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e^*-Essential Small Submodules and e^*-Hollow Modules

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Abstract. The purpose of this paper is to introduce the concepts of e^*-small essential submodules, e^*-radical submodules, and e^*-hollow modules as a generalizations of the concepts of small submodules, radical submodules, and hollow modules, respectively. We will prove some properties of these concepts.

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Key Words and Phrases: e^*-Small essential submodule, Small submodule, e^*-Radical submodule, e^*-Hollow module, Hollow module

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1. Introduction

Let R be a ring with identity, M is a right R-module and $E(M)$ the injective hull of M. A submodule N of M is called a small submodule of M denoted $(N \ll M)$ if for any submodule A of M such that $M = N + A$, we have $A = M$ [6] Recall that a submodule A of R-module B is called essential in B if every nonzero submodule of B has nonzero intersection with A [6], [4] and [5].

Oscan in [2] introduced the concept of cosingular submodule as follows: $Z^*(M) = \{m \in M | mR \ll E(M)\}$. An R-module M is called cosingular if $Z^*(M) = M$. Baanoon and Khaild in [1] introduced a type of submodule which called e^*-essential as follows. A submodule A of M is said to be e^*-essential in M if $A \cap B \neq 0$ for each nonzero cosingular submodule B of M. Denoted by $A \leq_{e^*} M$.

As in [7], we will used e^*-essential submodule that appeared in [1], to present a new generalization of a small sumodule namely e^*-essential small submodule. e^*-essential small submodules leads us to introduce e^*-hollow module as a generalization of hollow modules. In this paper main properties of these concepts are proved.

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2. e^*-Essential Small Submodules

In this section, so one generalization of small submodules are introduced with some properties. Recall that a submodule A of M is said to be e^*-essential denoted by $A \leq_{e^*} M$ if $A \cap B \neq 0$ for each nonzero cosingular submodule B of M [1].
The following gives some properties of e^*-essential submodules.

Lemma 1. *[1] Let M be an R-module.*

1. *If $A \leq B \leq M$, then $A \leq_{e^*} M$ if and only if $A \leq_{e^*} B \leq_{e^*} M$*
2. *Let $f : M \to M'$ be an R-homomorphism. If $A \leq_{e^*} M'$, then $f^{-1}(A) \leq_{e^*} M$.*
3. *If $A \leq_{e^*} B \leq M$ and $A' \leq_{e^*} B' \leq M$, then $A \cap A' \leq_{e^*} B \cap B'$.*

Definition 1. *Let M be an R-module, a submodule A of M is said to be e*-essential small in M denoted by $A \ll_{e^*} M$, if whenever $M = A + B$ (where B is an e*-essential submodule of M) implies that $M = B$.*

Examples and Remarks 1.

1. *Every small submodule is e*-essential small submodule, but the converse need not to be true in general. For example, in $\mathbb{Z}_6$ as a $\mathbb{Z}$-module, the only e*-essential submodule is $\mathbb{Z}_6$ [1]. So, every submodule of $\mathbb{Z}_6$ is e*-essential small. while $\langle\bar{2}\rangle$ is not a small submodule since $\langle\bar{2}\rangle + \langle\bar{3}\rangle = \mathbb{Z}_6$ but $\langle\bar{3}\rangle \neq \mathbb{Z}_6$.*
2. *Consider $\mathbb{Z}_4$ as a $\mathbb{Z}$-module, the submodles $\mathbb{Z}_4$ and $\langle\bar{2}\rangle$ are cosingular[2] and e*-essential, hence $\langle\bar{2}\rangle$ is an e*-essential small submodule.*
3. *Consider $\mathbb{Z}_6$ as a $\mathbb{Z}_6$-module. In this module every submodule is e*-essential [1], so $\langle\bar{2}\rangle + \langle\bar{3}\rangle = \mathbb{Z}_6$ but $\langle\bar{3}\rangle \neq \mathbb{Z}_6$. Therefore, $\langle\bar{2}\rangle$ is not e*-essential small submodule. Thus, e*-essential submodule need not to be e*-essential small.*
4. *Let M be an R-module, then:*
   - *The trivial submodule is always e*-essential small in M.*
   - *$M \ll_{e^*} M$ if and only if M is a simple module.*

In the following, we introduce the basic properties of e^*-essential small submodules.

Proposition 1. *Let M be an R-module, N a submodule of M and K a submodule of N.*

1. *If $N \ll_{e^*} M$, then $K \ll_{e^*} M$ and $\frac{N}{K} \ll_{e^*} \frac{M}{K}$.*
2. *If $K \ll_{e^*} N$, then $K \ll_{e^*} M$.*

Proof.

1. *Let L be an e^*-essential submodule of M such that $K + L = M$. Since $K \leq N$ and $N \ll_{e^*} M$, $L = M$. Thus $K \ll_{e^*} M$.*
   Now, to prove that $\frac{N}{K} \ll_{e^*} \frac{M}{K}$, let $\frac{M}{K} = \frac{A}{K} + \frac{N}{K}$ where $\frac{A}{K}$ is an e^*-essential submodule of $\frac{M}{K}$, hence A is e^*-essential submodule of M by lemma 1, and $M = A + N$, since $N \ll_{e^*} M$. Thus, $M = A$ implies that $\frac{A}{K} = \frac{M}{K}$. Therefore, $\frac{N}{K} \ll_{e^*} \frac{M}{K}$.

2. *Let L be an e^*-essential submodule of M such that $K+L = M$. Hence, $L \cap N \leq_{e^*} N$ by lemma 1, and $K+(L\cap N) = N\cap(K+L) = N$, since $K \ll_{e^*} N$. Thus, $L\cap N = N$, $N \leq L$. So $K \leq L$. Hence, $L = K + L = M$. Therefore, $K \ll_{e^*} M$.*

Proposition 2. *Let M be an R-module, K and N submodules of M such that $K \leq N$. If $K \ll_{e^*} M$ and N is a direct summand e^*-essential submodule of M , then $K \ll_{e^*} N$.*

Proof. *Let L be an e^*-essential submodule of N such that $K + L = N$. Since N is a direct summand of M, there exists a submodule N' of M such that $M = N \oplus N'$ and $M = (K + L) \oplus N' = K + (L + N')$. Since $L \leq_{e^*} N \leq_{e^*} M$, by lemma 1, this implies that $L \leq_{e^*} M$ and since $L \leq L + N' \leq M$ also by the same lemma, this implies that $L + N' leq_{e^*} M$. $K \ll_{e^*} N$ implies that $L + N' = M$. Now, for any $n \in N$, there exists $l \in L$ and $n' \in N'$ such that $n = l + n'$, so $n - l = n' \in N \cap N' = 0$, hence $n = l$ and $N \leq L$. Therefore, $N = L$ and $K \ll_{e^*} N$*

The following proposition shows that, the homorphic image of an e^*-essential small submodule is e^*-essential small submodule.

Proposition 3. *If $K \ll_{e^*} M$ and $f : M \to N$ is an R-homomorphism, then $f(K) \ll_{e^*} N$.*

Proof. *Let L be an e^*-essential submodule of N such that $f(K)+L = N$. hence $f^{-1}(L)$ is e^*-essential in M by lemma 1. Let $m \in M$, hence $f(m) \in N = f(K)+L$, so there exist $k \in K$ and $l \in L$ such that $f(m) = f(k) + l$. Thus, $l = f(m - k)$ so, $m - k \in f^{-1}(L)$ and $m = m - k + k \in K + f^{-1}(L)$. Hence, $K + f^{-1}(L) = M$ since $K \ll_{e^*} M$. Thus $f^{-1}(L) = M$ and $f(M) = f(f^{-1}(L)) = f(L) \cap L$, hence $f(M) \subseteq L$ i.e. $f(K) \subseteq L$. Therefore, $L = f(K) + L = N$ and $f(K) \ll_{e^*} N$.*

The sum of e^*-essential small submodules is e^*-essential small submodule as the following proposition shows.

Proposition 4. *Let N and L be submodules of an R-module M. Then $N + L \ll_{e^*} M$ if and only if $N \ll_{e^*} M$ and $L \ll_{e^*} M$.*

Proof. *$\Rightarrow$) Let K be e^*-essential in M such that $K + N = M$. So, $K + N + L = M$. By assumption, $K = M$ and $N \ll_{e^*} M$. Similarly for $L \ll_{e^*} M$.*
$\Leftarrow$) Let A be e^*-essential in M such that $N + L + A = M$, $M = N + (L + A) = M$, since $A \leq A + L \leq M$ and $A \leq_{e^*} M$ by lemma 1, this implies that $L + A = M$. Now, $N \ll_{e^*} M$ implies that $L + A + M$ and $L \ll_{e^*} M$ implies that $A = M$. Therefore, $N + L \ll_{e^*} M$.

The following corollary follows from Proposition 3 and Proposition 4.

Corollary 1. *Let $M = M_1 \oplus M_2$ and K_i a submodule of M_i, $i = 1, 2$. Then $K_i \ll_{e^*} M_i$, $i = 1, 2$ if and only if $K_1 \oplus K_2 \ll_{e^*} M_1 \oplus M_2$.*

3. e^*- Radical Submodule

Recall that for an R-module M, if M has maximal submodule, then the radical of M is the intersection of all maximal submodules of M dented by $Rad(M)$ [6]. We generalize this concept as the following:

Definition 2. *Let M be R-module. Then the intersection of all e^*-essential maximal submodule of M is called e^*- radical submodule denoted by $\underset{e^*}{Rad}(M)$.*

If M has no e^*-essential maximal submodule, then $\underset{e^*}{Rad}(M) = M$.

The following proposition gives the relationship between e^*-essential small submodules and e^*-essential maximal submodules.

Proposition 5. *Let M be an R-module and $m \in M$, then $\langle m \rangle$ is not e^*-essential small if and only if there exists an e^*-essential maximal submodule N of M with $m \notin N$.*

Proof. $\Rightarrow$) *Consider the set*
$\Gamma = \{B | B$ *is a proper e^*-essential submodule of M and* $\langle m \rangle + B = M\}$. *Since $\langle m \rangle$ is not e^*-essential small, there exists $B' \leq_{e^*} M$ such that $\langle m \rangle + B' = M$ and $B' \neq M$, hence $\Gamma \neq \phi$. Let $\{C_\alpha\}_{\alpha \in \lambda}$ be a chain in Γ, hence $\cup_{\alpha \in \lambda} C_\alpha$ is a proper submodule and since $C_\alpha \leq \cup_{\alpha \in \lambda} C_\alpha \leq M$ for each $\alpha \in \lambda$ with $C_\alpha \leq_{e^*} M$, then $\cup_{\alpha \in \lambda} C_\alpha \leq_{e^*} M$ with $\langle m \rangle + \cup_{\alpha \in \lambda} C_\alpha = M$. So, by Zorn's lemma, Γ has a maximal element say B_0. We claim that B_0 is maximal in M. Otherwise if $B_0 \gneqq C \leqslant M$, then $M = B_0 + \langle m \rangle \leq C + \langle m \rangle \leq M$. Thus, $\langle m \rangle + C = M$ and since $B_0 \leqslant_{e^*} M$, hence $C \leqslant_{e^*} M$. Now, if $C \neq M$, hence $C \in \Gamma$ which is a contradiction. Thus, $C = M$. So $B_0 \leqslant_{e^*} M$ which is maximal in M. Now, if $m \in B_0$, then $\langle m \rangle \subseteq B_0$ and since $\langle m \rangle + B_0 = M$, we have $B_0 = M$ which is a contradiction. So, $m \notin B_0$ i.e. there exists an e^*-essential maximal submodule of M that does not contain m.*

$\Leftarrow$) *To show that $\langle x \rangle$ is not e^*-essential small in M. If not, then as $x \notin N$ and N is as maximal submodule we have $\langle x \rangle + N = M$. Now, $\langle x \rangle \ll_{e^*} M$ and $N \leq_{e^*} M$ implies that $N = M$ which is a contraindication. Therefore, $\langle x \rangle$ is not e^*-essential small submodule of M.*

Examples and Remarks 2.

1. *Let M be an R-module, then $Rad(M) \leq \underset{e^*}{Rad}(M)$. But the converse need not to be true in general. For example: Consider $\mathbb{Z}_6$ as a $\mathbb{Z}$-module, $Rad(\mathbb{Z}_6) = \{\overline{0}\}$. When $\underset{e^*}{Rad}(\mathbb{Z}_6) = \mathbb{Z}_6$, since the maximal submodules of $\mathbb{Z}_6$ are $\langle \overline{2} \rangle$ and $\langle \overline{3} \rangle$ while the only e^*-essntial submodule is $\mathbb{Z}_6$ [1].*

2. *In $\mathbb{Z}_4$ as a $\mathbb{Z}$-module $\underset{e^*}{Rad}(\mathbb{Z}_4) = \{\overline{0}, \overline{2}\}$. Since all submodules of $\mathbb{Z}_4$ are: $\{\overline{0}\}, \{\overline{0}, \overline{2}\}$ and $\mathbb{Z}_4$. Hence, the e^*-essential submodule of $\mathbb{Z}_4$ are: $\{\overline{0}, \overline{2}\}$ and $\mathbb{Z}_4$. Thus, the only e^*-essential maximal submodule is $\{\overline{0}, \overline{2}\}$.*

Theorem 1. *Let M be an R-module, then* $\underset{e^*}{Rad}(M) = \sum\limits_{N \ll_{e^*} M} N$.

Proof. *Let* $m \notin \underset{e^*}{Rad}(M)$ *then there exists an* e^**-essential maximal* N *of* M *such that* $m \notin N$. *Hence by proposition 5, we have that* $\langle m \rangle$ *is not* e^**-essential small. Thus,* $m \notin \sum\{N | N \ll_{e^*} M\}$. *Therefore,* $\sum\{N | N \ll_{e^*} M\} \subseteq \underset{e^*}{Rad}(M)$.
Now, let $x \in \underset{e^*}{Rad}(M)$ *and* $x \notin \sum\{N | N \ll_{e^*} M\}$. *Hence,* $\langle x \rangle$ *is not* e^**-essential small and by proposition 5, there exists an* e^**-essential maximal submodule* K *of* M *such that* $x \notin K$ *but* $\underset{e^*}{Rad}(M) \leq K$ *which is a contradiction. Thus,* $x \in \sum\{N | N \ll_{e^*} M\}$ *and* $\underset{e^*}{Rad}(M) \leq \sum\{N | N \ll_{e^*} M\}$. *Therefore,* $\underset{e^*}{Rad}(M) = \sum\{N | N \ll_{e^*} M\}$.

Proposition 6. *If* $f : M \to M'$ *is an* R*-homomorphism, then* $f(\underset{e^*}{Rad}(M)) \leq \underset{e^*}{Rad}(M')$. *In particular,* $\underset{e^*}{Rad}(M)$ *is a fully invariant submodule of* M.

Proof. *By Therorm 1,* $\underset{e^*}{Rad}(M) = \sum\limits_{K \ll_{e^*} M} K$. *Hence,* $f(\underset{e^*}{Rad}(M)) = \sum\limits_{K \ll_{e^*} M} f(K)$. *By Proposition 3, Since* $K \ll_{e^*} M$ *then* $f(K) \ll_{e^*} M'$. *Thus,* $\sum\limits_{K \ll_{e^*} M} f(K) \leq \underset{e^*}{Rad}(M')$ *and* $f(\underset{e^*}{Rad}(M)) \leq \underset{e^*}{Rad}(M')$.

Corollary 2. *Let M be an R-module and N be a submodule of M, then:*

1. $\underset{e^*}{Rad}(N) \leq \underset{e^*}{Rad}(M)$.
2. $\dfrac{\underset{e^*}{Rad}(M)}{N} \leq \underset{e^*}{Rad}(\frac{M}{N})$.

4. e^*-Hollow Modules

Recall that a non-zero R-module M is called a hollow module if every proper submodule of M is small in M[3]. In this section we introduce e^*-hollow modules as a generalization of hollow modules and investigate some of their properties.

Definition 3. *A non zero R-module M is called e^*-hollow module if every proper submodule of M is e^*-essential small in M.*

Examples and Remarks 3.

1. *Every hollow module is e^*-hollow module. But the converse need not to be true in general. For example: in $\mathbb{Z}_6$ as $\mathbb{Z}$-module every proper submodule is e^*-essential small, hence $\mathbb{Z}_6$ is e^*-hollow module, but it is not hollow, since $\langle 2 \rangle$ is not small submodule.*

2. *Consider* $\mathbb{Z}_6$ *as a* $\mathbb{Z}_6$*-module. since* $\langle\overline{2}\rangle$ *is not an e***-essential small submodule. Thus,* $\mathbb{Z}_6$ *is not e***-hollow module.*

3. *The direct sum of two e***-hollow modules need not to be e***-hollow. For example:* $\mathbb{Z}_4$ *as a* $\mathbb{Z}$*-module is e***-hollow since* $\langle\overline{2}\rangle$ *and* $\mathbb{Z}_4$ *are the only e***-essntial submodules. So all the proper submodules are e***-essntial small. Also,* $\mathbb{Z}_3$ *as a* $\mathbb{Z}$*-module is e***-hollow since the only e***-essntial submodule is* $\mathbb{Z}_3$ *it self. But* $\mathbb{Z}_4 \oplus \mathbb{Z}_3 \simeq \mathbb{Z}_{12}$ *and* $\mathbb{Z}_{12}$ *is not e***-hollow. since the only e***-essntial submodule of* $\mathbb{Z}_{12}$ *are* $\langle\overline{2}\rangle$ *and* $\mathbb{Z}_{12}$ *with* $\langle\overline{3}\rangle + \langle\overline{2}\rangle = \mathbb{Z}_{12}$ *but* $\langle\overline{2}\rangle \neq \mathbb{Z}_{12}$.

4. *Any R-module which has no proper e***-essential submodule is e***-hollow.*

Proposition 7. *The epimorphic image of an e***-hollow module is e***-hollow.*

Proof. *Let* $f : M \to M'$ *be an R-epimorphism, with* M *an e***-hollow module. Let* B *be a proper submodule of* M'*. Hence* $f^{-1}(B)$ *is a proper submodule of* M*. since if not,* $f^{-1}(B) = M$ *implies that* $ff^{-1}(B) = B = M'$ *which is a contradiction. Since* M *is e***-hollow then* $f^{-1}(B)$ *is e***-essential small. By proposition 3,* $ff^{-1}(B) = B$ *is an e***-essential small submodule. Therefore,* M' *is e***-hollow.*

Corollary 3. *If* M *is an e***-hollow module, then* $\frac{M}{N}$ *is e***-hollow for any proper submodule* N *of* M.

Remark 1. *The converse of the above corollary need not to be true in general. For example: Consider* $\mathbb{Z}_{24}$ *as a* $\mathbb{Z}$*-module which is not e***-hollow. Since every submodule of* $\mathbb{Z}_{24}$ *is cosingular then the only e***-essntial submodule of* $\mathbb{Z}_{24}$ *are* $\langle\overline{0}\rangle, \langle\overline{2}\rangle, \langle\overline{4}\rangle$*, and* $\mathbb{Z}_{24}$*. Since* $\langle\overline{3}\rangle + \langle\overline{2}\rangle = \mathbb{Z}_{24}$ *and* $\langle\overline{2}\rangle \neq \mathbb{Z}_{24}$ *we have that* $\langle\overline{3}\rangle$ *is not e***-essential small. But* $\frac{\mathbb{Z}_{24}}{\langle\overline{4}\rangle}$ *is e***-hollow module since* $\frac{\mathbb{Z}_{24}}{\langle\overline{4}\rangle} \simeq \mathbb{Z}_4$.

The following proposition shows that under certain conditions the converse of corollary 3 is true. Recall that a submodule A of a module M is called e^*-closed if A has no proper e^*-essential extension inside M [1].

Lemma 2. *[1] If* $B \leq K$ *are submodules of an R-module* M *such that* B *is e***-closed in* M *and* K *is e***-essential in* M*, then* $\dfrac{K}{B} \leq_{e^*} \dfrac{M}{B}$.

Proposition 8. *Let* M *be an R-module. If* $\frac{M}{N}$ *is e***-hollow with* N *is a proper small e***-closed submodule, then* M *is e***-hollow.*

Proof. *Let* L *be a proper submodule of* M *and* K *an e***-essential submodule of* M *such that* $L + K = M$*. Then* $\frac{M}{N} = \frac{L+N}{N} + \frac{K+N}{N}$ *implies that* $M \neq L + N$*. For if* $M = L + N$ *with* N *a small submodule of* M *i.e.* $M = L$ *which is a contradiction. Thus,* $\frac{M}{N} \neq \frac{L+N}{N}$*. Since,* $N \leq_{e^*} M$ *then by lemma 1,* $N \leq_{ce^*} K + N \leq_{e^*} M$*, and by lemma 2,* $\frac{K+N}{N} \leq_{e^*} \frac{M}{N}$*. Since* $\frac{M}{N}$ *is e***-hollow, then* $\frac{K+N}{N} = \frac{M}{N}$*, and* $M = K + N$ *because* $N \ll M$*. Therefore,* $K = M$ *and* M *is e***-hollow.*

Proposition 9. *Let M be an e^*-hollow module, if M has proper a e^*-essential submodule N and $\frac{M}{N}$ is finitely generated then M is finitely generated.*

Proof. *Since $\frac{M}{N}$ is finitely generated there are $x_1, x_2, ..., x_n \in M$ such that $\frac{M}{N} = \langle x_1 + N, x_2 + N, ..., x_n + N\rangle$. We claim that $M = \langle x_1, x_2, ..., x_n\rangle$. Let $m \in M$, hence $m + N \in \frac{M}{N}$ and $m + N = (x_1r_1 + x_2r_2 + ... + x_nr_n) + N$ for some $r_1, r_2, ..., r_n \in R$. So, $m - (x_1r_1 + x_2r_2 + ... + x_nr_n) \in N$. Let $n = m - (x_1r_1 + x_2r_2 + ... + x_nr_n)$ where $n \in N$, hence $m = (x_1r_1 + x_2r_2 + ... + x_nr_n) + n$. Thus, $M = \langle x_1, x_2, ..., x_n\rangle + N$. If $\langle x_1, x_2, ..., x_n\rangle \neq M$, then $\langle x_1, x_2, ..., x_n\rangle \ll_{e^*} M$, since $N \lneqq_{e^*} M$. Hence, $M = N$ which is a contradiction. Therefore, $M = \langle x_1, x_2, ..., x_n\rangle$.*

The following proposition is a characterizes e^*-hollow modules.

Proposition 10. *An R-module M is e^*-hollow module if and only if every proper e^*-essential submodule of M is small in M.*

Proof. $\Rightarrow$) *Clear*

$\Leftarrow$) *Let A be a proper submodule of M and B an e^*-essential submodule of M such that $A+B = M$. If $B \neq M$ then B is a proper e^*-essential submodule of M and by assumption B is small. Hence $A = M$ which is a contradiction. Thus, $B = M$ and A is e^*-essential small in M. Therefore, M is e^*-hollow.*

Definition 4. *Let M be an R-module. A submodule A of M is called e^*-coclosed if whenever $B \leq A$, $\frac{A}{B} \ll_{e^*} \frac{M}{B}$, implies that $A = B$.*

One may ask a question. Is any submodule of an e^*-hollow module e^*-hollow?
The following proportion gives a partial answer.

Proposition 11. *Let M be an e^*-hollow R-module.*

1. *An e^*-essential direct summand of an e^*-hollow module is e^*-hollow.*

2. *An e^*-coclosed submodule of an e^*-hollow is e^*-hollow.*

Proof.

1. *Let A be an e^*-essential direct summand of M and B a proper submodule of A with $L \leq_{e^*} A$ such that $B + L = A$. Since $L \leq_{e^*} A \leq_{e^*} M$, then by lemma 1, $L \leq_{e^*} M$. Also, since A is a direct summand of M, there is a submodule $A^{'}$ of M such that $A \oplus A^{'} = M$. Thus, $M = B + L + A^{'}$ with $L + A^{'} \leq_{e^*} M$ and hence B is a proper submodule of M. This implies that B is e^*-essential small in M. Hence, $M = L+A^{'}$ and $A = A \cap M = A \cap (L + A^{'}) = L + (A \cap A^{'}) = L$. Therefore, B is e^*-essential small in A and A is e^*-hollow.*

2. *Let A be a e^*-coclosed submodule of M and B a proper submodule of A with C an e^*-essential submodule of A such that $B + C = A$. Since M is e^*-hollow then by corollary 3, $\frac{M}{C}$ is e^*-hollow. Now, $\frac{A}{C}$ is a proper submodule of $\frac{M}{C}$ implies that $\frac{A}{C}$ is e^*-essential small of $\frac{M}{C}$ since A is e^*-coclosed. Thus $A = C$ and B is e^*-essential small of A.The case $\frac{A}{C} = \frac{M}{C}$, implies that $A = M$. Thus A is e^*-hollow.*

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