On Generalized $\omega^*_e$-closed Sets

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Abstract. The aim of this paper is to introduce and study a new type of generalized closed sets, called generalized $\omega^*_e$-closed (briefly, $g\omega^*_e$-closed) sets, via $\omega^*_e$-closure operator. We examine the fundamental properties of the class of these sets. The notion of $g\omega^*_e$-closed set is weaker than the notions of $g\omega\beta$-closed set and $\omega^*_e$-closed set in the literature. Also, we define and discuss the notions of generalized $\omega^*_e$-continuous and generalized $\omega^*_e$-irresolute functions.

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1. Introduction

The notion of the generalized closed set is an important concept in the area of general topology. It was first introduced by Levine [14] in 1970. Since then, many forms of this notion such as $g\alpha$-closed [15], $gs$-closed [7], $gp$-closed [16], $gb$-closed [18], $g\beta$-closed [21], $ge$-closed [8], $\pi ge$-closed [9], $g\omega$-closed [5], and generalized $\omega\beta$-closed [4] have been defined and studied by many mathematicians. Moreover, the authors have introduced many new concepts via these new types of sets. They have also investigated some of their fundamental properties and characterizations of these concepts. Furthermore, they have not only discussed their fundamental properties but also put forth the relationships between them and the notions in the literature.

In this study, we define a new concept called generalized $\omega^*_e$-closed sets via the $\omega^*_e$-closure operator. We examine the relationships among this new concept and some other concepts existing in the literature such as generalized $\beta$-closed, generalized $e^*$-closed, generalized $\omega$-closed, and generalized $\omega\beta$-closed. In addition, by giving the notion of $\omega^*_e$-limit point, we prove that the union of two generalized $\omega^*_e$-closed sets is a generalized $\omega^*_e$-closed set under a special condition. Furthermore, the notions of generalized $\omega^*_e$-continuity and generalized $\omega^*_e$-irresoluteness have been introduced and finally many basic properties of such functions are obtained.

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2. Preliminaries

Throughout this present paper, X and Y represent topological spaces. For a subset A of a space X, cl(A) and int(A) denote the closure of A and the interior of A, respectively. The family of all closed (resp. open) sets of X is denoted C(X) (resp. O(X) or τ) and the family of all closed (resp. open) sets of X containing a point x of X is denoted C(X, x) (resp. O(X, x)). The family of all neighborhood of a point x ∈ X is denoted by N(x).

Definition 1. A subset A of a space X is called:

(a) regular open [20] if A = int(cl(A)). The complement of a regular open set is called regular closed. A point x ∈ X is said to be the δ-cluster point [22] of A if int(cl(U)) ∩ A ≠ ∅ for each open neighborhood U of x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by δ-cl(A). If A = δ-cl(A), then A is called δ-closed [22], and the complement of a δ-closed set is called δ-open. The set {x | (∃U ∈ O(x, x))(int(cl(U)) ⊆ A)} is called the δ-interior of A and is denoted by δ-int(A).

(b) β-open [1] if A ⊆ cl(int(cl(A))). The complement of a β-open set is called β-closed. The intersection of all β-closed sets containing A is called the β-closure of A and is denoted by β-cl(A). The union of all β-open sets of X contained in A is called the β-interior of A and is denoted by β-int(A).

(c) a-open [10] if A ⊆ int(cl(δ-int(A))). The complement of an a-open set is called a-closed [10]. The intersection of all a-closed sets containing A is called the a-closure [10] of A and is denoted by a-cl(A). The union of all a-open sets of X contained in A is called the a-interior [10] of A and is denoted by a-int(A).

(d) e*-open [11] if A ⊆ cl(int(δ-cl(A))). The complement of an e*-open set is called e*-closed [11]. The intersection of all e*-closed sets containing A is called the e*-closure [11] of A and is denoted by e*-cl(A). The union of all e*-open sets of X contained in A is called the e*-interior [11] of A and is denoted by e*-int(A).

(e) ω-open [6] (resp. ωβ-open [2]) if for every x ∈ A there exists an open (resp. β-open) set U containing x such that U \ A is countable. The complement of an ω-open set (resp. ωβ-open set) is said to be ω-closed (resp. ωβ-closed).

The family of all regular open (resp. regular closed, β-open, β-closed, a-open, a-closed, e*-open, e*-closed, ω-open, ω-closed, ωβ-open, ωβ-closed) subsets of X is denoted by RO(X) (resp. RC(X), βO(X), βC(X), aO(X), aC(X), e*O(X), e*C(X), ωO(X), ωC(X), ωβO(X), ωβC(X)). The family of all regular open (resp. regular closed, β-open, β-closed, a-open, a-closed, e*-open, e*-closed, ω-open, ω-closed, ωβ-open, ωβ-closed) subsets of X containing a point x of X is denoted by RO(X, x) (resp. RC(X, x), βO(X, x), βC(X, x), aO(X, x), aC(X, x), e*O(X, x), e*C(X, x), ωO(X, x), ωC(X, x), ωβO(X, x), ωβC(X, x)).

Definition 2. Let A be a subset of a space X. A is said to be ωe*-open [19] (resp. ωα-open [19]) if for every x ∈ A, there exists an e*-open (resp. a-open) set U containing x such that U \ A is countable. The complement of an ωe*-open (resp. ωα-open) set is called
The followings hold for subsets

\[ \text{Definition 3.} \quad [19] \text{Let } A \text{ be a subset of a space } X. \text{ The union of all } \omega^e\text{-open subsets of } X \text{ contained in } A \text{ is called the } \omega^e\text{-interior of } A \text{ and is denoted by } \omega^e\text{-int}(A). \]

\[ \text{Theorem 1.} \quad [19] \text{Let } A \text{ be a subset of a space } X. \text{ Then the following properties hold:} \]

- (a) \( \omega^e\text{-int}(A) \subseteq A \),
- (b) \( \omega^e\text{-int}(A) \in \omega^eO(X) \),
- (c) \( x \in \omega^e\text{-int}(A) \) if and only if there exists \( U \in \omega^eO(X, x) \) such that \( U \subseteq A \),
- (d) \( A \subseteq B \Rightarrow \omega^e\text{-int}(A) \subseteq \omega^e\text{-int}(B) \),
- (e) \( \omega^e\text{-int}(A) \cup \omega^e\text{-int}(B) \subseteq \omega^e\text{-int}(A \cup B) \),
- (f) \( \omega^e\text{-int}(A \cap B) \subseteq \omega^e\text{-int}(A) \cap \omega^e\text{-int}(B) \),
- (g) \( A \in \omega^eO(X) \) if and only if \( A = \omega^e\text{-int}(A) \),
- (h) \( \omega^e\text{-int}(\omega^e\text{-int}(A)) = \omega^e\text{-int}(A) \).

\[ \text{Definition 4.} \quad [19] \text{Let } A \text{ be a subset of a space } X. \text{ The intersection of all } \omega^e\text{-closed subsets of } X \text{ containing } A \text{ is called the } \omega^e\text{-closure of } A \text{ and is denoted by } \omega^e\text{-cl}(A). \]

\[ \text{Theorem 2.} \quad [19] \text{Let } A \text{ and } B \text{ be two subsets of a space } X. \text{ Then the following properties hold:} \]

- (a) \( A \subseteq \omega^e\text{-cl}(A) \),
- (b) \( \omega^e\text{-cl}(A) \in \omega^eC(X) \),
- (c) \( x \in \omega^e\text{-cl}(A) \) if and only if \( A \cap U \neq \emptyset \) for every \( U \in \omega^eO(X, x) \),
- (d) \( A \subseteq B \Rightarrow \omega^e\text{-cl}(A) \subseteq \omega^e\text{-cl}(B) \),
- (e) \( \omega^e\text{-cl}(A) \cup \omega^e\text{-cl}(B) \subseteq \omega^e\text{-cl}(A \cup B) \),
- (f) \( \omega^e\text{-cl}(A \cap B) \subseteq \omega^e\text{-cl}(A) \cap \omega^e\text{-cl}(B) \),
- (g) \( A \in \omega^eC(X) \) if and only if \( A = \omega^e\text{-cl}(A) \),
- (h) \( \omega^e\text{-cl}(\omega^e\text{-cl}(A)) = \omega^e\text{-cl}(A) \),
- (i) \( \omega^e\text{-cl}(X \setminus A) = X \setminus \omega^e\text{-int}(A) \).

\[ \text{Lemma 1.} \quad [19] \text{Let } X \text{ be a topological space. Then the following properties hold:} \]

- (a) The union of any family of } \omega^e\text{-open sets is } \omega^e\text{-open},
- (b) The intersection of an } \omega\text{-open set and an } \omega^e\text{-open set is } \omega^e\text{-open.}

\[ \text{Definition 5.} \quad \text{Let } A \text{ be a subset of a space } X. \text{ The intersection of all open sets in } X \text{ containing } A \text{ is called the kernel } [17] \text{ of } A \text{ and is denoted by ker}(A). \]

\[ \text{Lemma 2.} \quad [17] \text{The followings hold for subsets } A \text{ and } B \text{ of a space } X. \]

- (a) \( x \in \ker(A) \) if and only if \( A \cap F \neq \emptyset \) for any \( F \in C(X, x) \),
Definition 6. A function \( f : X \to Y \) is called:
(a) \( e^*\)-continuous \([11]\) if \( f^{-1}[V] \in e^*O(X) \) for each open set \( V \) of \( Y \),
(b) \( \beta\)-continuous \([1]\) if \( f^{-1}[V] \in \omega\beta O(X) \) for each open set \( V \) of \( Y \),
(c) \( \omega\)-continuous \([13]\) if \( f^{-1}[V] \in \omega O(X) \) for each open set \( V \) of \( Y \),
(d) \( \omega\beta\)-continuous \([3]\) if for each \( x \in X \) and each open set \( V \) in \( Y \) containing \( f(x) \), there exists an \( \omega\beta\)-open \( U \) in \( X \) containing \( x \) such that \( f[U] \subseteq V \),
(e) \( \omega^*\)-continuous \([19]\) at a point \( x \in X \) if for every open set \( V \) in \( Y \) containing \( f(x) \), there exists an \( \omega^*\)-open set \( U \) in \( X \) containing \( x \) such that \( f[U] \subseteq V \).

Definition 7. Let \( A \) be a subset of a space \( X \). \( A \) is said to be generalized closed \([14]\)/(briefly, \( g \)-closed) (resp. generalized \( \omega \)-closed \([5]\)/(briefly, \( g\omega \)-closed), generalized \( \beta \)-closed \([21]\)/(briefly, \( g\beta \)-closed), generalized \( e^* \)-closed \([12]\)/(briefly, \( ge^* \)-closed), generalized \( \omega \beta \)-closed \([4]\)/(briefly, \( gw\beta \)-closed)) if \( cl(A) \subseteq U \) (resp. \( \omega-cl(A) \subseteq U \), \( \beta-cl(A) \subseteq U \), \( e^*-cl(A) \subseteq U \), \( \omega\beta-cl(A) \subseteq U \)) whenever \( U \in O(X) \) and \( A \subseteq U \). The complement of a \( g \)-closed (resp. \( g\omega \)-closed \([5]\), \( g\beta \)-closed \([21]\), \( ge^* \)-closed \([12]\), \( gw\beta \)-closed \([4]\)) set is called a generalized open (briefly, \( g^* \)-open)/(resp. generalized \( \omega \)-open \([5]\)/(briefly, \( g\omega^* \)-open), generalized \( \beta \)-open \([21]\)/(briefly, \( g\beta^* \)-open), generalized \( e^* \)-open \([12]\)/(briefly, \( ge^* \)-open), generalized \( \omega \beta \)-open \([4]\)/(briefly, \( gw\beta^* \)-open)). The family of all \( g \)-closed (resp. \( g\omega \)-closed \([5]\), \( g\beta \)-closed \([21]\), \( ge^* \)-closed \([12]\), \( gw\beta \)-closed \([4]\)) sets of \( X \) will be denoted by \( gC(X) \) (resp. \( g\omega C(X) \), \( g\beta C(X) \), \( ge^* C(X) \), \( gw\beta C(X) \)). The family of all \( g \)-open (resp. \( g\omega \)-open \([5]\), \( g\beta \)-open \([21]\), \( ge^* \)-open \([12]\), \( gw\beta \)-open \([4]\)) sets of \( X \) will be denoted by \( gO(X) \) (resp. \( g\omega O(X) \), \( g\beta O(X) \), \( ge^* O(X) \), \( gw\beta O(X) \)).

3. Generalized \( \omega^* \)-closed Sets

Definition 8. A subset \( A \) of a space \( X \) is called generalized \( \omega^* \)-closed set (briefly, \( g\omega^* \)-closed set) if \( \omega^*-cl(A) \subseteq U \) whenever \( U \in O(X) \) and \( A \subseteq U \). We denote the family of all generalized \( \omega^* \)-closed subsets of a space \( X \) by \( g\omega^* C(X) \).

Proposition 1. Let \( X \) be a topological space. Then the followings hold:
(a) If \( X \) is a countable space, then \( g\omega^* C(X) = 2^X \),
(b) If \( \omega^* O(X) = \omega^* C(X) \), then \( g\omega^* C(X) = 2^X \).

Proof. (a) Let \( A \in 2^X \) and \( A \subseteq U \in O(X) \).
\[
|X| \leq 80 \Rightarrow \omega^* C(X) = 2^X \\
A \in 2^X \Rightarrow A \in \omega^* C(X) \Rightarrow \omega^*-cl(A) = A \\
A \subseteq U \in O(X) \Rightarrow \omega^*-cl(A) \subseteq U
\]
This means that \( A \in g\omega^* C(X) \). Then we have \( 2^X \subseteq g\omega^* C(X) \). On the other hand, we have always \( g\omega^* C(X) \subseteq 2^X \). Therefore \( g\omega^* C(X) = 2^X \).
(b) Let \( A \in 2^X \) and \( A \subseteq U \in O(X) \).
\[
A \subseteq U \in O(X) \quad \Rightarrow \quad \omega^{*}\text{-cl}(A) \subseteq \omega^{*}\text{-cl}(U) = U
\]

This is means that \( A \in g\omega^{*}C(X) \). Then we have \( 2^X \subseteq g\omega^{*}C(X) \).

On the other hand, we have always \( g\omega^{*}C(X) \subseteq 2^X \). Therefore \( g\omega^{*}C(X) = 2^X \).

**Remark 1.** The following diagram follows immediately from the definitions in which none of the implications is reversible. Also, examples for the other implications are shown in the related papers.

\[
\begin{array}{ccc}
\text{closed} & \rightarrow & \omega\text{-closed} \\
\downarrow & & \downarrow \\
g\text{-closed} & \rightarrow & g\omega\text{-closed} \\
\downarrow & & \downarrow \\
g\beta\text{-closed} & \rightarrow & g\omega\beta\text{-closed} \\
\downarrow & & \downarrow \\
ge^{*}\text{-closed} & \rightarrow & g\omega e^{*}\text{-closed} \\
\downarrow & & \downarrow \\
e^{*}\text{-closed} & \rightarrow & \omega e^{*}\text{-closed}
\end{array}
\]

**Figure 1: Relationships between some types of closed sets**

**Example 1.** Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) and \( A = \{a, b\} \). Then \( A \) is \( g\omega^{*}\text{-closed} \) since \( X \) is countable. But \( A \) is not \( g e^{*}\text{-closed} \) since \( A \subseteq \{a, b\} \in O(X) \) but \( e^{*}\text{-cl}(A) = X \not\subseteq \{a, b\} \).

**Question:** Is there an example of \( g\omega^{*}\text{-closed} \) set which is not \( \omega^{*}\text{-closed} \)?

**Theorem 3.** Let \( A \) be a subset of a space \( X \). If \( A \) is \( g\omega^{*}\text{-closed} \), then \( \omega^{*}\text{-cl}(A) \setminus A \) does not contain any non-empty closed sets.

**Proof.** Suppose that \( F \in C(X) \setminus \{\emptyset\} \) and \( F \subseteq \omega^{*}\text{-cl}(A) \setminus A \).
\[
(F \in C(X) \setminus \{\emptyset\})(F \subseteq \omega^{*}\text{-cl}(A) \setminus A) \Rightarrow A \subseteq X \setminus F \in O(X) \quad \Rightarrow \quad A \in g\omega^{*}C(X) \\
\Rightarrow \omega^{*}\text{-cl}(A) \subseteq X \setminus F \Rightarrow F \subseteq X \setminus \omega^{*}\text{-cl}(A) \setminus A \quad \Rightarrow \quad F = \emptyset
\]

This contradicts with \( F \neq \emptyset \).

**Theorem 4.** Let \( A \) be a \( g\omega^{*}\text{-closed} \) subset of a space \( X \). Then \( A \) is \( \omega^{*}\text{-closed} \) if and only if \( \omega^{*}\text{-cl}(A) \setminus A \) is closed.

**Proof.** \((\Rightarrow)\) : It is obvious.
\((\Leftarrow)\) : Let \( \omega^{*}\text{-cl}(A) \setminus A \in C(X) \).
Definition 9. A space $X$ is called an $\omega^*$-locally indiscrete space if every open set is $\omega^*$-closed.

Proposition 2. Let $X$ be a topological space. Then the following are equivalent.
(a) $X$ is $\omega^*$-locally indiscrete;
(b) Every subset of $X$ is $\omega^*$-closed.

Proof. (a) $\Rightarrow$ (b): Let $A \subseteq U \in O(X)$.
$A \subseteq U \in O(X) \quad \Rightarrow \quad A \subseteq U \in \omega^*C(X) \Rightarrow \omega^*\text{-cl}(A) \subseteq \omega^*\text{-cl}(U) = U$.

(b) $\Rightarrow$ (a): Let $U \in O(X)$.
$U \in O(X) \quad \Rightarrow \quad U \in \omega^*C(X) \quad \Rightarrow \quad \omega^*\text{-cl}(U) \subseteq U \Rightarrow U \in \omega^*C(X)$.

Theorem 5. Let $A$ be a subset of a space $X$. If $A$ is both $\omega^*$-closed and open, then $\omega^*\text{-cl}(A) \setminus A = \emptyset$.

Proof. Let $A \in O(X) \cap \omega^*C(X)$.
$A \in O(X) \cap \omega^*C(X) \Rightarrow (A \in O(X))(A \in \omega^*C(X))$
$\Rightarrow (A \in O(X))(\forall U \in O(X))(A \subseteq U \Rightarrow \omega^*\text{-cl}(A) \subseteq U)$
$\Rightarrow \omega^*\text{-cl}(A) \subseteq A$
$\Rightarrow \omega^*\text{-cl}(A) \setminus A = \emptyset$.

Theorem 6. Let $A$ and $B$ be subsets of a space $X$. If $A$ is $\omega^*$-closed and $B$ is any set such that $A \subseteq B \subseteq \omega^*\text{-cl}(A)$, then $B$ is $\omega^*$-closed.

Proof. Let $B \subseteq U \in O(X)$.
$B \subseteq U \in O(X) \quad \Rightarrow \quad (A \subseteq B \subseteq U)(A \subseteq B \subseteq \omega^*\text{-cl}(A) \subseteq U)$
$\Rightarrow \omega^*\text{-cl}(A) \subseteq \omega^*\text{-cl}(B) \subseteq \omega^*\text{-cl}(\omega^*\text{-cl}(A)) = \omega^*\text{-cl}(A) \subseteq U$.

Definition 10. Let $A$ be a subset of a space $X$. A point $x \in X$ is said to be an $\omega^*$-limit point of $A$ if for each $\omega^*$-open set $U$ containing $x$, we have $U \cap (A \setminus \{x\}) \neq \emptyset$. The set of all $\omega^*$-limit points of $A$ is called the $\omega^*$-derived set of $A$ and is denoted by $D_{\omega^*}(A)$.

Lemma 3. Let $A$ be a subset of a space $X$. If $D(A) = D_{\omega^*}(A)$, then $cl(A) = \omega^*\text{-cl}(A)$.

Proof. It is clear.
Proposition 3. Let $A$ and $B$ be two subsets of a space $X$. Then the following properties hold:
(a) $D_{\omega^*}(A) \subseteq \omega^*\text{-}cl(A)$,
(b) $A \subseteq B \Rightarrow D_{\omega^*}(A) \subseteq D_{\omega^*}(B)$,
(c) $D_{\omega^*}(A) \cup D_{\omega^*}(B) \subseteq D_{\omega^*}\text{-}cl(A \cup B)$,
(d) $D_{\omega^*}\text{-}cl(A \cap B) \subseteq D_{\omega^*}(A) \cap D_{\omega^*}(B)$,
(e) $D_{\omega^*}(A) \subseteq D(A)$,
(f) $A \in \omega^*C(X)$ if and only if $D_{\omega^*}(A) \subseteq A$,
(g) $A \cup D_{\omega^*}(A) \in \omega^*C(X)$,
(h) $\omega^*\text{-}cl(A) = A \cup D_{\omega^*}(A)$.

Proof. The proofs of above results are standard. Hence, they are omitted.

Corollary 1. Let $A$ be a subset of a space $X$. If $D(A) \subseteq D_{\omega^*}(A)$, then for any subsets $F$ and $B$ of $X$, we have $\omega^*\text{-}cl(F \cup B) = \omega^*\text{-}cl(F) \cup \omega^*\text{-}cl(B)$.

Proof. It is obvious.

Proposition 4. Let $A$ and $B$ be subsets of a space $X$. If $A$ and $B$ are $\omega^*$-closed sets such that $D(A) \subseteq D_{\omega^*}(A)$ and $D(B) \subseteq D_{\omega^*}(B)$, then $A \cup B$ is $\omega^*$-closed.

Proof. Let $A \cup B \subseteq U \in O(X)$.
$A \cup B \subseteq U \in O(X) \Rightarrow (A \subseteq U \in O(X))(B \subseteq U \in O(X)) \Rightarrow A, B \in \omega^*C(X) \Rightarrow$
$(\omega^*\text{-}cl(A) \subseteq U)(\omega^*\text{-}cl(B) \subseteq U) \Rightarrow (D(A) \subseteq D_{\omega^*}(A))(D(B) \subseteq D_{\omega^*}(B)) \Rightarrow cl(A \cup B) = cl(A) \cup cl(B) = \omega^*\text{-}cl(A) \cup \omega^*\text{-}cl(B) = \omega^*\text{-}cl(A \cup B) \subseteq U$.

Proposition 5. Let $A$ and $B$ be subsets of a space $X$. Then the following properties hold:
(a) If $A$ is open and $\omega^*$-closed and $B$ is $\omega^*$-closed, then $A \cap B$ is $\omega^*$-closed,
(b) If $A$ is $\omega^*$-closed and $B$ is closed, then $A \cap B$ is $\omega^*$-closed.

Proof. (a) Let $A \in O(X) \cap g\omega^*C(X)$.
$A \in O(X) \cap g\omega^*C(X) \Rightarrow \omega^*\text{-}cl(A) \setminus A = \emptyset \Rightarrow \omega^*\text{-}cl(A) = A$
$A \in \omega^*C(X) \Rightarrow A \cap B \in \omega^*C(X) \subseteq g\omega^*C(X)$.

(b) Let $A \cap B \subseteq U \in O(X)$.
$A \cap B \subseteq U \in O(X) \Rightarrow X \setminus B \in O(X) \Rightarrow$
$A \subseteq (A \cap B) \cup (X \setminus B) \subseteq U \cup (X \setminus B) \in O(X) \Rightarrow A \in g\omega^*C(X)$.
\[ \Rightarrow \omega^*\text{-cl}(A) \subseteq U \cup (X \setminus B) \Rightarrow \omega^*\text{-cl}(A \cap B) \subseteq \omega^*\text{-cl}(A) \cap \omega^*\text{-cl}(B) \]

\[ \subseteq \omega^*\text{-cl}(A) \cap \omega^*\text{-cl}(B) \]

\[ = \omega^*\text{-cl}(A) \cap B \]

\[ \subseteq (U \cup (X \setminus B)) \cap B \]

\[ = U \cap B \]

\[ \subseteq U. \]

**Theorem 7.** Let \( A \) be a subset of a space \( X \). \( A \) is \( \omega^* \)-closed if and only if \( \text{cl}(\{x\}) \cap A \neq \emptyset \) for every \( x \in \omega^*\text{-cl}(A) \).

**Proof.** (\( \Rightarrow \)): Suppose that \( x \in \omega^*\text{-cl}(A) \) and \( \text{cl}(\{x\}) \cap A = \emptyset \).

\[ \text{cl}(\{x\}) \cap A = \emptyset \Rightarrow A \subseteq X \setminus \text{cl}(\{x\}) \in O(X) \]

\[ A \in g\omega^*C(X) \]

\[ \Rightarrow \omega^*\text{-cl}(A) \subseteq X \setminus \text{cl}(\{x\}) \]

\[ \Rightarrow x \notin \omega^*\text{-cl}(A) \]

This contradicts with \( x \in \omega^*\text{-cl}(A) \).

\[ (\Leftarrow): \text{Let } A \subseteq U \in O(X) \text{ and } x \in \omega^*\text{-cl}(A). \]

\[ x \in \omega^*\text{-cl}(A) \]

\[ \Rightarrow A \cap \text{cl}(\{x\}) \neq \emptyset \Rightarrow (\exists y \in X)(y \in A \cap \text{cl}(\{x\})) \]

\[ \Rightarrow (y \in A)(y \in \text{cl}(\{x\})) \]

\[ A \subseteq U \in O(X) \]

\[ \Rightarrow (y \in A \subseteq U \in O(X))(y \in \text{cl}(\{x\})) \]

\[ \Rightarrow (U \in O(X,y))(y \in \text{cl}(\{x\})) \Rightarrow U \cap \{x\} \neq \emptyset \Rightarrow x \in U. \]

**Theorem 8.** Let \( X \) be a space. For an element \( x \in X \), either \( \{x\} \) is closed or \( X \setminus \{x\} \) is \( \omega^* \)-closed.

**Proof.** Suppose that \( \{x\} \notin C(X) \).

\[ \{x\} \notin C(X) \Rightarrow X \setminus \{x\} \notin O(X) \]

\[ X \setminus \{x\} \subseteq X \in O(X) \]

\[ \Rightarrow \omega^*\text{-cl}(X \setminus \{x\}) \subseteq \omega^*\text{-cl}(X) = X. \]

**Definition 11.** A space \( X \) is said to be an \( \omega^*\text{-T}_1 \) space if for every generalized \( \omega^* \)-closed set is \( \omega^* \)-closed.

**Example 2.** Any set with indiscrete topology is an example for an \( \omega^*\text{-T}_1 \) space.

**Theorem 9.** Let \( X \) be a space. \( X \) is an \( \omega^*\text{-T}_1 \) space if and only if every singleton is either closed or \( \omega^* \)-open.

**Proof.** (\( \Rightarrow \)): Suppose that \( \{x\} \notin C(X) \).

\[ \{x\} \notin C(X) \Rightarrow X \setminus \{x\} \notin g\omega^*C(X) \]

\[ \Rightarrow X \setminus \{x\} \in g\omega^*\text{-cl}(X) \]

\[ \Rightarrow X \setminus \{x\} \in \omega^*\text{-cl}(X) \]

(\( \Leftarrow \)): Let \( A \in g\omega^*C(X) \) and \( x \in \omega^*\text{-cl}(A) \).
This result contradicts with Theorem 3. Hence, \( x \notin A \).

Then, we have \( A \in \omega^e\)-closed.

2nd case: Let \( \{x\} \in \omega^eO(X) \).

\[ \Rightarrow \{x\} \in \omega^eO(X) \]

\[ \Rightarrow \{x\} \in \omega^eO(X) \]

This means that \( A \) is \( \omega^e \)-closed.

Definition 12. A space \( X \) is said to be an \( e^* \)-anti-locally countable if each \( U \in \omega^eO(X) \setminus \{\emptyset\} \) is uncountable.

Theorem 10. Let \( X \) be a space. If \( X \) is \( e^* \)-anti-locally countable and \( \omega^e\)-\( T_1 \) space, then \( X \) is \( T_1 \) space.

Proof. Let \( x \in X \) and suppose that \( \{x\} \notin C(X) \).

\[ \{x\} \notin C(X) \Rightarrow X \setminus \{x\} \in g\omega^eC(X) \]

\[ \Rightarrow X \setminus \{x\} \in \omega^eC(X) \]

\[ \Rightarrow \exists U \in \omega^eO(X, x) \mid \{x\} \cap \{x\} \neq \emptyset \Rightarrow x \in A \]

This contradicts the fact that \( X \) is \( e^* \)-anti-locally countable. Then, \( \{x\} \in C(X) \) for all \( x \in X \). Namely, \( X \) is \( T_1 \) space.

Proposition 6. Let \( A \) be a subset of a space \( X \). \( A \) is \( g\omega^e \)-closed set if and only if \( \omega^e\)-cl\( (A) \subseteq \ker(A) \).

Proof. \((\Rightarrow)\) : Let \( A \in g\omega^eC(X) \).

\[ A \in g\omega^eC(X) \Rightarrow (\forall U \in O(X))(A \subseteq U \Rightarrow \omega^e\)-cl\( (A) \subseteq U \)

\[ \ker(A) := \cap\{U|(A \subseteq U)(U \in O(X))\} \Rightarrow \omega^e\)-cl\( (A) \subseteq \ker(A) \).

\((\Leftarrow)\) : Let \( A \subseteq U \in O(X) \).

\[ A \subseteq U \Rightarrow \ker(A) \subseteq \ker(U) \]

\[ \operatorname{Hypothesis} \Rightarrow \omega^e\)-cl\( (A) \subseteq \ker(A) \subseteq \ker(U) \]

\[ U \in O(X) \Rightarrow U = \ker(U) \]

\[ \Rightarrow \omega^e\)-cl\( (A) \subseteq U \).

4. Generalized \( \omega^e \)-open Sets and Generalized \( \omega^e \)-neighborhoods

Definition 13. A subset \( A \) of a space \( X \) is called generalized \( \omega^e \)-open if its complement is generalized \( \omega^e \)-closed. We denote the family of all generalized \( \omega^e \)-open subsets of a space \( X \) by \( g\omega^eO(X) \).
Corollary 2. Let $A$ be a subset of a space $X$. $A$ is $g\omega^*$-open if and only if $F \subseteq g\omega^*$-$\text{int}(A)$, where $F$ is closed set and $F \subseteq A$.

Proof. $(\Rightarrow)$: Let $A \in g\omega^*O(X)$ and $F \in C(X)$ such that $F \subseteq A$.
$A \in g\omega^*O(X) \Rightarrow X \setminus A \in g\omega^*C(X)$ \(\quad\) \(\Rightarrow\)
$A \supseteq F \in C(X) \Rightarrow X \setminus A \subseteq X \setminus F \in O(X)$ \(\Rightarrow\)
$\Rightarrow g\omega^*$-$\text{cl}(X \setminus A) = X \setminus g\omega^*$-$\text{int}(A) \subseteq X \setminus F \Rightarrow F \subseteq g\omega^*$-$\text{int}(A)$.

\((\Leftarrow)\): Let $X \setminus A \subseteq U \in O(X)$.
$X \setminus A \subseteq U \in O(X) \Rightarrow (X \setminus U \in C(X))(X \setminus U \subseteq A)$ \(\Rightarrow\)
Hypothesis \(\Rightarrow\)
$\Rightarrow X \setminus U \subseteq g\omega^*$-$\text{int}(A)$

$\Rightarrow g\omega^*$-$\text{cl}(X \setminus A) = X \setminus g\omega^*$-$\text{int}(A) \subseteq U$.

Proposition 7. Let $A$ and $B$ be subsets of a space $X$. If $g\omega^*$-$\text{int}(A) \subseteq B \subseteq A$ and $A$ is $g\omega^*$-open, then $B$ is $g\omega^*$-open.

Proof. It is clear from Theorem 6.

Proposition 8. Let $A$ be a subset of a space $X$. If $A$ is $g\omega^*$-closed, then $g\omega^*$-$\text{cl}(A) \setminus A$ is $g\omega^*$-open.

Proof. It is clear from Theorem 4.

Remark 2. Let $A$ be a subset of a space $X$. Then $g\omega^*$-$\text{int}(g\omega^*$-$\text{cl}(A) \setminus A) = \emptyset$.

Proposition 9. Let $A$ and $B$ be two subsets of a space $X$. If $A \subseteq B \subseteq X$ and $g\omega^*$-$\text{cl}(A) \setminus A$ is $g\omega^*$-open, then $g\omega^*$-$\text{cl}(A) \setminus B$ is $g\omega^*$-open.

Proof. Let $F \in C(X)$ such that $F \subseteq g\omega^*$-$\text{cl}(A) \setminus B$.
$A \subseteq B \Rightarrow g\omega^*$-$\text{cl}(A) \setminus A \subseteq g\omega^*$-$\text{cl}(A) \setminus B$ \(\Rightarrow\)
$(F \in C(X))(F \subseteq g\omega^*$-$\text{cl}(A) \setminus B)$ \(\Rightarrow\)
$\Rightarrow (F \in C(X))(F \subseteq g\omega^*$-$\text{cl}(A) \setminus A \subseteq g\omega^*$-$\text{cl}(A) \setminus B)$ \(\Rightarrow\)
$g\omega^*$-$\text{cl}(A) \setminus A \in g\omega^*O(X)$ \(\Rightarrow\)
$F \subseteq g\omega^*$-$\text{int}(g\omega^*$-$\text{cl}(A) \setminus A) \subseteq g\omega^*$-$\text{int}(g\omega^*$-$\text{cl}(A) \setminus B)$.

Proposition 10. Let $A$ be a subset of a space $X$. If $A$ is $g\omega^*$-open, then $U = X$ whenever $U$ is open in $X$ and $g\omega^*$-$\text{int}(A) \cup (X \setminus A) \subseteq U$.

Proof. Let $g\omega^*$-$\text{int}(A) \cup (X \setminus A) \subseteq U \in O(X)$.
$g\omega^*$-$\text{int}(A) \cup (X \setminus A) \subseteq U \in O(X) \Rightarrow (X \setminus U \in C(X))(X \setminus U \subseteq X \setminus [g\omega^*$-$\text{int}(A) \cup (X \setminus A)]$ \(\Rightarrow\)
$(X \setminus U \in C(X))(X \setminus U \subseteq X \setminus [g\omega^*$-$\text{int}(A) \cup (X \setminus A)] = g\omega^*$-$\text{cl}(X \setminus A) \setminus (X \setminus A)$ \(\Rightarrow\)
$A \in g\omega^*O(X) \Rightarrow X \setminus A \in g\omega^*C(X)$ \(\Rightarrow\)
Theorem 3 \(\Rightarrow\)
$X \setminus U = \emptyset \Rightarrow X \subseteq U$ \(\Rightarrow\)
$U = X$. 


Theorem 11. Let $A$ and $B$ be two subsets of a space $X$. Then the following properties hold:

(a) If $A$ is $gw\omega^{*}$-open and $B$ is $\omega a$-open, then $A \cap B$ is $gw\omega^{*}$-open,

(b) If $B$ is $gw\omega^{*}$-open and $\omega^{*} \text{int}(B) \subseteq A$, then $A \cap B$ is $gw\omega^{*}$-open.

Proof. (a) Let $F \in C(X)$ such that $F \subseteq A \cap B$.

\[
(F \in C(X))(F \subseteq A \cap B) \Rightarrow (F \in C(X))(F \subseteq A \cap B \subseteq A) \quad \text{Corollary 2}
\]

\[
A \in gw\omega^{*}O(X) \quad \Rightarrow
\]

\[
\Rightarrow F \subseteq \omega^{*} \text{int}(A) \quad \text{Proposition 7}
\]

\[
B \in \omega aO(X) \quad \Rightarrow
\]

\[
F = F \cap B \subseteq \omega^{*} \text{int}(A) \cap B = \omega^{*} \text{int}(A \cap B).
\]

(b) Let $B \in gw\omega^{*}O(X)$ and $\omega^{*} \text{int}(B) \subseteq A$.

\[
\omega^{*} \text{int}(B) \subseteq A \Rightarrow B \cap \omega^{*} \text{int}(B) \subseteq A \cap B \subseteq B \quad \text{Proposition 7}
\]

\[
B \in gw\omega^{*}O(X) \quad \Rightarrow
\]

\[
A \cap B \in gw\omega^{*}O(X).
\]

Definition 14. Let $X$ be a space and $x \in X$. A subset $N$ of $X$ is called a $gw\omega^{*}$-neighborhood of $x$ if there exists a $gw\omega^{*}$-open set $U$ such that $x \in U \subseteq N$. The set of all $gw\omega^{*}$-neighborhoods of $x$ is called the $gw\omega^{*}$-neighborhood system at $x$, and is denoted by $N_{gw\omega^{*}}(x)$.

Definition 15. Let $X$ be a space and $A \subseteq X$. A subset $N$ of $X$ is called a $gw\omega^{*}$-neighborhood of $A$ if there exists a $gw\omega^{*}$-open set $U$ such that $A \subseteq U \subseteq N$.

Corollary 3. Let $X$ be a space and $x \in X$. Every neighborhood $N$ of $x$ in a space $X$ need not be a neighborhood of $x$ as shown by the following example.

Example 3. Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $A = \{a, c\}$. Since $X$ is countable, $gw\omega^{*}O(X) = 2^{X}$. Then, $A$ is a $gw\omega^{*}$-neighborhood of the point $c$, since $\{c\}$ is $gw\omega^{*}$-open set such that $c \in \{c\} \subseteq \{a, c\}$. However, the set $\{a, c\}$ is not a neighborhood of the point $c$, since there exists no open set $U$ such that $c \in U \subseteq \{a, c\}$.

Theorem 12. Let $N$ be a subset of a space $X$ and $x \in X$. If $N$ is $gw\omega^{*}$-open, then $N$ is a $gw\omega^{*}$-neighborhood of $x$.

Proof. It is clear.

Theorem 13. Let $N$ and $F$ be two subsets of a space $X$ and $x \in X$. If $F$ is $gw\omega^{*}$-closed and $x \in X \setminus F$, then there exists a $gw\omega^{*}$-neighborhood $N$ of $x$ such that $N \cap F = \emptyset$.

Proof. It is clear.
Theorem 14. Let $N$ be a subset of a space $X$ and $x \in X$. Then the following properties hold:
(a) For all $x \in X$, $N_{g\omega^*}(x) \neq \emptyset$,
(b) If $N \in N_{g\omega^*}(x)$, then $x \in N$,
(c) If $N \subseteq N_{g\omega^*}(x)$ and $N \subseteq M \subseteq X$, then $M \in N_{g\omega^*}(x)$,
(d) If $N \in N_{g\omega^*}(x)$, then there exists $M \in N_{g\omega^*}(x)$ such that $M \subseteq N$ and $N \in N_{g\omega^*}(y)$ for every $y \in M$.

Proof. Straightforward.

Definition 16. Let $A$ be a subset of a space $X$. The intersection of all generalized $\omega^*$-closed (resp. generalized closed [14]) subsets of $X$ containing $A$ is called the generalized $\omega^*$-closure (resp. generalized closure [14]) of $A$ and is denoted by $g\omega^*\text{-cl}(A)$ (resp. $g\text{-cl}(A)$).

The proofs of the following results are standard, hence they are omitted.

Theorem 15. Let $A$ and $B$ be subsets of a space $X$ and $x \in X$. Then the following properties hold:
(a) $x \in g\omega^*\text{-cl}(A)$ iff $V \cap A \neq \emptyset$ for every $g\omega^*$-open set $V$ containing $x$,
(b) $g\omega^*\text{-cl}(\emptyset) = \emptyset$ and $g\omega^*\text{-cl}(X) = X$,
(c) If $A \subseteq B$, then $g\omega^*\text{-cl}(A) \subseteq g\omega^*\text{-cl}(B)$,
(d) $A \subseteq g\omega^*\text{-cl}(A) \subseteq \omega^*\text{-cl}(A) \subseteq \text{cl}(A)$,
(e) $A \subseteq g\omega^*\text{-cl}(A) \subseteq g\text{-cl}(A) \subseteq \text{cl}(A)$,
(f) $g\omega^*\text{-cl}(A) \cup g\omega^*\text{-cl}(B) \subseteq g\omega^*\text{-cl}(A \cup B)$,
(g) $g\omega^*\text{-cl}(A \cap B) \subseteq g\omega^*\text{-cl}(A) \cap g\omega^*\text{-cl}(B)$,
(h) $A \in g\omega^*\text{-cl}(X)$ if and only if $A = g\omega^*\text{-cl}(A)$,
(i) $g\omega^*\text{-cl}(A) = g\omega^*\text{-cl}(g\omega^*\text{-cl}(A))$,
(j) $g\omega^*\text{-cl}(A) \in g\omega^*\text{-cl}(X)$.

Definition 17. Let $X$ be a topological space.
(a) $[14] \tau^* = \{U \subseteq X | \text{cl}^*(X \setminus U) = X \setminus U\}$
(b) $\tau_{\omega^*}^* = \{V \subseteq X | g\omega^*\text{-cl}(X \setminus V) = X \setminus V\}$

Proposition 11. For a subset $A$ of $X$, the following properties hold:
(a) $\tau \subseteq \omega^*\text{-cl}(O(X)) \subseteq \tau_{\omega^*}^*$,
(b) $\tau \subseteq gO(X) \subseteq \tau^* \subseteq \tau_{\omega^*}^*$.

Theorem 16. Let $X$ be a topological space. If the family $g\omega^*\text{-cl}(O(X))$ is a topology on $X$, then the family $\tau_{\omega^*}^*$ is a topology on $X$.

Proof. It is obvious that $\emptyset, X \in \tau_{\omega^*}^*$. Let $A, B \in \tau_{\omega^*}^*$. Then $A, B \in \tau_{\omega^*}^* \Rightarrow (g\omega^*\text{-cl}(X \setminus A) = X \setminus A) (g\omega^*\text{-cl}(X \setminus B) = X \setminus B) \Rightarrow g\omega^*\text{-cl}(O(X) \text{ is a topology on } X) \Rightarrow g\omega^*\text{-cl}(X \setminus A) \cup g\omega^*\text{-cl}(X \setminus B) = (X \setminus A) \cup (X \setminus B) \Rightarrow g\omega^*\text{-cl}(X \setminus (A \cap B)) = g\omega^*\text{-cl}(X \setminus (A \setminus B)) = X \setminus (A \cap B)$
\[ A \cap B \in \tau^* \]

Now, let \( A \subseteq \tau^* \)

\[ A \in \tau^* \Rightarrow g\omega^*\text{-cl}(X \setminus A) = X \setminus A \Rightarrow X \setminus A \in g\omega^*C(X) \]  
\[ g\omega^*O(X) \text{ is a topology on } X \]

\[ \Rightarrow X \setminus (\bigcup A) = \bigcap_{A \in A} (X \setminus A) \in g\omega^*C(X) \]
\[ g\omega^*\text{-cl}(\bigcap_{A \in A} (X \setminus A)) = X \setminus (\bigcup A) \]
\[ \Rightarrow \bigcup A \in \tau^* \]

Theorem 17. Let \( X \) be a topological space. Then the following properties hold:

(a) A space \( X \) is \( \omega^*T_\frac{1}{2} \) if and only if \( \tau^* = \omega^*O(X) \),

(b) Every \( g\omega^* \)-closed is closed if and only if \( \tau^* = \tau \).

Proof. (a) (\( \Rightarrow \)) : Let \( A \in \tau^* \).

\[ A \in \tau^* \Rightarrow X \setminus A = g\omega^*\text{-cl}(X \setminus A) \Rightarrow X \setminus A \in g\omega^*C(X) \]
\[ X \text{ is } \omega^*T_\frac{1}{2} \]

\[ \Rightarrow X \setminus A \in \omega^*O(X) \Rightarrow A \in \omega^*O(X) \]

\[ \Rightarrow A \in \omega^*C(X) \]

\[ (\Leftarrow) : \text{Let } A \in g\omega^*C(X). \]
\[ A \in g\omega^*C(X) \Rightarrow X \setminus A = g\omega^*\text{-cl}(X \setminus A) \Rightarrow X \setminus A \in \tau^* \text{ Hypothesis} \]
\[ \Rightarrow X \setminus A \in \omega^*O(X) \]
\[ \Rightarrow A \in \omega^*C(X) \]

(b) (\( \Rightarrow \)) : Let \( A \in \tau^* \).

\[ A \in \tau^* \Rightarrow X \setminus A = g\omega^*\text{-cl}(X \setminus A) \Rightarrow X \setminus A \in g\omega^*C(X) \]
\[ \text{Hypothesis} \]

\[ \Rightarrow X \setminus A \in C(X) \Rightarrow A \in \tau \]

\[ (\Leftarrow) : \text{Let } A \in g\omega^*C(X). \]
\[ A \in g\omega^*C(X) \Rightarrow X \setminus A = g\omega^*\text{-cl}(X \setminus A) \Rightarrow X \setminus A \in \tau^* \text{ Hypothesis} \]
\[ \Rightarrow X \setminus A \in \tau \]
\[ \Rightarrow A \in C(X) \]

5. \( g\omega^* \)-continuity, \( g\omega^* \)-irresoluteness and \( g\omega^* \)-closedness

Definition 18. A function \( f : X \to Y \) is said to be \( g\omega^* \)-continuous (resp. \( g\omega^* \)-continuous \( \{4\} \)) if \( f^{-1}[V] \) is \( g\omega^* \)-closed (resp. \( g\omega^* \)-closed \( \{4\} \)) in \( X \) for every closed set \( V \) of \( Y \).

Corollary 4. Let \( f : X \to Y \) be a function. \( f \) is \( g\omega^* \)-continuous if and only if the inverse image of every open set in \( Y \) is \( g\omega^* \)-open in \( X \).
Remark 4. Every continuous function is $gω^*-continuous$ but the converse need not to be true as shown by the following example.

Example 4. Consider the real numbers $\mathbb{R}$ with usual topology and let $Y = \{1, 2\}$ with the topology $\tau = \{\emptyset, Y, \{1\}\}$. Define the function $f : \mathbb{R} \rightarrow Y$ by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 2, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Then the function $f$ is $gω^*$-continuous but not continuous since $f^{-1}[\{2\}] = \mathbb{R} \setminus \mathbb{Q}$ is not closed in $\mathbb{R}$.

Remark 5. Let $f : X \rightarrow Y$ be a function. Then the following properties hold:

(a) If $τ_{ω^*} = τ$ in $X$, then the notion of continuity and the notion of $gω^*$-continuity coincide.

(b) Every $gω^*$-continuous function defined on $ω^*-T_2$ space is $ω^*$-continuous.

Remark 6. The following diagram follows immediately from the definitions in which none of the implications is reversible.

$$
\begin{align*}
\text{continuous} & \rightarrow \omega\beta\text{-continuous} \rightarrow gω\beta\text{-continuous} \\
\downarrow & \downarrow \downarrow \\
\text{e*-continuous} & \rightarrow \omega^*\text{-continuous} \rightarrow gω^*\text{-continuous}
\end{align*}
$$

Theorem 18. Let $f : X \rightarrow Y$ be a function. If $f$ is $gω^*$-continuous, then $f[gω^*-cl(A)] \subseteq cl(f[A])$ for every subset $A$ of $X$.

Proof. Let $A \subseteq X$.

$A \subseteq X \Rightarrow cl(f[A]) \subseteq C(Y)$

\(f\) is $gω^*$-continuous \(\Rightarrow f^{-1}[cl(f[A])] \subseteq gω^*C(X)

\(\Rightarrow gω^*-cl(f^{-1}[cl(f[A])]) = f^{-1}[cl(f[A])] \)

\(\Rightarrow gω^*-cl(A) \subseteq f^{-1}[cl(f[A])] \)

\(\Rightarrow f[gω^*-cl(A)] \subseteq cl(f[A]). \)

Theorem 19. Let $f : X \rightarrow Y$ be a function. If for each point $x \in X$ and each open set $V$ containing $f(x)$ there exists a $gω^*$-open set $U$ containing $x$ such that $f[U] \subseteq V$, then $f[gω^*-cl(A)] \subseteq cl(f[A])$ for every subset $A$ of $X$.

Proof. Let $y \in f[gω^*-cl(A)]$.

\(y \in f[gω^*-cl(A)] \Rightarrow (\exists x \in gω^*-cl(A))(f(x) = y) \)

\(\Rightarrow (\forall U \in gω^*O(X, x))(U \cap A \neq \emptyset)(f(x) = y) \) \(\text{Hypothesis} \)

\(\Rightarrow (\forall V \in O(Y, f(x)))(U \in gω^*O(X, x))(\emptyset \neq f[U] \cap A) \subseteq f[U] \cap f[A] \subseteq V \cap f[A] \)

\(\Rightarrow y = f(x) \in cl(f[A]). \)
Theorem 20. Let \( f : X \to Y \) be a function. Then the following statements are equivalent:
(a) \( f[^*\omega\text{-}\text{cl}(A)] \subseteq \text{cl}(f[A]) \) for every subset \( A \) of \( X \);
(b) If \( \tau[^*\omega\text{-}] \) is a topology on \( X \), then \( f : (X, \tau[^*\omega\text{-}]) \to (Y, \sigma) \) is continuous.

Proof. (a) \( \Rightarrow \) (b): Let \( A \subseteq C(Y) \).
\[
A \subseteq C(Y) \Rightarrow f^{-1}[A] \subseteq X \quad \text{Hypothesis}
\]
\[
\Rightarrow f[^*\omega\text{-}\text{cl}(f^{-1}[A])] \subseteq \text{cl}(f[f^{-1}[A]]) \subseteq \text{cl}(A) = A
\]
\[
\Rightarrow \quad \text{g}[^*\omega\text{-}\text{cl}(f^{-1}[A]) \subseteq f^{-1}[A] \quad \text{Hypothesis}
\]
\[
\Rightarrow \quad f^{-1}[A] = \text{g}[^*\omega\text{-}\text{cl}(f^{-1}[A])]
\]
\[
\Rightarrow X \setminus f^{-1}[A] \in \tau[^*\omega\text{-}] \Rightarrow f^{-1}[A] \in C(X, \tau[^*\omega\text{-}]).
\]

(b) \( \Rightarrow \) (a): Let \( A \subseteq X \).
\[
A \subseteq X \Rightarrow \text{cl}(f[A]) \subseteq C(Y) \quad \text{Hypothesis}
\]
\[
\Rightarrow X \setminus f^{-1}[\text{cl}(f[A])] \in \tau[^*\omega\text{-}]
\]
\[
\Rightarrow \quad f^{-1}[\text{cl}(f[A])] \subseteq \text{cl}(f[A]) = A
\]
\[
\Rightarrow \quad \text{g}[^*\omega\text{-}\text{cl}(f^{-1}[A]) \subseteq \text{cl}(f[f^{-1}[A]]) = f^{-1}[\text{cl}(f[A])]]
\]
\[
\Rightarrow \quad f[^*\omega\text{-}\text{cl}(A)] \subseteq \text{cl}(f[A]).
\]

Definition 19. A function \( f : X \to Y \) is said to be pre-\(^*\omega\text{-}\text{closed} \) if \( f[F] \) is \(^*\omega\text{-}\text{closed} \) in \( Y \) for every \(^*\omega\text{-}\text{closed} \) set \( F \) of \( X \).

Definition 20. A function \( f : X \to Y \) is said to be pre-\(^*\omega\text{-}\text{closed} \) if \( f[U] \) is \(^*\omega\text{-}\text{closed} \) in \( Y \) for every \(^*\omega\text{-}\text{closed} \) set \( U \) of \( X \).

Theorem 21. Let \( f : X \to Y \) be a function. If \( f \) is continuous and pre-\(^*\omega\text{-}\text{closed} \), then \( f \) is pre-\(^*\omega\text{-}\text{open} \).

Proof. Let \( A \in \text{g}[^*\omega\text{-}\text{C}(X) \text{ and } f[A] \subseteq U \in O(Y) \).
\[
(A \in \text{g}[^*\omega\text{-}\text{C}(X)(f[A] \subseteq U \in O(Y))) \quad \Rightarrow
\]
\[
\Rightarrow \quad (A \subseteq f^{-1}[U] \in O(X))(\text{g}[^*\omega\text{-}\text{cl}(A) \subseteq f^{-1}[U]) \Rightarrow f[\text{g}[^*\omega\text{-}\text{cl}(A)] \subseteq U \quad \text{f is pre-\(^*\omega\text{-}\text{closed} \text{function})}
\]
\[
\Rightarrow \quad \text{g}[^*\omega\text{-}\text{cl}(f[A)] \subseteq \text{g}[^*\omega\text{-}\text{cl}(f[\text{g}[^*\omega\text{-}\text{cl}(A)]]) = f[\text{g}[^*\omega\text{-}\text{cl}(A)] \subseteq U.
\]

Definition 21. A function \( f : X \to Y \) is said to be \(^*\omega\text{-}\text{irresolute} \) if \( f^{-1}[V] \) is \(^*\omega\text{-}\text{closed} \) in \( X \) for every \(^*\omega\text{-}\text{closed} \) set \( V \) of \( Y \).

Corollary 5. Let \( f : X \to Y \) be a function. \( f \) is \(^*\omega\text{-}\text{irresolute} \) if \( f^{-1}[V] \) is \(^*\omega\text{-}\text{open} \) in \( X \) for every \(^*\omega\text{-}\text{open} \) set \( V \) of \( Y \).

Proposition 12. Let \( f : (X, \tau) \to (Y, \sigma) \) be a function. If \( f \) is \(^*\omega\text{-}\text{continuous} \) and \( \sigma[^*\omega\text{-}] = \sigma \) holds, then \( f \) is \(^*\omega\text{-}\text{irresolute} \).

The proof follows from Remark 5.
Theorem 22. Let $f : X \to Y$ be a function. If $f$ is an $\omega^e$-irresolute open bijection, then $f$ is $g\omega^e$-irresolute.

Proof. Let $F \in g\omega^e C(Y)$ and $f^{-1}[F] \subseteq U \subseteq O(X)$.

If $f$ is open bijection

$$f^{-1}[F] \subseteq U \subseteq O(X) \quad \Rightarrow \quad F \subseteq f[U] \in O(Y) \quad \Rightarrow \quad \omega^e-cl(F) \subseteq f[U]$$

$$\Rightarrow f^{-1}[\omega^e-cl(F)] \subseteq U \quad \Rightarrow \quad \omega^e-cl(f^{-1}[F]) \subseteq \omega^e-cl(f^{-1}[\omega^e-cl(F)]) = f^{-1}[\omega^e-cl(F)] \subseteq U.$$

Theorem 23. Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then the following properties hold:

(a) If $g$ is continuous and $f$ is $g\omega^e$-continuous, then $g \circ f$ is $g\omega^e$-continuous,
(b) If $g$ is $g\omega^e$-irresolute and $f$ is $g\omega^e$-irresolute, then $g \circ f$ is $g\omega^e$-irresolute,
(c) If $g$ is $g\omega^e$-irresolute and $f$ is $g\omega^e$-irresolute, then $g \circ f$ is $g\omega^e$-continuous,
(d) If $g$ is $g\omega^e$-continuous and $f$ is $\omega^e$-irresolute and $Y$ is $\omega^e-T_{1\frac{1}{2}}$ space, then $g \circ f$ is $\omega^e$-continuous,
(e) If $g$ and $f$ are $g\omega^e$-continuous and $\sigma^{\omega^e} = \sigma$, then $g \circ f$ is $\omega^e$-continuous.

Proof. Straightforward.

Theorem 24. Let $f : X \to Y$ be a function. Then the following properties hold:

(a) If $f$ is $g\omega^e$-irresolute and $X$ is $\omega^e-T_{1\frac{1}{2}}$ space, then $f$ is $\omega^e$-irresolute,
(b) If $f$ is $g\omega^e$-continuous and $X$ is $\omega^e-T_{1\frac{1}{2}}$ space, then $f$ is $\omega^e$-continuous.

Proof. Straightforward.

Theorem 25. Let $f : X \to Y$ be a pre-$\omega^e$-closed and $g\omega^e$-irresolute surjection. If $X$ is $\omega^e-T_{1\frac{1}{2}}$ space, then $Y$ is $\omega^e-T_{1\frac{1}{2}}$ space.

Proof. Let $F \in g\omega^e C(Y)$.

$$F \in g\omega^e C(Y) \quad \Rightarrow \quad f^{-1}[F] \in g\omega^e C(X) \quad \Rightarrow \quad f^{-1}[F] \in \omega^e C(X)$$

$$\Rightarrow f^{-1}[F] \subseteq \omega^e C(X) \quad \Rightarrow \quad f[f^{-1}[F]] = F \in \omega^e C(Y).$$

Definition 22. A function $f : X \to Y$ is said to be $g^*\omega^e$-continuous if $f^{-1}[V]$ is $g\omega^e$-closed in $X$ for every $\omega^e$-closed set $V$ of $Y$. 
Remark 7. Recall that every gω*-irresolute function is gω*-continuous function and every gω*-continuous function is gω*-continuous function.

Proposition 13. Let \( f : X \rightarrow Y \) be a function. If \( f \) is an open bijection and gω*-continuous, then \( f \) is gω*-irresolute.

Proof. Let \( A \in g\omega^* C(Y) \) and \( f^{-1}[A] \subseteq U \in O(X) \).

\[
\begin{align*}
  f^{-1}[A] \subseteq U &\in O(X) \quad \text{f is open bijection} \\
  \Rightarrow f[f^{-1}[A]] = A \subseteq f[U] &\in O(Y) \\
  \quad A \in g\omega^* C(Y) &\Rightarrow \omega^* -cl(A) \subseteq f[U] \\
  \Rightarrow f^{-1}[\omega^* -cl(A)] \subseteq U &\Rightarrow f^{-1}[\omega^* -cl(A)] \in g\omega^* C(X) \\
  \Rightarrow \omega^* -cl(f^{-1}[A]) \subseteq \omega^* -cl(f^{-1}[\omega^* -cl(A)]) &\subseteq U.
\end{align*}
\]

Proposition 14. Let \( f : X \rightarrow Y \) be a pre-gω*-closed and gω*-continuous bijection open function. If \( X \) is gω*-\( T_{\frac{1}{2}} \) space, then \( Y \) is gω*-\( T_{\frac{1}{2}} \) space.

Proof. Let \( A \in g\omega^* C(Y) \).

\[
\begin{align*}
  A \in g\omega^* C(Y) &\quad \text{Proposition 13} \\
  f \text{ is g}^\ast \omega^* \text{-continuous open bijection} &\Rightarrow f^{-1}[A] \in g\omega^* C(X) \\
  X \text{ is g}^\ast \omega^* -T_{\frac{1}{2}} \text{ space} &\Rightarrow \Rightarrow \\
  f^{-1}[A] \in \omega^* C(X) &\Rightarrow f[f^{-1}[A]] = A \in \omega^* C(Y).
\end{align*}
\]

Definition 23. A function \( f : X \rightarrow Y \) is said to be gω*-closed if \( f[F] \) is gω*-closed in \( Y \) for every closed set \( F \) of \( X \).

Remark 8. Every closed function is gω*-closed function but not conversely.

Example 5. Let \( X = \{1, 2\} \) with the topologies \( \tau = \{X, \emptyset, \{1\}\} \) and \( \sigma = \{X, \emptyset, \{2\}\} \). Let \( f : (X, \tau) \rightarrow (X, \sigma) \) be the identity function. Then \( f \) is gω*-closed but not closed since \( f[\{2\}] = \{2\} \) is not closed in \( X \).

Theorem 26. Let \( f : X \rightarrow Y \) be a function. Then, \( f \) is gω*-closed if and only if for each subset \( S \) of \( Y \) and for each open set \( U \) containing \( f^{-1}[S] \), there exists a gω*-open set \( V \) of \( Y \) such that \( S \subseteq V \) and \( f^{-1}[V] \subseteq U \).

Proof. (\( \Rightarrow \)) : Let \( S \subseteq Y \) and \( f^{-1}[S] \subseteq U \in O(X) \).

\[
\begin{align*}
  f^{-1}[S] \subseteq U &\in O(X) \\
  \Rightarrow (X \setminus U) \in O(X) &\Rightarrow (X \setminus U) \subseteq X \setminus f^{-1}[S] \\
  \Rightarrow f \text{ is g}^\ast \omega^* -closed &\Rightarrow \\
  \Rightarrow (f[X \setminus U] \in g\omega^* C(Y)) &\Rightarrow f[X \setminus U] \subseteq f[X \setminus f^{-1}[S]] = f[f^{-1}[Y \setminus S]] \subseteq Y \setminus S \\
  \Rightarrow (Y \setminus f[X \setminus U] \in g\omega^* O(Y)) &\Rightarrow V := Y \setminus f[X \setminus U] \\
  \Rightarrow (V \in g\omega^* O(Y)) &\Rightarrow (V \in g\omega^* O(Y))(S \subseteq V)(f^{-1}[V] \subseteq U).
\end{align*}
\]
Theorem 27. Let \( f : X \to Y \) be a function. If \( f \) is \( g \omega e^*\)-closed, then \( g \omega e^*\)-cl(\([fA]\)) \( \subseteq \) \([fcl(A)]\) for every subset \( A \) of \( X \).

**Proof.** Let \( A \subseteq X \).
\[
A \subseteq X \Rightarrow cl(A) \subseteq C(X) \quad \Rightarrow \quad f[cl(A)] \subseteq g \omega e^*C(Y)
\]
\[
g \omega e^*\text{-cl}(f[A]) \subseteq g \omega e^*\text{-cl}(f[cl(A)]) = f[cl(A)].
\]

Theorem 28. Let \( f : X \to Y \) be a function. If \( f \) is continuous, \( g \omega e^*\)-closed and \( A \) is a \( g \)-closed subset of \( X \), then \([fA]\) is \( g \omega e^*\)-closed.

**Proof.** Let \([fA]\) \( \subseteq U \in O(Y) \).
\[
\begin{align*}
f[\text{is continuous}] & \quad \Rightarrow \quad A \subseteq f^{-1}[fA] \subseteq f^{-1}[U] \in O(X) \\
& \quad A \in gC(X) \quad \Rightarrow \quad cl(A) \subseteq f^{-1}[U]
\end{align*}
\]
\[
(\text{cl}(A) \in C(X))(f[A] \subseteq f[\text{cl}(A)]) \subseteq f[f^{-1}[U]] \subseteq U
\]
\[
\begin{align*}
& \quad f \text{ is } g \omega e^*\text{-closed} \\
& \quad \Rightarrow \quad f[\text{cl}(A)] \in g \omega e^*C(Y))(f[A] \subseteq f[\text{cl}(A)]) \subseteq U
\end{align*}
\]
\[
\omega e^*\text{-cl}(f[A]) \subseteq \omega e^*\text{-cl}(f[\text{cl}(A)]) \subseteq U.
\]

Theorem 29. Let \( f : X \to Y \) be an open bijection. If \( f \) is \( g \omega e^*\)-continuous, then \( f \) is \( g \omega e^*\)- irresolute.

**Proof.** Let \( V \in g \omega e^*C(Y) \) and \( f^{-1}[V] \subseteq U \in O(X) \).
\[
\begin{align*}
(V \in g \omega e^*C(Y))(f^{-1}[V] \subseteq U \in O(X))
& \quad \Rightarrow \quad f \text{ is open bijection}
\end{align*}
\]
\[
\begin{align*}
& \quad \Rightarrow \quad (f[f^{-1}[V]] = V \subseteq f[U] \in O(Y))(\omega e^*\text{-cl}(V) \subseteq f[U])
\end{align*}
\]
\[
\begin{align*}
& \quad \Rightarrow \quad f^{-1}[\omega e^*\text{-cl}(V)] \in g \omega e^*C(X)(f^{-1}[\omega e^*\text{-cl}(V)]) \subseteq U
\end{align*}
\]
\[
\begin{align*}
& \quad \Rightarrow \quad \omega e^*\text{-cl}(f^{-1}[V]) \subseteq \omega e^*\text{-cl}(f^{-1}[\omega e^*\text{-cl}(V)]) \subseteq U.
\end{align*}
\]

Theorem 30. Let \( f : X \to Y \) be a function. If \( f \) is a continuous pre-\( \omega e^* \)-closed bijection, then the inverse function of \( f \) is \( g \omega e^*\)- irresolute.

**Proof.** Let \( A \in g \omega e^*C(X) \) and \( (f^{-1})^{-1}[A] = f[A] \subseteq U \in O(Y) \).
\[
\begin{align*}
(f[A] \subseteq U \in O(Y) \quad f \text{ is continuous})
& \quad \Rightarrow \quad f^{-1}[U] \in O(X) \quad A \in g \omega e^*C(X)
\end{align*}
\]
\[ \Rightarrow \omega^*\text{-cl}(A) \subseteq f^{-1}[U] \quad \Rightarrow \]

\( f \) is a pre-\( \omega^* \)-closed bijection \( \Rightarrow \)

\( (\{ f[\omega^*\text{-cl}(A)] \in \omega^*\text{C}(Y) \}) \cap \{ f[A] \subseteq \omega^*\text{-cl}(A) \} \subseteq f[f^{-1}[U]] = U \) \( \Rightarrow \omega^*\text{-cl}(f[A]) \subseteq \omega^*\text{-cl}(f[\omega^*\text{-cl}(A)]) = f[\omega^*\text{-cl}(A)] \subseteq U. \)

**Theorem 31.** Let \( f : X \to Y \) and \( g : Y \to Z \) be two functions. If \( f \) is a continuous surjection and \( g \circ f \) is \( \omega^* \)-closed, then \( g \) is \( \omega^* \)-closed.

**Proof.** Let \( V \in \text{C}(Y). \)

\[
\begin{align*}
\text{f is continuous} \quad &\Rightarrow \quad f^{-1}[V] \in \text{C}(X) \quad \text{f is surjective} \quad \Rightarrow \\
\Rightarrow \quad \{ g[f^{-1}[V]] = (g \circ f)[f^{-1}[V]] = g[V] \} \quad \Rightarrow \quad g[V] \in \omega^*\text{C}(X).
\end{align*}
\]

**Theorem 32.** Let \( f : X \to Y \) be a function. If \( f \) is \( \omega^* \)-closed continuous and \( X \) is normal, then \( Y \) is \( \omega^* \)-normal.

**Proof.** Let \( A, B \subseteq \text{C}(Y) \) and \( A \cap B = \emptyset. \)

\( (A, B \subseteq \text{C}(Y)) | (A \cap B = \emptyset) \quad \Rightarrow \quad \)

\( (f^{-1}[A], f^{-1}[B] \subseteq \text{C}(X)) | (f^{-1}[A] \cap f^{-1}[B] = f^{-1}[\emptyset] = \emptyset) \quad \Rightarrow \quad X \text{ is normal} \quad \Rightarrow \\
\Rightarrow \quad (\exists U \subseteq \text{O}(X, f^{-1}[A])) | (\exists V \subseteq \text{O}(X, f^{-1}[B])) | (U \cap V = \emptyset) \quad \text{Theorem 26} \quad \Rightarrow \\
\Rightarrow \quad (\exists G, H \subseteq \omega^*\text{C}(Y)) | (A \subseteq G) | (B \subseteq H) | (f^{-1}[G] \subseteq U) | (f^{-1}[H] \subseteq V) | (U \cap V = \emptyset) \quad \Rightarrow \\
\Rightarrow \quad (A \subseteq \omega^*\text{-int}(G)) | (B \subseteq \omega^*\text{-int}(H)) | (A \cap B \subseteq G \cap H) | (f^{-1}[G] \cap f^{-1}[H] = \emptyset) \quad \Rightarrow \\
\Rightarrow \quad (U' := \omega^*\text{-int}(G)) | (V' := \omega^*\text{-int}(H)) \quad \Rightarrow \\
\Rightarrow \quad (U' \subseteq \omega^*\text{O}(Y, A)) | (V' \subseteq \omega^*\text{O}(Y, B)) | (U' \cap V' = \emptyset).
\]

**Theorem 33.** Let \( f : X \to Y \) be a function. If \( f \) is a \( \omega^* \)-closed continuous surjection, \( \omega^* \)-open and \( X \) is regular, then \( Y \) is \( \omega^* \)-regular.

**Proof.** Let \( y \in Y \) and \( U \subseteq \text{O}(Y, y). \)

\( (y \in Y) | (U \subseteq \text{O}(Y, y)) \quad \Rightarrow \quad \{ \exists x \subseteq X | y = f(x) \} | (f^{-1}[U] \subseteq \text{O}(X, x)) \quad \text{X is regular} \quad \Rightarrow \\
\Rightarrow \quad (\exists V \subseteq \text{O}(X, x)) | (V \subseteq \text{cl}(V) \subseteq f^{-1}[U]) \quad \text{f is \( \omega^* \)-closed surjection} \quad \Rightarrow \)

\( (\exists V \subseteq \text{O}(X, x)) | (f[\text{cl}(V)] \in \omega^*\text{C}(X)) | (y \in f[V] \subseteq f[\text{cl}(V)] \subseteq U) \quad \text{f is \( \omega^* \)-open} \quad \Rightarrow \\
\Rightarrow \quad (f[V] \subseteq \omega^*\text{O}(Y, y)) | (\omega^*\text{-cl}(f[V]) \subseteq \omega^*\text{-cl}(f[\text{cl}(V)]) \subseteq U). \)
Conclusion

Many forms of generalized closed sets which are first defined by Levine [14] have been studied by many authors in recent years. This paper is concerned with the notion of generalized $\omega e^*$-closed sets which are defined by utilizing the concept of $\omega e^*$-open set. We have seen that this concept is weaker than many generalized closed set forms in the literature as will be seen in Figure 1. In addition, we gave some examples related to the concept but we could not find an example generalized $\omega e^*$-closed set which is not $\omega e^*$-closed. We believe that this study will help researchers to upgrade and support further studies related to compactness and connectedness etc. Also, the objects considered in the article may find an application in the area of both pure and applied sciences such as computational topology and digital topology.

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References


REFERENCES


