1. \( f^{-1}(B^{\theta(\Lambda,p)}) \) is \( \theta(\Lambda,p) \)-closed in \( X \) for every subset \( B \) of \( Y \);
2. \( f \) is weakly \((\Lambda,p)\)-continuous;
3. \( f^{-1}(F) \) is \((\Lambda,p)\)-closed in \( X \) for every \( \theta(\Lambda,p) \)-closed subset \( F \) of \( Y \);
(4) \( f^{-1}(V) \) is \((\Lambda, p)\)-open in \( X \) for every \( \theta(\Lambda, p) \)-open subset \( V \) of \( Y \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( B \) be any subset of \( Y \). Then,

\[
[f^{-1}(B)]^{(\Lambda, p)} \subseteq [f^{-1}(B^{\theta(\Lambda, p)})]^{(\Lambda, p)} = f^{-1}(B^{\theta(\Lambda, p)}),
\]

by Theorem 26, \( f \) is weakly \((\Lambda, p)\)-continuous.

(2) \( \Rightarrow \) (3): Let \( F \) be any \( \theta(\Lambda, p) \)-closed subset of \( Y \). By Theorem 26, we have

\[
[f^{-1}(F)]^{(\Lambda, p)} \subseteq f^{-1}(F^{\theta(\Lambda, p)}) = f^{-1}(F)
\]

and hence \( f^{-1}(F) \) is \((\Lambda, p)\)-closed in \( X \).

(3) \( \Rightarrow \) (4): Let \( V \) be any \( \theta(\Lambda, p) \)-open subset of \( Y \). Then, \( Y - V \) is \( \theta(\Lambda, p) \)-closed, by (3), \( X - f^{-1}(V) = F^{-1}(Y - V) \) is \((\Lambda, p)\)-closed in \( X \). Thus, \( f^{-1}(V) \) is \((\Lambda, p)\)-open.

(4) \( \Rightarrow \) (1): Let \( B \) be any subset of \( Y \). By Lemma 14, \( B^{\theta(\Lambda, p)} \) is \((\Lambda, p)\)-closed in \( Y \) and by Lemma 16, \( Y - B^{\theta(\Lambda, p)} \) is \( \theta(\Lambda, p) \)-open in \( Y \). Thus, by (4), we have

\[
X - f^{-1}(B^{\theta(\Lambda, p)}) = f^{-1}(Y - B^{\theta(\Lambda, p)})
\]

is \((\Lambda, p)\)-open in \( X \) and hence \( f^{-1}(B^{\theta(\Lambda, p)}) \) is \( \theta(\Lambda, p) \)-closed.

7. Conclusion

Closedness and openness are fundamental with respect to the investigation of general topological spaces. Various types of generalizations of closed sets and open sets in topological spaces have been researched by many mathematicians. This article is devoted to introducing and discussing the concepts of \((\Lambda, p)\)-closed sets and \((\Lambda, p)\)-open sets. Moreover, some characterizations of \( \Lambda_p-R_0 \) spaces are explored. Additionally, several characterizations of weakly \((\Lambda, p)\)-continuous functions are established. The ideas and results of this article may motivate further research.

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References


