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On some closed sets and low separation axioms via topological ideals

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Abstract. This paper deals with the concepts of $\Lambda_{p(\star)}$-sets and $(\Lambda, p(\star))$-closed sets which are defined by utilizing the notions of pre-$\mathscr{I}$-open sets and pre-$\mathscr{I}$-closed sets. Moreover, we investigate some properties of $(\Lambda, p(\star))$-extremally disconnected ideal topological spaces. Several characterizations of $(\Lambda, p(\star))$-continuous functions are discussed. Especially, we introduce and characterize some low separation axioms of ideal topologies constructed by the concepts of pre-$\mathscr{I}$-open sets and the pre-$\mathscr{I}$-closure operator.

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Key Words and Phrases: pre-$\mathscr{I}$-open set, $\Lambda_{p(\star)}$-set, $(\Lambda, p(\star))$-closed set, $(\Lambda, p(\star))$-extremally disconnected, pre-$\mathscr{I}$-T_0, pre-$\mathscr{I}$-T_1, pre-$\mathscr{I}$-R_0, $(\Lambda, p(\star))$-continuous function

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1. Introduction

Openness and closedness are fundamental concept for the study and investigation in topological spaces. Many mathematicians introduced and studied the various types of generalizations of open sets. In 1982, Mashhour et al. [28] introduced the notion of preopen sets which is also known under the name of locally dense sets [12] in the literature. Kar and Bhattacharya [25] introduced new separation axioms pre-T_0, pre-T_1 and pre-T_2 by using preopen sets due to Mashhour et al. [28]. Caldas [5] and Jafari [22] introduced independently the notions of p-D-sets and a separation axiom p-D_1 which is strictly between pre-T_0 and pre-T_1. In [7], the present authors introduced two new classes of topological spaces called pre-R_0 and pre-R_1 spaces in terms of concept of preopen sets and investigated some of their fundamental properties. In 1986, Maki [27] introduced the concept of Λ-sets in topological spaces as the sets that coincide with their kernel. The kernel of a set A is the intersection of all open superset A. Arenas et al. [4] introduced and investigated the concept of λ-closed sets by involving Λ-sets and closed sets. Caldas et al. [10] introduced the concept of λ-closure of a set by utilizing the notion of λ-open sets defined in [4]. In [9], the present authors introduced and studied two new low

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separation axioms called λ-R_0 and λ-R_1 by utilizing the notions of λ-open sets and the λ-closure operator. Jafari et al. [23] introduced a new class of functions between topological spaces, namely almost λ-continuous functions and investigated several characterizations of almost λ-continuous functions. Ekici et al. [16] introduced a new class of generalization of continuous functions via λ-open sets called weakly λ-continuous functions and investigated some fundamental properties of such functions. Ganster et al. [19] introduced the notions of a pre-Λ-set and a pre-V-set in a topological space and investigated the topologies defined by these families of sets. Veličko [31] introduced and studied the concepts of δ-open sets, δ-closure operator and δ-closed sets. Georgiou et al. [20] by considering the notion of δ-closed sets, introduced and investigated Λ_δ-sets, (Λ, δ)-closed, (Λ, δ)-open sets and the (Λ, δ)-closure operator. Caldas and Jafari [6] introduced and investigated some new low separation axioms by using the notions of (Λ, δ)-open sets and the (Λ, δ)-closure operator. Cammaroto and Noiri [11] introduced and investigated three topological spaces (X, Λ_m), (X, Λ^*_{mc}) and $(X, \Lambda_{g\Lambda m})$ by using Λ_m-sets, (Λ, m)-closed sets and generalized Λ_m-sets, respectively. Caldas et al. [8] introduced and studied two new weak separation axioms called Λ_θ-R_0 and Λ_θ-R_1 spaces by using the notions of (Λ, θ)-open sets and the (Λ, θ)-closure operator.

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [26] and Vaidyanathswamy [30]. The topology τ of a space is enlarged to a topology $\tau^\star$ using an ideal $\mathscr{I}$ whose members are disjoint with the members of τ. Every topological space is an ideal topological space and all the results of ideal topological spaces are generalizations of the results established in topological spaces. In 1990, Janković and Hamlett [24] introduced the notion of $\mathscr{I}$-open sets in ideal topologial spaces. Abd El-Monsef et al. [18] further investigated $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açıkgöz et al. [2] introduced and investigated the notions of weakly-$\mathscr{I}$-continuous and weak*-$\mathscr{I}$-continuous functions in ideal topological spaces. Donthev [15] introduced the notion of pre-$\mathscr{I}$-open sets and obtained a decomposition of $\mathscr{I}$-continuity. Hatır and Noiri [21] introduced α-$\mathscr{I}$-open, semi-$\mathscr{I}$-open and β-$\mathscr{I}$-open sets via idealization and using these sets obtained new decompositions of continuity. In [3], the present authors studied the concepts of α-$\mathscr{I}$-continuity and α-$\mathscr{I}$-openness in ideal topological spaces and obtained several characterizations of these functions. Açıkgöz et al. [1] introduced two new classes of functions called α-$\mathscr{I}$-preirresolute functions and β-$\mathscr{I}$-preirresolute functions in ideal topologial spaces and investigated the relationships between these classes of functions and other classes of non-continuous functions. In 2009, Ekici and T. Noiri [17] introduced the concept of $\star$-extremally disconnected ideal topological spaces and investigated several characterizations of $\star$-extremally disconnected ideal topological spaces.

The paper is organized as follows. In Section 3, we introduce the notions of $\Lambda_{p(\star)}$-sets and $\Lambda_{p(\star)}$-sets. Moreover, some properties of $\Lambda_{p(\star)}$-sets and $\Lambda_{p(\star)}$-sets are investigated. In Section 4, we define $(\Lambda, p(\star))$-closed sets and investigate several characterizations of $(\Lambda, p(\star))$-extremally disconnected. In Section 5, we introduce the concept of $(\Lambda, p(\star))$-continuous functions and investigate some characterizations of such functions. In the last section, we introduce and study some new low separation axioms by using the concepts of pre-$\mathscr{I}$-open sets and the pre-$\mathscr{I}$-closure operator.

2. Preliminaries

We begin with some definitions and known results which will be used throughout this paper. In the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ represent the closure and the interior of A, respectively. A nonempty collection $\mathscr{I}$ of subsets of a set X is said to be an *ideal* on X if $\mathscr{I}$ satisfies the following two properties: (i) $A \in \mathscr{I}$ and $B \subseteq A \Rightarrow B \in \mathscr{I}$; (ii) $A \in \mathscr{I}$ and $B \in \mathscr{I} \Rightarrow A \cup B \in \mathscr{I}$. For a topological space (X, τ) with an ideal $\mathscr{I}$ on X, a set operator $(.)^\star : \mathscr{P}(X) \to \mathscr{P}(X)$ where $\mathscr{P}(X)$ is the set of all subsets of X, called a *local function* [26] of A with respect to $\mathscr{I}$ and τ is defined as follows: for $A \subseteq X$, $A^\star(\mathscr{I}, \tau) = \{x \in X \mid G \cap A \notin \mathscr{I}$ for every $G \in \tau(x)\}$ where $\tau(x) = \{G \in \tau \mid x \in G\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\mathscr{I}, \tau)$, called the $\star$-*topology* and finer than τ, is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star$ [24]. We shall simply write $A^\star$ for $A^\star(\mathscr{I}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathscr{I}, \tau)$. A basis $\mathscr{B}(\mathscr{I}, \tau)$ for $\tau^\star$ can be described as follows: $\mathscr{B}(\mathscr{I}, \tau) = \{V - I' \mid V \in \tau$ and $I' \in \mathscr{I}\}$. However, $\mathscr{B}(\mathscr{I}, \tau)$ is not always a topology [24]. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called $\star$-*closed* ($\tau^\star$-*closed*) [24] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}, \tau))$ is denoted by $\mathrm{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *pre-$\mathscr{I}$-open* [15] if $A \subseteq \mathrm{Int}(\mathrm{Cl}^\star(A))$. The complement of a pre-$\mathscr{I}$-open set is called *pre-$\mathscr{I}$-closed*. The family of all pre-$\mathscr{I}$-open sets of an ideal topological space $(X, \tau, \mathscr{I})$ is denoted by $p\mathscr{I}O(X, \tau)$. For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all pre-$\mathscr{I}$-closed sets of X containing A is called the *pre-$\mathscr{I}$-closure* [13] of A and is denoted by $p\imath\mathrm{Cl}(A)$. The union of all pre-$\mathscr{I}$-open sets contained in A is called the *pre-$\mathscr{I}$-interior* of A and is denoted by $p\imath\mathrm{Int}(A)$.

Lemma 1. [13] *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$ and $x \in X$. Then, the following properties hold:*

(1) $x \in p\imath Cl(A)$ if and only if $U \cap A \neq \emptyset$ for every pre-$\mathscr{I}$-open set U of X containing x.

(2) A is pre-$\mathscr{I}$-closed if and only if $A = p\imath Cl(A)$.

(3) $X - p\imath Cl(A) = p\imath Int(X - A)$.

(4) $X - p\imath Int(A) = p\imath Cl(X - A)$.

Lemma 2. [14, 29] *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. Then, $p\imath Cl(A) = A \cup Cl(Int^\star(A))$.*

3. Some properties of $\Lambda_{p(\star)}$-sets and $\delta_{p(\star)}$-sets

In this section, we introduce the notions of $\Lambda_{p(\star)}$-sets and $\delta_{p(\star)}$-sets. Moreover, several properties of $\Lambda_{p(\star)}$-sets and $\delta_{p(\star)}$-sets are discussed.

Definition 1. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. A subset $\Lambda_{p(\star)}(A)$ is defined as follows: $\Lambda_{p(\star)}(A) = \cap\{U \mid A \subseteq U; U$ is pre-$\mathscr{I}$-open$\}$.*

Proposition 1. *For subsets A, B and $C_\gamma(\gamma \in \Gamma)$ of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $A \subseteq \Lambda_{p(\star)}(A)$.

(2) If $A \subseteq B$, then $\Lambda_{p(\star)}(A) \subseteq \Lambda_{p(\star)}(B)$.

(3) $\Lambda_{p(\star)}(\Lambda_{p(\star)}(A)) = \Lambda_{p(\star)}(A)$.

(4) If A is a pre-$\mathscr{I}$-open set, then $\Lambda_{p(\star)}(A) = A$.

(5) $\Lambda_{p(\star)}(\cup\{C_\gamma|\gamma \in \Gamma\}) = \cup\{\Lambda_{p(\star)}(C_\gamma)|\gamma \in \Gamma\}$.

(6) $\Lambda_{p(\star)}(\cap\{C_\gamma|\gamma \in \Gamma\}) \subseteq \cap\{\Lambda_{p(\star)}(C_\gamma)|\gamma \in \Gamma\}$.

Proof. We prove only properties (5) and (6) since the others are immediate consequences of Definition 1.

(5) First for each $\gamma \in \Gamma$, $\Lambda_{p(\star)}(C_\gamma) \subseteq \Lambda_{p(\star)}(\cup_{\gamma\in\Gamma}C_\gamma)$. Thus,

$$\cup_{\gamma\in\Gamma}\Lambda_{p(\star)}(C_\gamma) \subseteq \Lambda_{p(\star)}(\cup_{\gamma\in\Gamma}C_\gamma).$$

On the other hand, let $x \notin \cup_{\gamma\in\Gamma}\Lambda_{p(\star)}(C_\gamma)$. Then, $x \notin \Lambda_{p(\star)}(C_\gamma)$ for each $\gamma \in \Gamma$ and so there exists a pre-$\mathscr{I}$-open set V_γ such that $C_\gamma \subseteq V_\gamma$ and $x \notin V_\gamma$ for each $\gamma \in \Gamma$. Thus, $\cup_{\gamma\in\Gamma}C_\gamma \subseteq \cup_{\gamma\in\Gamma}V_\gamma$ and hence $\cup_{\gamma\in\Gamma}V_\gamma$ is a pre-$\mathscr{I}$-open set which does not contain x. This implies that $x \notin \Lambda_{p(\star)}(\cup_{\gamma\in\Gamma}C_\gamma)$. Therefore, $\Lambda_{p(\star)}(\cup_{\gamma\in\Gamma}C_\gamma) \subseteq \cup_{\gamma\in\Gamma}\Lambda_{p(\star)}(C_\gamma)$. Consequently, we obtain $\Lambda_{p(\star)}(\cup_{\gamma\in\Gamma}C_\gamma) = \cup_{\gamma\in\Gamma}\Lambda_{p(\star)}(C_\gamma)$.

(6) Suppose that $x \notin \cap_{\gamma\in\Gamma}\Lambda_{p(\star)}(C_\gamma)$. There exists $\gamma_0 \in \Gamma$ such that $x \notin \Lambda_{p(\star)}(C_{\gamma_0})$ and there exists a pre-$\mathscr{I}$-open set V such that $x \notin V$ and $C_{\gamma_0} \subseteq V$. Therefore,

$$\cap_{\gamma\in\Gamma}C_\gamma \subseteq C_{\gamma_0} \subseteq V.$$

Thus, $x \notin \Lambda_{p(\star)}(\cap_{\gamma\in\Gamma}C_\gamma)$ and hence $\Lambda_{p(\star)}(\cap_{\gamma\in\Gamma}C_\gamma) \subseteq \cap_{\gamma\in\Gamma}\Lambda_{p(\star)}(C_\gamma)$.

Remark 1. *In Proposition 1(6), the converse is not always true as the following example shows.*

Example 1. *Let $X = \{-1, 1\}$ with a topology $\tau = \{\emptyset, \{-1\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{1\}\}$. Let $A = \{-1\}$ and $B = \{1\}$. Then, $\Lambda_{p(\star)}(A \cap B) = \Lambda_{p(\star)}(\emptyset) = \emptyset$ and*

$$\Lambda_{p(\star)}(A) \cap \Lambda_{p(\star)}(B) = \{-1\}.$$

Definition 2. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called a $\Lambda_{p(\star)}$-set if $A = \Lambda_{p(\star)}(A)$. The family of all $\Lambda_{p(\star)}$-sets of an ideal topological space $(X, \tau, \mathscr{I})$ is denoted by $\Lambda_{p(\star)}(X)$.*

Proposition 2. *For subsets A and $B_\gamma(\gamma \in \Gamma)$ of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $\Lambda_{p(\star)}(A)$ is a $\Lambda_{p(\star)}$-set.

(2) If A is a pre-$\mathscr{I}$-open set, then A is a $\Lambda_{p(\star)}$-set.

(3) If B_γ is a $\Lambda_{p(\star)}$-set for each $\gamma \in \Gamma$, then $\cup_{\gamma\in\Gamma} B_\gamma$ is a $\Lambda_{p(\star)}$-set.

(4) If B_γ is a $\Lambda_{p(\star)}$-set for each $\gamma \in \Gamma$, then $\cap_{\gamma\in\Gamma} B_\gamma$ is a $\Lambda_{p(\star)}$-set.

Proof. (1) and (2) are obvious.

(3) Let B_γ be a $\Lambda_{p(\star)}$-set for each $\gamma \in \Gamma$. Then, by Proposition 1(5), we have

$$\cup_{\gamma\in\Gamma} B_\gamma = \cup_{\gamma\in\Gamma} \Lambda_{p(\star)}(B_\gamma) = \Lambda_{p(\star)}(\cup_{\gamma\in\Gamma} B_\gamma) \supseteq \cup_{\gamma\in\Gamma} B_\gamma.$$

Thus, $\cup_{\gamma\in\Gamma} B_\gamma = \Lambda_{p(\star)}(\cup_{\gamma\in\Gamma} B_\gamma)$ and hence $\cup_{\gamma\in\Gamma} B_\gamma$ is a $\Lambda_{p(\star)}$-set.

(4) Let B_γ be a $\Lambda_{p(\star)}$-set for each $\gamma \in \Gamma$. Thus, by Proposition 1(6),

$$\cap_{\gamma\in\Gamma} B_\gamma = \cap_{\gamma\in\Gamma} \Lambda_{p(\star)}(B_\gamma) \supseteq \Lambda_{p(\star)}(\cap_{\gamma\in\Gamma} B_\gamma) \supseteq \cap_{\gamma\in\Gamma} B_\gamma$$

and hence $\cap_{\gamma\in\Gamma} B_\gamma = \Lambda_{p(\star)}(\cap_{\gamma\in\Gamma} B_\gamma)$. This shows that $\cap_{\gamma\in\Gamma} B_\gamma$ is a $\Lambda_{p(\star)}$-set.

Proposition 3. *For an ideal topological space $(X, \tau, \mathscr{I})$, the pair $(X, \Lambda_{p(\star)}(X))$ is an Alexandroff space.*

Proof. (1) $\emptyset, X \in \Lambda_{p(\star)}(X)$ since $\emptyset, X \in p\mathscr{I}O(X, \tau)$ and $p\mathscr{I}O(X, \tau) \subseteq \Lambda_{p(\star)}(X)$.

(2) Let $V_\gamma \in \Lambda_{p(\star)}(X)$ for each $\gamma \in \Gamma$. Then, we have $\cup_{\gamma\in\Gamma} V_\gamma \in \Lambda_{p(\star)}(X)$ by Proposition 2(3).

(3) Let $V_\gamma \in \Lambda_{p(\star)}(X)$ for each $\gamma \in \Gamma$. Then, we have $\cap_{\gamma\in\Gamma} V_\gamma \in \Lambda_{p(\star)}(X)$ by Proposition 2(4).

Proposition 4. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, $\Lambda_{p(\star)}(X) = \Lambda_{\Lambda_{p(\star)}}(X)$.*

Proof. By Proposition 2(2), $p_{\mathscr{I}}O(X) \subseteq \Lambda_{p(\star)}(X)$. For any subset A of X, we have $\Lambda_{\Lambda_{p(\star)}}(A) = \cap\{U \mid A \subseteq U; U \in \Lambda_{p(\star)}(X)\} \subseteq \cap\{U \mid A \subseteq U; U \in p_{\mathscr{I}}O(X)\} = \Lambda_{p(\star)}(A)$ and hence $\Lambda_{\Lambda_{p(\star)}}(A) \subseteq \Lambda_{p(\star)}(A)$. On the other hand, suppose that $x \notin \Lambda_{\Lambda_{p(\star)}}(A)$. Then, there exists $U \in \Lambda_{p(\star)}(X)$ such that $A \subseteq U$ and $x \notin U$. Since $x \notin U$, there exists a pre-$\mathscr{I}$-open set V such that $U \subseteq V$ and $x \notin V$. Thus, $x \notin \Lambda_{p(\star)}(A)$ and hence $\Lambda_{\Lambda_{p(\star)}}(A) \supseteq \Lambda_{p(\star)}(A)$. Consequently, we obtain $\Lambda_{\Lambda_{p(\star)}}(A) = \Lambda_{p(\star)}(A)$.

Definition 3. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. A subset $\delta_{p(\star)}(A)$ is defined as follows: $\delta_{p(\star)}(A) = \cup\{F \mid F \subseteq A; F$ is pre-$\mathscr{I}$-closed$\}$.*

Definition 4. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called a $\delta_{p(\star)}$-set if $A = \delta_{p(\star)}(A)$. The family of all $\delta_{p(\star)}$-sets of an ideal topological space $(X, \tau, \mathscr{I})$ is denoted by $\delta_{p(\star)}(X)$.*

Proposition 5. *For subsets A, B and $C_\gamma(\gamma \in \Gamma)$ of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $\delta_{p(\star)}(A) \subseteq A$.

(2) If $A \subseteq B$, *then* $\delta_{p(\star)}(A) \subseteq \delta_{p(\star)}(B)$.

(3) $\delta_{p(\star)}(\delta_{p(\star)}(A)) = \delta_{p(\star)}(A)$.

(4) If A is a pre-$\mathscr{I}$-closed set, then $\delta_{p(\star)}(A) = A$.

(5) $\delta_{p(\star)}(\cap\{C_\gamma | \gamma \in \Gamma\}) = \cap\{\delta_{p(\star)}(C_\gamma) | \gamma \in \Gamma\}$.

(6) $\delta_{p(\star)}(\cup\{C_\gamma | \gamma \in \Gamma\}) \supseteq \cup\{\delta_{p(\star)}(C_\gamma) | \gamma \in \Gamma\}$.

(7) $\Lambda_{p(\star)}(X - A) = X - \delta_{p(\star)}(A)$ *and* $\delta_{p(\star)}(X - A) = X - \Lambda_{p(\star)}(A)$.

Proposition 6. *For subsets A and $B_\gamma(\gamma \in \Gamma)$ of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $\delta_{p(\star)}(A)$ *is a* $\delta_{p(\star)}$*-set.*

(2) If A is a pre-$\mathscr{I}$-closed set, then A is a $\delta_{p(\star)}$-set.

(3) If B_γ is a $\delta_{p(\star)}$-set for each $\gamma \in \Gamma$, then $\cap_{\gamma\in\Gamma} B_\gamma$ is a $\delta_{p(\star)}$-set.

(4) If B_γ is a $\delta_{p(\star)}$-set for each $\gamma \in \Gamma$, then $\cup_{\gamma\in\Gamma} B_\gamma$ is a $\delta_{p(\star)}$-set.

Proposition 7. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. Then,*

$$\Lambda_{p(\star)}(A) = \{x \in X \mid p\imath Cl(\{x\}) \cap A \neq \emptyset\}.$$

Proof. Let $x \in \Lambda_{p(\star)}(A)$. Suppose that $p\imath\mathrm{Cl}(\{x\}) \cap A = \emptyset$. Then, $x \notin X - p\imath\mathrm{Cl}(\{x\})$ which is a pre-$\mathscr{I}$-open set containing A. This is a contradiction. Thus, $p\imath\mathrm{Cl}(\{x\}) \cap A \neq \emptyset$. On the other hand, let $x \in X$ such that $p\imath\mathrm{Cl}(\{x\}) \cap A \neq \emptyset$ and suppose that $x \notin \Lambda_{p(\star)}(A)$. Then, there exists a pre-$\mathscr{I}$-open set U containing A and $x \notin U$. Let $y \in p\imath\mathrm{Cl}(\{x\}) \cap A$. Then, we have $y \in p\imath\mathrm{Cl}(\{x\})$ and $y \in A \subseteq U$. Therefore, $U \cap \{x\} \neq \emptyset$. This is a contradiction. Consequently, we obtain $x \in \Lambda_{p(\star)}(A)$.

Definition 5. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$ and $x \in X$. Then $\prec x \succ_{p(\star)}$ is defined by $\prec x \succ_{p(\star)} = p\imath Cl(\{x\}) \cap \Lambda_{p(\star)}(\{x\})$.*

Proposition 8. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) For each $x \in X$, $\Lambda_{p(\star)}(\prec x \succ_{p(\star)}) = \Lambda_{p(\star)}(\{x\})$.

(2) For each $x \in X$, $p\imath Cl(\prec x \succ_{p(\star)}) = p\imath Cl(\{x\})$.

(3) For each pre-$\mathscr{I}$-open set V and each $x \in V$, $\prec x \succ_{p(\star)} \subseteq V$.

(4) For each pre-$\mathscr{I}$-closed set F and each $x \in F$, $\prec x \succ_{p(\star)} \subseteq F$.

Proof. (1) Let $x \in X$. Then, $\{x\} \subseteq p\imath\text{Cl}(\{x\}) \cap \Lambda_{p(\star)}(\{x\}) = \prec x \succ_{p(\star)}$, by Proposition 1(2), $\Lambda_{p(\star)}(\prec x \succ_{p(\star)}) \supseteq \Lambda_{p(\star)}(\{x\})$. On the other hand, suppose that $y \notin \Lambda_{p(\star)}(\{x\})$. Then, there exists a pre-$\mathscr{I}$open set U such that $x \in U$ and $y \notin U$. Since

$$\prec x \succ_{p(\star)} \subseteq \Lambda_{p(\star)}(\{x\}) \subseteq \Lambda_{p(\star)}(U) = U,$$

we have $\Lambda_{p(\star)}(\prec x \succ_{p(\star)}) \subseteq U$ and hence $y \notin \Lambda_{p(\star)}(\prec x \succ_{p(\star)})$. Thus,

$$\Lambda_{p(\star)}(\prec x \succ_{p(\star)}) \subseteq \Lambda_{p(\star)}(\{x\}).$$

Consequently, we obtain $\Lambda_{p(\star)}(\prec x \succ_{p(\star)}) = \Lambda_{p(\star)}(\{x\})$.

(2) Let $x \in X$. Since $\{x\} \subseteq \prec x \succ_{p(\star)}$, we have $p\imath\text{Cl}(\{x\}) \subseteq p\imath\text{Cl}(\prec x \succ_{p(\star)})$. On the other hand, we have $\prec x \succ_{p(\star)} \subseteq p\imath\text{Cl}(\{x\})$ and so

$$p\imath\text{Cl}(\prec x \succ_{p(\star)}) \subseteq p\imath\text{Cl}(p\imath\text{Cl}(\{x\})) = p\imath\text{Cl}(\{x\}).$$

Thus, $p\imath\text{Cl}(\prec x \succ_{p(\star)}) = p\imath\text{Cl}(\{x\})$.

(3) Let V be any pre-$\mathscr{I}$-open set and $x \in V$. Then, $\Lambda_{p(\star)}(\prec x \succ_{p(\star)}) \subseteq V$ and hence $\prec x \succ_{p(\star)} \subseteq V$.

(4) Let F be any pre-$\mathscr{I}$-closed set and $x \in F$. Therefore, we have

$$\prec x \succ_{p(\star)} = p\imath\text{Cl}(\{x\}) \cap \Lambda_{p(\star)}(\{x\}) \subseteq p\imath\text{Cl}(\{x\}) \subseteq p\imath\text{Cl}(F) = F.$$

4. $(\Lambda, p(\star))$-closed sets

In this section, we introduce the concept of $(\Lambda, p(\star))$-closed sets and investigate some properties of $(\Lambda, p(\star))$-closed sets.

Definition 6. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be $(\Lambda, p(\star))$-closed if $A = T \cap C$, where T is a $\Lambda_{p(\star)}$-set and C is a pre-$\mathscr{I}$-closed set. The collection of all $(\Lambda, p(\star))$-closed sets in an ideal topological space $(X, \tau, \mathscr{I})$ is denoted by $(\Lambda, p(\star))C(X)$.*

Theorem 1. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) A is $(\Lambda, p(\star))$-closed.

(2) $A = T \cap p\imath Cl(A)$, where T is a $\Lambda_{p(\star)}$-set.

(3) $A = \Lambda_{p(\star)}(A) \cap p\imath Cl(A)$.

Proof. (1) ⇒ (2): Suppose that $A = T \cap C$, where T is a $\Lambda_{p(\star)}$-set and C is a pre-$\mathscr{I}$-closed set. Since $A \subseteq C$, we have $p\imath\text{Cl}(A) \subseteq C$ and $A = T \cap C \supseteq T \cap p\imath\text{Cl}(A) \supseteq A$. Consequently, we obtain $A = T \cap p\imath\text{Cl}(A)$.

(2) ⇒ (3): Suppose that $A = T \cap p\imath\text{Cl}(A)$, where T is a $\Lambda_{p(\star)}$-set. Since $A \subseteq T$, we have $\Lambda_{p(\star)}(A) \subseteq \Lambda_{p(\star)}(T) = T$ and hence $A \subseteq \Lambda_{p(\star)}(A) \cap p\imath\text{Cl}(A) \subseteq T \cap p\imath\text{Cl}(A) = A$. Thus, $A = \Lambda_{p(\star)}(A) \cap p\imath\text{Cl}(A)$.

(3) ⇒ (1): Since $\Lambda_{p(\star)}(A)$ is $\Lambda_{p(\star)}$-set, $p\imath\text{Cl}(A)$ is a pre-$\mathscr{I}$-closed set and

$$A = \Lambda_{p(\star)}(A) \cap p\imath\text{Cl}(A).$$

This shows that A is $(\Lambda, p(\star))$-closed.

Remark 2. *Every $\Lambda_{p(\star)}$-set (resp. pre-$\mathscr{I}$-closed set) is $(\Lambda, p(\star))$-closed.*

The converse of Remark 2 is not true in general as shown by the following example.

Example 2. *Let $X = \{1, 2\}$ with a topology $\tau = \{\emptyset, \{1\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{1\}\}$. Let $A = \{1\}$. Then, A is $(\Lambda, p(\star))$-closed but it is not pre-$\mathscr{I}$-closed. Moreover, let $B = \{2\}$, then B is $(\Lambda, p(\star))$-closed but it is not a $\Lambda_{p(\star)}$-set.*

Definition 7. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be $(\Lambda, p(\star))$-open if the complement of A is $(\Lambda, p(\star))$-closed. The collection of all $(\Lambda, p(\star))$-open sets in an ideal topological space $(X, \tau, \mathscr{I})$ is denoted by $\Lambda_{p(\star)}O(X)$.*

Theorem 2. *Let $A_\gamma (\gamma \in \Gamma)$ be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. Then, the following properties hold:*

(1) If A_γ is $(\Lambda, p(\star))$-closed for each $\gamma \in \Gamma$, then $\cap\{A_\gamma \mid \gamma \in \Gamma\}$ is $(\Lambda, p(\star))$-closed.

(2) If A_γ is $(\Lambda, p(\star))$-open for each $\gamma \in \Gamma$, then $\cup\{A_\gamma \mid \gamma \in \Gamma\}$ is $(\Lambda, p(\star))$-open.

Proof. (1) Suppose that A_γ is $(\Lambda, p(\star))$-closed for each $\gamma \in \Gamma$. Then, for each γ, there exist a $\Lambda_{p(\star)}$-set T_γ and a pre-$\mathscr{I}$-closed set C_γ such that $A_\gamma = T_\gamma \cap C_\gamma$. Thus, $\cap_{\gamma\in\Gamma} A_\gamma = \cap_{\gamma\in\Gamma}(T_\gamma \cap C_\gamma) = (\cap_{\gamma\in\Gamma} T_\gamma) \cap (\cap_{\gamma\in\Gamma} C_\gamma)$. Since $\cap_{\gamma\in\Gamma} C_\gamma$ is pre-$\mathscr{I}$-closed and $\cap_{\gamma\in\Gamma} T_\gamma$ is a $\Lambda_{p(\star)}$-set by Proposition 2(4), we have $\cap_{\gamma\in\Gamma} A_\gamma$ is $(\Lambda, p(\star))$-closed.

(2) Let A_γ is $(\Lambda, p(\star))$-open for each $\gamma \in \Gamma$. Then, $X - A_\gamma$ is $(\Lambda, p(\star))$-closed. Since $X - \cup_{\gamma\in\Gamma} A_\gamma = \cap_{\gamma\in\Gamma}(X - A_\gamma)$ and by (1), $\cup_{\gamma\in\Gamma} A_\gamma$ is $(\Lambda, p(\star))$-open.

Theorem 3. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) A is $(\Lambda, p(\star))$-open.

(2) $A = S \cup V$, where S is a $\delta_{p(\star)}$-set and V is a pre-$\mathscr{I}$-open set.

(3) $A = S \cup p\imath Int(A)$, where S is a $\delta_{p(\star)}$-set.

(4) $A = \delta_{p(\star)}(A) \cup p\iota Int(A)$.

Proof. The proof follows from Theorem 1.

Definition 8. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. A point $x \in X$ is called a $(\Lambda, p(\star))$-cluster point of A if $A \cap U \neq \emptyset$ for every $(\Lambda, p(\star))$-open set U containing x. The set of all $(\Lambda, p(\star))$-cluster points of A is called the $(\Lambda, p(\star))$-closure of A and is denoted by $A^{(\Lambda,p(\star))}$.*

Lemma 3. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathscr{I})$. For the $(\Lambda, p(\star))$-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,p(\star))}$ *and* $[A^{(\Lambda,p(\star))}]^{(\Lambda,p(\star))} = A^{(\Lambda,p(\star))}$.

(2) $A^{(\Lambda,p(\star))} = \cap\{F \mid A \subseteq F \text{ and } F \text{ is } (\Lambda, p(\star))\text{-closed}\}$.

(3) If $A \subseteq B$, *then* $A^{(\Lambda,p(\star))} \subseteq B^{(\Lambda,p(\star))}$.

(4) A *is* $(\Lambda, p(\star))$*-closed if and only if* $A^{(\Lambda,p(\star))} = A$.

(5) $A^{(\Lambda,p(\star))}$ *is* $(\Lambda, p(\star))$*-closed.*

Definition 9. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. The union of all $(\Lambda, p(\star))$-open sets contained in A is called the $(\Lambda, p(\star))$-interior of A and is denoted by $A_{(\Lambda,p(\star))}$.*

Lemma 4. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathscr{I})$. For the $(\Lambda, p(\star))$-interior, the following properties hold:*

(1) $[A_{(\Lambda,p(\star))}]_{(\Lambda,p(\star))} = A_{(\Lambda,p(\star))}$.

(2) If $A \subseteq B$, *then* $A_{(\Lambda,p(\star))} \subseteq B_{(\Lambda,p(\star))}$.

(4) A *is* $(\Lambda, p(\star))$*-open if and only if* $A_{(\Lambda,p(\star))} = A$.

(5) $A_{(\Lambda,p(\star))}$ *is* $(\Lambda, p(\star))$*-open.*

Definition 10. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called semi-$(\Lambda, p(\star))$-open (resp. pre-$(\Lambda, p(\star))$-open, α-$(\Lambda, p(\star))$-open, β-$(\Lambda, p(\star))$-open) if $A \subseteq [A_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))}$ (resp. $A \subseteq [A^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$, $A \subseteq [[A_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$, $A \subseteq [[A^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))}$). The complement of a semi-$(\Lambda, p(\star))$-open (resp. pre-$(\Lambda, p(\star))$-open, α-$(\Lambda, p(\star))$-open, β-$(\Lambda, p(\star))$-open) set is called semi-$(\Lambda, p(\star))$-closed (resp. pre-$(\Lambda, p(\star))$-closed, α-$(\Lambda, p(\star))$-closed, β-$(\Lambda, p(\star))$-closed).*

Definition 11. *An ideal topological space $(X, \tau, \mathscr{I})$ is called $(\Lambda, p(\star))$-extremally disconnected if the $(\Lambda, p(\star))$-closure of every $(\Lambda, p(\star))$-open set of X is $(\Lambda, p(\star))$-open.*

Example 3. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{c\}\}$. Then $(X, \tau, \mathscr{I})$ is a $(\Lambda, p(\star))$-extremally disconnected space.*

Theorem 4. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ *is* $(\Lambda, p(\star))$*-extremally disconnected.*

(2) $F_{(\Lambda,p(\star))}$ *is* $(\Lambda, p(\star))$*-closed for every* $(\Lambda, p(\star))$*-closed set* F *of* X*.*

(3) $[A_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))} \subseteq [A^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$ *for every subset* A *of* X*.*

(4) Every semi-$(\Lambda, p(\star))$*-open set is pre-*$(\Lambda, p(\star))$*-open.*

(5) The $(\Lambda, p(\star))$*-closure of every* β*-*$(\Lambda, p(\star))$*-open set of* X *is* $(\Lambda, p(\star))$*-open.*

(6) Every β*-*$(\Lambda, p(\star))$*-open set is pre-*$(\Lambda, p(\star))$*-open.*

(7) For every subset A *of* X*,* A *is* α*-*$(\Lambda, p(\star))$*-open if and only if it is semi-*$(\Lambda, p(\star))$*-open.*

Proof. (1) ⇒ (2): Let A be any $(\Lambda, p(\star))$-closed set. Then, $X - A$ is $(\Lambda, p(\star))$-open and by (1), we have $(X - A)^{(\Lambda,p(\star))} = X - A_{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open. Thus, $A_{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-closed.

(2) ⇒ (3): Let A be any subset of X. Then, $X - A_{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-closed and by (2), $[X - A_{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-closed. Therefore, we have $[A_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open and hence $[A_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))} \subseteq [A^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$.

(3) ⇒ (4): Let V be any semi-$(\Lambda, p(\star))$-open set. By (3), we have

$$V \subseteq [V_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))} \subseteq [V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$$

and hence V is pre-$(\Lambda, p(\star))$-open.

(4) ⇒ (5): Let V be any β-$(\Lambda, p(\star))$-open set. Then, $V^{(\Lambda,p(\star))}$ is semi-$(\Lambda, p(\star))$-open and by (4), we have $V^{(\Lambda,p(\star))}$ is pre-$(\Lambda, p(\star))$-open. Thus, $V^{(\Lambda,p(\star))} \subseteq [V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$ and hence $V^{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open.

(5) ⇒ (6): Let V be any β-$(\Lambda, p(\star))$-open set. By (5), $V^{(\Lambda,p(\star))} = [V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$. Therefore, $V \subseteq V^{(\Lambda,p(\star))} = [V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$ and hence V is pre-$(\Lambda, p(\star))$-open.

(6) ⇒ (7): Let V be any semi-$(\Lambda, p(\star))$-open set. Then, V is β-$(\Lambda, p(\star))$-open and by (6), V is pre-$(\Lambda, p(\star))$-open. Since V is semi-$(\Lambda, p(\star))$-open and pre-$(\Lambda, p(\star))$-open, V is α-$(\Lambda, p(\star))$-open.

(7) ⇒ (1): Let V be any $(\Lambda, p(\star))$-open set. Then, $V^{(\Lambda,p(\star))}$ is semi-$(\Lambda, p(\star))$-open and by (7), we have $V^{(\Lambda,p(\star))}$ is α-$(\Lambda, p(\star))$-open. Thus,

$$V^{(\Lambda,p(\star))} \subseteq [[[V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))} = [V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$$

and hence $V^{(\Lambda,p(\star))} = [V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$. Therefore, $V^{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open. This shows that $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-extremally disconnected.

Definition 12. *An ideal topological space $(X, \tau, \mathscr{I})$ is said to be $(\Lambda, p(\star))$-normal if, for any pair disjoint $(\Lambda, p(\star))$-open sets U and V, there exist disjoint $(\Lambda, p(\star))$-closed sets F and H such that $U \subseteq F$ and $V \subseteq H$.*

Theorem 5. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-normal.

(2) $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-extremally disconnected.

Proof. (1) $\Rightarrow$ (2): Let U be any $(\Lambda, p(\star))$-open set. Then, we have U and $V = X - U^{(\Lambda,p(\star))}$ are disjoint $(\Lambda, p(\star))$-open sets. There exist disjoint $(\Lambda, p(\star))$-closed sets F and H such that $U \subseteq F$ and $V \subseteq H$. Since

$$U^{(\Lambda,p(\star))} \subseteq F^{(\Lambda,p(\star))} = F \subseteq X - H \subseteq X - V = U^{(\Lambda,p(\star))},$$

we have $U^{(\Lambda,p(\star))} = F$. Since $V \subseteq H \subseteq X - F = V$, $V = H$. Thus, $U^{(\Lambda,p(\star))} = X - H$ is $(\Lambda, p(\star))$-open. This shows that $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-extremally disconnected.

(2) $\Rightarrow$ (1): Let U and V be any two disjoint $(\Lambda, p(\star))$-open sets. Then, $U^{(\Lambda,p(\star))}$ and $X - U^{(\Lambda,p(\star))}$ are disjoint $(\Lambda, p(\star))$-closed sets containing U and V, respectively. Thus, $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-normal.

Definition 13. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(1) regular $(\Lambda, p(\star))$-open if $A = [A^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$;

(2) regular $(\Lambda, p(\star))$-closed if $A = [A_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))}$.

Theorem 6. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-extremally disconnected.

(2) Every regular $(\Lambda, p(\star))$-open set is $(\Lambda, p(\star))$-closed.

(3) Every regular $(\Lambda, p(\star))$-closed set is $(\Lambda, p(\star))$-open.

Proof. (1) $\Rightarrow$ (2): Suppose that $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-extremally disconnected. Let V be any regular $(\Lambda, p(\star))$-open set. Then, we have $V = [V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$. Since $V^{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open, $V = [V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))} = V^{(\Lambda,p(\star))}$ and hence V is $(\Lambda, p(\star))$-closed.

(2) $\Rightarrow$ (1): Suppose that every regular $(\Lambda, p(\star))$-open set is $(\Lambda, p(\star))$-closed. Let V be any $(\Lambda, p(\star))$-open set. Then, we have $[V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$ is regular $(\Lambda, p(\star))$-open and by (2), $[V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$ is closed. Thus, $V^{(\Lambda,p(\star))} \subseteq [[V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))} = [V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$ and hence $V^{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open. Therefore, $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-extremally disconnected.

(2) $\Leftrightarrow$ (3): This is obvious.

Theorem 7. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-extremally disconnected.

(2) The $(\Lambda, p(\star))$-closure of every semi-$(\Lambda, p(\star))$-open set is $(\Lambda, p(\star))$-open.

(3) The $(\Lambda, p(\star))$-closure of every pre-$(\Lambda, p(\star))$-open set is $(\Lambda, p(\star))$-open.

(4) The $(\Lambda, p(\star))$-closure of every regular $(\Lambda, p(\star))$-open set is $(\Lambda, p(\star))$-open.

Proof. (1) $\Rightarrow$ (2): Let V be any semi-$(\Lambda, p(\star))$-open set. Then, V is β-$(\Lambda, p(\star))$-open and by Theorem 4, $V^{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open.

(2) $\Rightarrow$ (4): Let V be any regular $(\Lambda, p(\star))$-open set. Then, $V^{(\Lambda,p(\star))}$ is semi-$(\Lambda, p(\star))$-open and by (2), we have $[V^{(\Lambda,p(\star))}]^{(\Lambda,p(\star))} = V^{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open.

(4) $\Rightarrow$ (1): Let V be any $(\Lambda, p(\star))$-open set. Then, we have $[V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$ is regular $(\Lambda, p(\star))$-open and by (4), $[[V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open. Thus,

$$V^{(\Lambda,p(\star))} \subseteq [[V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))} = [[[V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}]^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))} = [V^{(\Lambda,p(\star))}]_{(\Lambda,p(\star))}$$

and hence $V^{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open. Therefore, $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-extremally disconnected.

(1) $\Rightarrow$ (3): Let V be any pre-$(\Lambda, p(\star))$-open set. Then, V is β-$(\Lambda, p(\star))$-open and by Theorem 4, $V^{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open.

(3) $\Rightarrow$ (4): Let V be any regular $(\Lambda, p(\star))$-open set. Then, we have V is pre-$(\Lambda, p(\star))$-open, by (3), $V^{(\Lambda,p(\star))}$ is $(\Lambda, p(\star))$-open.

5. Characterizations of $(\Lambda, p(\star))$-continuous functions

In this section, we introduce the notion of $(\Lambda, p(\star))$-continuous functions. In particular, several characterizations of $(\Lambda, p(\star))$-continuous functions are investigated.

Definition 14. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called $(\Lambda, p(\star))$-continuous at a point $x \in X$ if, for each $(\Lambda, p(\star))$-open set V of Y containing $f(x)$, there exists a $(\Lambda, p(\star))$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called $(\Lambda, p(\star))$-continuous if f has this property at each point $x \in X$.*

Example 4. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, \{a\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{a\}\}$. Let $Y = \{1, 2, 3\}$ with a topology $\sigma = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, Y\}$ and an ideal $\mathscr{J} = \{\emptyset, \{2\}\}$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is defined as follows: $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$. Then, f is $(\Lambda, p(\star))$-continuous.*

Lemma 5. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. Then, $x \in A^{(\Lambda,p(\star))}$ if and only if $A \cap U \neq \emptyset$ for every $(\Lambda, p(\star))$-open set U of X containing x.*

Theorem 8. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) f *is* $(\Lambda, p(\star))$*-continuous at* $x \in X$.

(2) $x \in [f^{-1}(V)]_{(\Lambda,p(\star))}$ *for every* $(\Lambda, p(\star))$*-open set* V *of* Y *containing* $f(x)$.

(3) $x \in f^{-1}([f(A)]^{(\Lambda,p(\star))})$ *for every subset* A *of* X *such that* $x \in A^{(\Lambda,p(\star))}$.

(4) $x \in f^{-1}(B^{(\Lambda,p(\star))})$ *for every subset* B *of* Y *such that* $x \in [f^{-1}(B)]^{(\Lambda,p(\star))}$.

(5) $x \in [f^{-1}(B)]_{(\Lambda,p(\star))}$ *for every subset* B *of* Y *such that* $x \in f^{-1}(B_{(\Lambda,p(\star))})$.

(6) $x \in f^{-1}(K)$ *for every* $(\Lambda, p(\star))$*-closed set* K *of* Y *such that*

$$x \in [f^{-1}(K)]^{(\Lambda,p(\star))}.$$

Proof. (1) ⇒ (2): Let V be any $(\Lambda, p(\star))$-open set of Y containing $f(x)$. Then, there exists a $(\Lambda, p(\star))$-open set U of X containing x such that $f(U) \subseteq V$. Thus, $U \subseteq f^{-1}(V)$. Since U is $(\Lambda, p(\star))$-open in X, we have $x \in [f^{-1}(V)]_{(\Lambda,p(\star))}$.

(2) ⇒ (3): Let A be any subset of X and $x \in A^{(\Lambda,p(\star))}$. Let V be any $(\Lambda, p(\star))$-open set of Y containing $f(x)$. By (2), $x \in [f^{-1}(V)]_{(\Lambda,p(\star))}$ and there exists a $(\Lambda, p(\star))$-open set U of X such that $x \in U \subseteq f^{-1}(V)$. Since $x \in A^{(\Lambda,p(\star))}$, by Lemma 5, $U \cap A \neq \emptyset$ and

$$\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A).$$

Thus, $f(x) \in [f(A)]^{(\Lambda,p(\star))}$ and hence $x \in f^{-1}([f(A)]^{(\Lambda,p(\star))})$.

(3) ⇒ (4): Let B be any subset of Y and $x \in [f^{-1}(B)]^{(\Lambda,p(\star))}$. By (3),

$$x \in f^{-1}([f(f^{-1}(B))]^{(\Lambda,p(\star))}) \subseteq f^{-1}(B^{(\Lambda,p(\star))})$$

and hence $x \in f^{-1}(B^{(\Lambda,p(\star))})$.

(4) ⇒ (5): Let B be any subset of Y such that $x \notin [f^{-1}(B)]_{(\Lambda,p(\star))}$. Then,

$$x \in X - [f^{-1}(B)]_{(\Lambda,p(\star))} = [X - f^{-1}(B)]^{(\Lambda,p(\star))} = [f^{-1}(Y - B)]^{(\Lambda,p(\star))}$$

and by (4), $x \in f^{-1}([Y - B]^{(\Lambda,p(\star))}) = f^{-1}(Y - B_{(\Lambda,p(\star))}) = X - f^{-1}(B_{(\Lambda,p(\star))})$. Thus, $x \notin f^{-1}(B_{(\Lambda,p(\star))})$.

(5) ⇒ (6): Let K be any $(\Lambda, p(\star))$-closed set of Y such that $x \notin f^{-1}(K)$. Then, we have $x \in X - f^{-1}(K) = f^{-1}(Y - K) = f^{-1}((Y - K)_{(\Lambda,p(\star))})$, by (5),

$$x \in [f^{-1}(Y - K)]_{(\Lambda,p(\star))} = [X - f^{-1}(K)]_{(\Lambda,p(\star))} = X - [f^{-1}(K)]^{(\Lambda,p(\star))}$$

and hence $x \notin [f^{-1}(K)]^{(\Lambda,p(\star))}$.

(6) ⇒ (2): Let $x \in X$ and let V be any $(\Lambda, p(\star))$-open set of Y containing $f(x)$. Suppose that $x \notin [f^{-1}(V)]_{(\Lambda,p(\star))}$. Then,

$$x \in X - [f^{-1}(V)]_{(\Lambda,p(\star))} = [X - f^{-1}(V)]^{(\Lambda,p(\star))} = [f^{-1}(Y - V)]^{(\Lambda,p(\star))}.$$

By (6), we have $x \in f^{-1}(Y - V) = X - f^{-1}(V)$ and hence $x \notin f^{-1}(V)$. This contraries to the hypothesis.

(2) $\Rightarrow$ (1): Let V be any $(\Lambda, p(\star))$-open set of Y containing $f(x)$. By (2), we have $x \in [f^{-1}(V)]_{(\Lambda,p(\star))}$ and so there exists a $(\Lambda, p(\star))$-open set U of X containing x such that $x \in U \subseteq f^{-1}(V)$; hence $f(U) \subseteq V$. This shows that f is $(\Lambda, p(\star))$-continuous at x.

Theorem 9. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) f is $(\Lambda, p(\star))$-continuous.

(2) $f^{-1}(V)$ is $(\Lambda, p(\star))$-open in X for every $(\Lambda, p(\star))$-open set V of Y.

(3) $f(A^{(\Lambda,p(\star))}) \subseteq [f(A)]^{(\Lambda,p(\star))}$ for every subset A of X.

(4) $[f^{-1}(B)]^{(\Lambda,p(\star))} \subseteq f^{-1}(B^{(\Lambda,p(\star))})$ for every subset B of Y.

(5) $f^{-1}(B_{(\Lambda,p(\star))}) \subseteq [f^{-1}(B)]_{(\Lambda,p(\star))}$ for every subset B of Y.

(6) $f^{-1}(K)$ is $(\Lambda, p(\star))$-closed in X for every $(\Lambda, p(\star))$-closed set K of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $(\Lambda, p(\star))$-open set of Y such that $x \in f^{-1}(V)$. Then, $f(x) \in V$ and there exists a $(\Lambda, p(\star))$-open set U of X containing x such that $f(U) \subseteq V$. Since U is $(\Lambda, p(\star))$-open in X, $x \in [f^{-1}(V)]_{(\Lambda,p(\star))}$ and hence $f^{-1}(V) \subseteq [f^{-1}(V)]_{(\Lambda,p(\star))}$. Thus, $f^{-1}(V)$ is $(\Lambda, p(\star))$-open.

(2) $\Rightarrow$ (3): Let A be any subset of X. Let $x \in A^{(\Lambda,p(\star))}$ and let V be any $(\Lambda, p(\star))$-open set of Y containing $f(x)$. By (2), we have $x \in [f^{-1}(V)]_{(\Lambda,p(\star))}$ and there exists a $(\Lambda, p(\star))$-open set U of X such that $x \in U \subseteq f^{-1}(V)$. Since $x \in A^{(\Lambda,p(\star))}$, by Lemma 5, $U \cap A \neq \emptyset$ and $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. Thus, $f(x) \in [f(A)]^{(\Lambda,p(\star))}$ and hence $f(A^{(\Lambda,p(\star))}) \subseteq [f(A)]^{(\Lambda,p(\star))}$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. By (3),

$$f([f^{-1}(B)]^{(\Lambda,p(\star))}) \subseteq [f(f^{-1}(B))]^{(\Lambda,p(\star))} \subseteq B^{(\Lambda,p(\star))}.$$

Therefore, $[f^{-1}(B)]^{(\Lambda,p(\star))} \subseteq f^{-1}(B^{(\Lambda,p(\star))})$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4), we have

$$\begin{aligned} X - [f^{-1}(B)]_{(\Lambda,p(\star))} &= [X - f^{-1}(B)]^{(\Lambda,p(\star))} \\ &= [f^{-1}(Y - B)]^{(\Lambda,p(\star))} \\ &\subseteq f^{-1}([Y - B]^{(\Lambda,p(\star))}) \\ &= f^{-1}(Y - B_{(\Lambda,p(\star))}) \\ &= X - f^{-1}(B_{(\Lambda,p(\star))}) \end{aligned}$$

and hence $f^{-1}(B_{(\Lambda,p(\star))}) \subseteq [f^{-1}(B)]_{(\Lambda,p(\star))}$.

(5) ⇒ (6): Let K be any $(\Lambda, p(\star))$-closed set of Y. Then, $Y - K = [Y - K]_{(\Lambda,p(\star))}$ and by (5),

$$\begin{aligned} X - f^{-1}(K) &= f^{-1}(Y - K) \\ &= f^{-1}([Y - K]_{(\Lambda,p(\star))}) \\ &\subseteq [f^{-1}(Y - K)]_{(\Lambda,p(\star))} \\ &= [X - f^{-1}(K)]_{(\Lambda,p(\star))} \\ &= X - [f^{-1}(K)]^{(\Lambda,p(\star))}. \end{aligned}$$

Thus, $[f^{-1}(K)]^{(\Lambda,p(\star))} \subseteq f^{-1}(K)$ and hence $f^{-1}(K)$ is $(\Lambda, p(\star))$-closed.

(6) ⇒ (2): The proof is obvious.

(2) ⇒ (1): Let $x \in X$ and let V be any $(\Lambda, p(\star))$-open set of Y containing $f(x)$. By (2), $x \in [f^{-1}(V)]_{(\Lambda,p(\star))}$ and so there exists a $(\Lambda, p(\star))$-open set U of X containing x such that $x \in U \subseteq f^{-1}(V)$; hence $f(U) \subseteq V$. Thus, f is $(\Lambda, p(\star))$-continuous at x. This shows that f is $(\Lambda, p(\star))$-continuous.

Definition 15. *An ideal topological space $(X, \tau, \mathscr{I})$ is said to be $(\Lambda, p(\star))$-connected if X cannot be written as a disjoint union of two nonempty $(\Lambda, p(\star))$-open sets of X.*

Example 5. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, \{a, b\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then, $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-connected.*

Proposition 9. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is a $(\Lambda, p(\star))$-continuous surjection and $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-connected, then $(Y, \sigma, \mathscr{J})$ is $(\Lambda, p(\star))$-connected.*

Proof. Suppose that $(Y, \sigma, \mathscr{J})$ is not $(\Lambda, p(\star))$-connected. There exist nonempty $(\Lambda, p(\star))$-open sets U and V of Y such that $U \cap V = \emptyset$ and $U \cup V = Y$. Then, we have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(V) = X$. Moreover, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty $(\Lambda, p(\star))$-open sets of X. This shows that $(X, \tau, \mathscr{I})$ is not $(\Lambda, p(\star))$-connected.

Definition 16. *An ideal topological space $(X, \tau, \mathscr{I})$ is said to be $(\Lambda, p(\star))$-compact if every cover of X by $(\Lambda, p(\star))$-open sets of X has a finite subcover.*

Proposition 10. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is a $(\Lambda, p(\star))$-continuous surjection and $(X, \tau, \mathscr{I})$ is $(\Lambda, p(\star))$-compact, then $(Y, \sigma, \mathscr{J})$ is $(\Lambda, p(\star))$-compact.*

Proof. Let $\{V_\gamma \mid \gamma \in \Gamma\}$ be any cover of Y by $(\Lambda, p(\star))$-open sets of Y. Since f is $(\Lambda, p(\star))$-continuous, by Theorem 9, $\{f^{-1}(V_\gamma) \mid \gamma \in \Gamma\}$ is a cover of X by $(\Lambda, p(\star))$-open sets of X. Thus, there exists a finite subset Γ_0 of Γ such that $X = \cup\{f^{-1}(V_\gamma) \mid \gamma \in \Gamma_0\}$. Since f is surjective, $Y = f(X) = \cup\{V_\gamma \mid \gamma \in \Gamma_0\}$. This shows that $(Y, \sigma, \mathscr{J})$ is $(\Lambda, p(\star))$-compact.

Definition 17. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be a $(\Lambda, p(\star))$-neighbourhood of x if there exists a $(\Lambda, p(\star))$-open set U of X such that $x \in U \subseteq A$.*

Definition 18. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. A subset $\Lambda_{(\Lambda,p(\star))}(A)$ is defined as follows: $\Lambda_{(\Lambda,p(\star))}(A) = \cap\{U \mid A \subseteq U; U$ is $(\Lambda, p(\star))$-open$\}$.*

Proposition 11. *For subsets A, B and $C_\gamma(\gamma \in \Gamma)$ of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $A \subseteq \Lambda_{(\Lambda,p(\star))}(A)$.

(2) If $A \subseteq B$, then $\Lambda_{(\Lambda,p(\star))}(A) \subseteq \Lambda_{(\Lambda,p(\star))}(B)$.

(3) $\Lambda_{(\Lambda,p(\star))}(\Lambda_{(\Lambda,p(\star))}(A)) = \Lambda_{(\Lambda,p(\star))}(A)$.

(4) If A is a $(\Lambda, p(\star))$-open set, then $\Lambda_{(\Lambda,p(\star))}(A) = A$.

(5) $\Lambda_{(\Lambda,p(\star))}(\cap\{C_\gamma|\gamma \in \Gamma\}) \subseteq \cap\{\Lambda_{(\Lambda,p(\star))}(C_\gamma)|\gamma \in \Gamma\}$.

(6) $\Lambda_{(\Lambda,p(\star))}(\cup\{C_\gamma|\gamma \in \Gamma\}) = \cup\{\Lambda_{(\Lambda,p(\star))}(C_\gamma)|\gamma \in \Gamma\}$.

Lemma 6. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$ and $x \in X$. Then, $x \in \Lambda_{(\Lambda,p(\star))}(A)$ if and only if $A \cap F \neq \emptyset$ for every $(\Lambda, p(\star))$-closed set F of X with $x \in F$.*

Theorem 10. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) f is $(\Lambda, p(\star))$-continuous.

(2) For each $x \in X$ and each $(\Lambda, p(\star))$-open set V of Y such that $f(x) \in V$, $f^{-1}(V)$ is a $(\Lambda, p(\star))$-neighbourhood of x.

(3) $f(A_{(\Lambda,p(\star))}) \subseteq \Lambda_{(\Lambda,p(\star))}(f(A))$ for every subset A of X.

(4) $[f^{-1}(B)]_{(\Lambda,p(\star))} \subseteq f^{-1}(\Lambda_{(\Lambda,p(\star))}(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and let V be any $(\Lambda, p(\star))$-open set of Y such that $f(x) \in V$. Since f is $(\Lambda, p(\star))$-continuous, there exists a $(\Lambda, p(\star))$-open set U of X containing x such that $f(U) \subseteq V$. Thus, $x \in U \subseteq f^{-1}(V)$ and hence $f^{-1}(V)$ is a $(\Lambda, p(\star))$-neighbourhood of x.

(2) $\Rightarrow$ (1): Let $x \in X$ and let V be any $(\Lambda, p(\star))$-open set of Y containing $f(x)$. By (2), $f^{-1}(V)$ is a $(\Lambda, p(\star))$-neighbourhood of x and there exists a $(\Lambda, p(\star))$-open set U of X such that $x \in U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V$ and hence f is $(\Lambda, p(\star))$-continuous.

(1) $\Rightarrow$ (3): Let A be any subset of X and let $y \notin \Lambda_{(\Lambda,p(\star))}(f(A))$. By Lemma 6, there exists a $(\Lambda, p(\star))$-closed set F of Y such that $y \in F$ and $f(A) \cap F = \emptyset$. Thus, $A \cap f^{-1}(F) = \emptyset$ and hence $f^{-1}(F) \cap A_{(\Lambda,p(\star))} = \emptyset$. Therefore, $f(A_{(\Lambda,p(\star))}) \cap F = \emptyset$. This shows that $y \notin f(A_{(\Lambda,p(\star))})$. Consequently, we obtain $f(A_{(\Lambda,p(\star))}) \subseteq \Lambda_{(\Lambda,p(\star))}(f(A))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. By (3) and Proposition 11(2), we have

$$f([f^{-1}(B)]_{(\Lambda,p(\star))}) \subseteq \Lambda_{(\Lambda,p(\star))}(f(f^{-1}(B))) \subseteq \Lambda_{(\Lambda,p(\star))}(B)$$

and hence $[f^{-1}(B)]_{(\Lambda,p(\star))} \subseteq f^{-1}(\Lambda_{(\Lambda,p(\star))}(B))$.

(4) $\Rightarrow$ (1): Let V be any $(\Lambda, p(\star))$-open set of Y. By (4) and Proposition 11(4), $[f^{-1}(V)]_{(\Lambda,p(\star))} \subseteq f^{-1}(\Lambda_{(\Lambda,p(\star))}(V)) = f^{-1}(V)$ and hence $[f^{-1}(V)]_{(\Lambda,p(\star))} = f^{-1}(V)$. Thus, $f^{-1}(V)$ is $(\Lambda, p(\star))$-open, by Theorem 9, f is $(\Lambda, p(\star))$-continuous.

6. Some low separation axioms

We begin this section by introducing some low separation axioms.

Definition 19. *An ideal topological space* $(X, \tau, \mathscr{I})$ *is said to be:*

(i) pre-$\mathscr{I}$*-*T_0 *if, for each pair of distinct points of* X*, there exists a pre-*$\mathscr{I}$*-open set containing one of the points but not the other;*

(ii) pre-$\mathscr{I}$*-*T_1 *if, for each pair of distinct points* x *and* y *of* X*, there exists a pair of pre-*$\mathscr{I}$*-open sets one containing* x *but not* y *and the other containing* y *but not* x*;*

(iii) pre-$\mathscr{I}$*-*R_0 *if every pre-*$\mathscr{I}$*-open set contains the pre-*$\mathscr{I}$*-closure of each of its singletons.*

Example 6. *Let* $X = \{a, b, c\}$ *with a topology* $\tau = \{\emptyset, \{b\}, \{b, c\}, X\}$ *and an ideal* $\mathscr{I} = \{\emptyset, \{b\}\}$*. Then,* $(X, \tau, \mathscr{I})$ *is a pre-*T_1 *space.*

Remark 3. *For an ideal topological space* $(X, \tau, \mathscr{I})$*, the following implications hold:*

$$\text{pre-}\mathscr{I}\text{-}R_0 \Leftarrow \text{pre-}\mathscr{I}\text{-}T_1 \Rightarrow \text{pre-}\mathscr{I}\text{-}T_0.$$

The following examples show that these implications are not reversible.

Example 7. *Let* $(X, \tau, \mathscr{I})$ *be the same ideal topological space as in Example 2. Then,* $(X, \tau, \mathscr{I})$ *is a pre-*$\mathscr{I}$*-*T_0 *space but* $(X, \tau, \mathscr{I})$ *is not pre-*$\mathscr{I}$*-*T_1*.*

Example 8. *Let* $X = \{a, b\}$ *with a topology* $\tau = \{\emptyset, X\}$ *and an ideal* $\mathscr{I} = \{\emptyset, \{a\}, \{b\}, X\}$*. Then,* $(X, \tau, \mathscr{I})$ *is a pre-*$\mathscr{I}$*-*R_0 *space but* $(X, \tau, \mathscr{I})$ *is not pre-*$\mathscr{I}$*-*T_1*.*

Theorem 11. *An ideal topological space* $(X, \tau, \mathscr{I})$ *is pre-*$\mathscr{I}$*-*T_0 *if and only if, for each* $x \in X$*, the singleton* $\{x\}$ *is* $(\Lambda, p(\star))$*-closed.*

Proof. Suppose that $(X, \tau, \mathscr{I})$ is a pre-$\mathscr{I}$-T_0 space. For each $x \in X$, we have

$$\{x\} \subseteq \Lambda_{p(\star)}(\{x\}) \cap p\imath\mathrm{Cl}(\{x\}).$$

If $y \neq x$, (i) there exists a pre-$\mathscr{I}$-open set U such that $y \notin U$ and $x \in U$ or (ii) there exists a pre-$\mathscr{I}$-open set V such that $x \notin V$ and $y \in V$. In case of (i), $y \notin \Lambda_{p(\star)}(\{x\})$ and $y \notin \Lambda_{p(\star)}(\{x\}) \cap p\imath\mathrm{Cl}(\{x\})$. This shows that $\{x\} \supseteq \Lambda_{p(\star)}(\{x\}) \cap p\imath\mathrm{Cl}(\{x\})$. In case (ii), $y \notin p\imath\mathrm{Cl}(\{x\})$ and $y \notin \Lambda_{p(\star)}(\{x\}) \cap p\imath\mathrm{Cl}(\{x\})$. Thus, $\{x\} \supseteq \Lambda_{p(\star)}(\{x\}) \cap p\imath\mathrm{Cl}(\{x\})$ and hence $\{x\} = \Lambda_{p(\star)}(\{x\}) \cap p\imath\mathrm{Cl}(\{x\})$.

Conversely, suppose that $(X, \tau, \mathscr{I})$ is not pre-$\mathscr{I}$-T_0. There exist two distinct points x, y of X such that (i) $y \in U$ for every pre-$\mathscr{I}$-open set U containing x and (ii) $x \in V$ for every pre-$\mathscr{I}$-open set V containing y. From (i) and (ii), we obtain $y \in \Lambda_{p(\star)}(\{x\})$ and $y \in p\imath\mathrm{Cl}(\{x\})$, respectively. Thus, $y \in \Lambda_{p(\star)}(\{x\}) \cap p\imath\mathrm{Cl}(\{x\})$. By Theorem 1, $\{x\} = \Lambda_{p(\star)}(\{x\}) \cap p\imath\mathrm{Cl}(\{x\})$ since $\{x\}$ is $(\Lambda, p(\star))$-closed. This is contrary to $x \neq y$.

Theorem 12. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-T_1.

(2) For each $x \in X$, the singleton $\{x\}$ is a pre-$\mathscr{I}$-closed set.

(3) For each $x \in X$, the singleton $\{x\}$ is a $\Lambda_{p(\star)}$-set.

Proof. (1) $\Rightarrow$ (2): Let $y \in X$ and $x \in X - \{y\}$. There exists a pre-$\mathscr{I}$-open set G_x such that $x \in G_x$ and $y \notin G_x$. Therefore, we have $X - \{y\} = \cup_{x \in X-\{y\}} G_x$ and hence $\{y\}$ is a pre-$\mathscr{I}$-closed set.

(2) $\Rightarrow$ (3): Let $x \in X$ and $y \in X - \{x\}$. Then, we have $x \in X - \{y\}$ and

$$\Lambda_{p(\star)}(\{x\}) \subseteq X - \{y\}.$$

Thus, $y \notin \Lambda_{p(\star)}(\{x\})$ and hence $\Lambda_{p(\star)}(\{x\}) \subseteq \{x\}$. This implies that $\Lambda_{p(\star)}(\{x\}) = \{x\}$. Consequently, we obtain $\{x\}$ is a $\Lambda_{p(\star)}$-set.

(3) $\Rightarrow$ (1): Let x and y be any distinct points of X. Then, we have $y \notin \Lambda_{p(\star)}(\{x\})$ and so there exists a pre-$\mathscr{I}$-open set U such that $x \in U$ and $y \notin U$. Similarly, $x \notin \Lambda_{p(\star)}(\{y\})$ and there exists a pre-$\mathscr{I}$-open set V such that $y \in V$ and $x \notin V$. This shows that $(X, \tau, \mathscr{I})$ is a pre-$\mathscr{I}$-T_1 space.

Corollary 1. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-T_1;

(2) $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-T_0 and pre-$\mathscr{I}$-R_0.

Proof. (1) $\Rightarrow$ (2): Suppose that $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-T_1. By Remark 3 and Theorem 12, every pre-$\mathscr{I}$-T_1 space is pre-$\mathscr{I}$-T_0 and pre-$\mathscr{I}$-R_0.

(2) $\Rightarrow$ (1): Suppose that $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-T_0 and pre-$\mathscr{I}$-R_0. Since $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-T_0, for any distinct points x and y of X, there exists a pre-$\mathscr{I}$-open set V such that $x \in V$ and $y \notin V$. Since $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0, we have $p\imath\mathrm{Cl}(\{x\}) \subseteq V$. Thus, $x \notin X - p\imath\mathrm{Cl}(\{x\})$ and hence $y \in X - V \subseteq X - p\imath\mathrm{Cl}(\{x\})$. Therefore, $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-T_1.

Lemma 7. *An ideal topological space $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0 if and only if, for each pre-$\mathscr{I}$-open set U, $x \in U$ implies $Cl(Int^\star(\{x\})) \subseteq U$.*

Proof. Let U be any pre-$\mathscr{I}$-open set and $x \in U$. Then, we have $p\imath\mathrm{Cl}(\{x\}) \subseteq U$ and by Lemma 2, $\mathrm{Cl}(\mathrm{Int}^\star(\{x\})) \subseteq U$.

Conversely, let U be any pre-$\mathscr{I}$-open set and $x \in U$. By the hypothesis, we have $\mathrm{Cl}(\mathrm{Int}^\star(\{x\})) \subseteq U$ and by Lemma 2, $p\imath\mathrm{Cl}(\{x\}) \subseteq U$. This shows that $(X, \tau, \mathscr{I})$ is a pre-$\mathscr{I}$-R_0 space.

Theorem 13. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0.

(2) For each pre-$\mathscr{I}$-closed set F and each $x \in X - F$, there exists a pre-$\mathscr{I}$-open set U such that $F \subseteq U$ and $x \notin U$.

(3) For each pre-$\mathscr{I}$-closed set F and each $x \in X - F$, $p\imath Cl(\{x\}) \cap F = \emptyset$.

(4) For any distinct points x and y of X, either $p\imath Cl(\{x\}) = p\imath Cl(\{y\})$ or

$$p\imath Cl(\{x\}) \cap p\imath Cl(\{y\}) = \emptyset.$$

Proof. (1) $\Rightarrow$ (2): Let F be any pre-$\mathscr{I}$-closed set and $x \in X - F$. Then, we have $x \in X - F$ and by (1), $p\imath\text{Cl}(\{x\}) \subseteq X - F$. Put $U = X - p\imath\text{Cl}(\{x\})$, then U is a pre-$\mathscr{I}$-open set such that $F \subseteq U$ and $x \notin U$.

(2) $\Rightarrow$ (3): Let F be any pre-$\mathscr{I}$-closed set and $x \in X - F$. Then by (2), there exists a pre-$\mathscr{I}$-open set V such that $F \subseteq V$ and $x \notin V$. Since V is a pre-$\mathscr{I}$-open set, we have $p\imath\text{Cl}(\{x\}) \cap V = \emptyset$ and hence $p\imath\text{Cl}(\{x\}) \cap F = \emptyset$.

(3) $\Rightarrow$ (4): Let x, y be any points of X. Suppose that $p\imath\text{Cl}(\{x\}) \cap p\imath\text{Cl}(\{y\}) \neq \emptyset$. By (3), $x \notin p\imath\text{Cl}(\{y\})$ and $y \notin p\imath\text{Cl}(\{x\})$. This implies that $p\imath\text{Cl}(\{x\}) \subseteq p\imath\text{Cl}(\{y\}) \subseteq p\imath\text{Cl}(\{x\})$. Consequently, we obtain $p\imath\text{Cl}(\{x\}) = p\imath\text{Cl}(\{y\})$.

(4) $\Rightarrow$ (1): Let U be any pre-$\mathscr{I}$-open set and $x \in U$. For each $y \notin U$, we have $p\imath\text{Cl}(\{y\}) \cap U = \emptyset$ and so $x \notin p\imath\text{Cl}(\{y\})$. Therefore, $p\imath\text{Cl}(\{x\}) \neq p\imath\text{Cl}(\{y\})$. By (4), for each $y \notin U$, $p\imath\text{Cl}(\{x\}) \cap p\imath\text{Cl}(\{y\}) = \emptyset$. Since $X - U$ is pre-$\mathscr{I}$-closed, $y \in p\imath\text{Cl}(\{y\}) \subseteq X - U$ and $X - U = \cup_{y \in X-U} p\imath\text{Cl}(\{y\})$. Thus,

$$\begin{aligned} p\imath\text{Cl}(\{x\}) \cap (X - U) &= p\imath\text{Cl}(\{x\}) \cap [\cup_{y \in X-V} p\imath\text{Cl}(\{y\})] \\ &= \cup_{y \in X-U} [p\imath\text{Cl}(\{x\}) \cap p\imath\text{Cl}(\{y\})] \\ &= \emptyset \end{aligned}$$

and hence $p\imath\text{Cl}(\{x\}) \subseteq U$. This shows that $(X, \tau, \mathscr{I})$ is a pre-$\mathscr{I}$-R_0 space.

Corollary 2. *An ideal topological space $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0 if and only if, for each $x, y \in X$, $p\imath Cl(\{x\}) \neq p\imath Cl(\{y\})$ implies $p\imath Cl(\{x\}) \cap p\imath Cl(\{y\}) = \emptyset$.*

Proof. This is obvious by Theorem 13(4).

Conversely, let U be any pre-$\mathscr{I}$-open set and $x \in U$. For each $y \notin U$, we have $p\imath\text{Cl}(\{y\}) \cap U = \emptyset$. Thus, $x \notin p\imath\text{Cl}(\{y\})$ and hence $p\imath\text{Cl}(\{x\}) \neq p\imath\text{Cl}(\{y\})$. By the hypothesis, $p\imath\text{Cl}(\{x\}) \cap p\imath\text{Cl}(\{y\}) = \emptyset$ and so $y \notin p\imath\text{Cl}(\{x\})$. Therefore, $p\imath\text{Cl}(\{x\}) \subseteq U$. This shows that $(X, \tau, \mathscr{I})$ is a pre-$\mathscr{I}$-R_0 space.

Lemma 8. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $x, y \in X$. Then, $y \in \Lambda_{p(\star)}(\{x\})$ if and only if $x \in p\imath Cl(\{y\})$.*

Proof. Suppose that $y \notin \Lambda_{p(\star)}(\{x\})$. Then, there exists a pre-$\mathscr{I}$-open set V containing x such that $y \notin V$. Therefore, we have $x \notin p\imath\mathrm{Cl}(\{y\})$.

Conversely, suppose that $x \notin p\imath\mathrm{Cl}(\{y\})$. Then by Lemma 1, there exists a pre-$\mathscr{I}$-open set V containing x such that $V \cap \{y\} = \emptyset$. Therefore, we have $y \notin V$ and hence $y \notin \Lambda_{p(\star)}(\{x\})$.

Lemma 9. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $x, y \in X$. Then, $\Lambda_{p(\star)}(\{x\}) = \Lambda_{p(\star)}(\{y\})$ if and only if $p\imath Cl(\{x\}) = p\imath Cl(\{y\})$.*

Proof. Let x, y be any points of X. Suppose that $\Lambda_{p(\star)}(\{x\}) = \Lambda_{p(\star)}(\{y\})$. Since $x \in \Lambda_{p(\star)}(\{x\})$, we have $x \in \Lambda_{p(\star)}(\{y\})$ and by Lemma 8, $y \in p\imath\mathrm{Cl}(\{x\})$. Therefore, $p\imath\mathrm{Cl}(\{y\}) \subseteq p\imath\mathrm{Cl}(\{x\})$. Similarly, we have $p\imath\mathrm{Cl}(\{x\}) \subseteq p\imath\mathrm{Cl}(\{y\})$ and hence

$$p\imath\mathrm{Cl}(\{y\}) = p\imath\mathrm{Cl}(\{x\}).$$

Conversely, suppose that $p\imath\mathrm{Cl}(\{x\}) = p\imath\mathrm{Cl}(\{y\})$. Since $x \in p\imath\mathrm{Cl}(\{x\})$, $x \in p\imath\mathrm{Cl}(\{y\})$ and by Lemma 8, $y \in \Lambda_{p(\star)}(\{x\})$. Thus, $\Lambda_{p(\star)}(\{y\}) \subseteq \Lambda_{p(\star)}(\Lambda_{p(\star)}(\{x\})) = \Lambda_{p(\star)}(\{x\})$. Similarly, we have $\Lambda_{p(\star)}(\{x\}) \subseteq \Lambda_{p(\star)}(\{y\})$ and hence $\Lambda_{p(\star)})(\{x\}) = \Lambda_{p(\star)}(\{y\})$.

Theorem 14. *An ideal topological space $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0 if and only if, for each $x, y \in X$, $\Lambda_{p(\star)}(\{x\}) \neq \Lambda_{p(\star)}(\{y\})$ implies $\Lambda_{p(\star)}(\{x\}) \cap \Lambda_{p(\star)}(\{y\}) = \emptyset$.*

Proof. Let x, y be any points of X. Suppose that $\Lambda_{p(\star)}(\{x\}) \cap \Lambda_{p(\star)}(\{y\}) \neq \emptyset$. Let $z \in \Lambda_{p(\star)}(\{x\}) \cap \Lambda_{p(\star)}(\{y\})$. Then, $z \in \Lambda_{p(\star)}(\{x\})$ and by Lemma 8, we have $x \in p\imath\mathrm{Cl}(\{z\})$. Therefore, $x \in p\imath\mathrm{Cl}(\{z\}) \cap p\imath\mathrm{Cl}(\{x\})$ and by Corollary 2, $x \in p\imath\mathrm{Cl}(\{z\}) = p\imath\mathrm{Cl}(\{x\})$. Similarly, we have $p\imath\mathrm{Cl}(\{z\}) = p\imath\mathrm{Cl}(\{y\})$ and hence $p\imath\mathrm{Cl}(\{x\}) = p\imath\mathrm{Cl}(\{y\})$. By Lemma 9, $\Lambda_{p(\star)}(\{x\}) = \Lambda_{p(\star)}(\{y\})$.

Conversely, let x, y be any points of X. Suppose that $p\imath\mathrm{Cl}(\{x\}) \neq p\imath\mathrm{Cl}(\{y\})$. By Lemma 9, $\Lambda_{p(\star)}(\{x\}) \neq \Lambda_{p(\star)}(\{y\})$ and hence $\Lambda_{p(\star)}(\{x\}) \cap \Lambda_{p(\star)}(\{y\}) = \emptyset$. Therefore, $p\imath\mathrm{Cl}(\{x\}) \cap p\imath\mathrm{Cl}(\{y\}) = \emptyset$. In fact, assume that $z \in p\imath\mathrm{Cl}(\{x\}) \cap p\imath\mathrm{Cl}(\{y\})$. Then, $z \in p\imath\mathrm{Cl}(\{x\})$ implies $x \in \Lambda_{p(\star)}(\{z\})$ and hence $x \in \Lambda_{p(\star)}(\{z\}) \cap \Lambda_{p(\star)}(\{x\})$. By the hypothesis, $\Lambda_{p(\star)}(\{z\}) = \Lambda_{p(\star)}(\{x\})$ and by Lemma 9, $p\imath\mathrm{Cl}(\{z\}) = p\imath\mathrm{Cl}(\{x\})$. Similarly, we have $p\imath\mathrm{Cl}(\{z\}) = p\imath\mathrm{Cl}(\{y\})$ and hence $p\imath\mathrm{Cl}(\{x\}) = p\imath\mathrm{Cl}(\{y\})$. This contradicts that $p\imath\mathrm{Cl}(\{x\}) \neq p\imath\mathrm{Cl}(\{y\})$. Thus, $p\imath\mathrm{Cl}(\{x\}) \cap p\imath\mathrm{Cl}(\{y\}) = \emptyset$. This shows that $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0.

Theorem 15. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0;

(2) $x \in p_{\mathscr{I}} Cl(\{y\})$ if and only if $y \in p\imath Cl(\{x\})$.

Proof. (1) $\Rightarrow$ (2): Suppose that $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0 and $x \in p\imath\mathrm{Cl}(\{y\})$. By Lemma 8, we have $y \in \Lambda_{p(\star)}(\{x\})$. Thus, $\Lambda_{p(\star)}(\{x\}) \cap \Lambda_{p(\star)}(\{y\}) \neq \emptyset$ and by Theorem 14,

$\Lambda_{p(\star)}(\{x\}) = \Lambda_{p(\star)}(\{y\})$. Therefore, $x \in \Lambda_{p(\star)}(\{y\})$ and by Lemma 8, $y \in p\imath\text{Cl}(\{x\})$. The converse is similarly shown.

(2) $\Rightarrow$ (1): Let V be any pre-$\mathscr{I}$-open set and $x \in V$. For each $y \notin V$, we have $p\imath\text{Cl}(\{x\}) \cap V = \emptyset$. This implies that $x \notin p\imath\text{Cl}(\{y\})$ and $y \notin p\imath\text{Cl}(\{x\})$. Thus,

$$p\imath\text{Cl}(\{x\}) \subseteq V$$

and hence $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0.

Theorem 16. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0.

(2) For each nonempty subset A of X and each pre-$\mathscr{I}$-open set V such that $A \cap V \neq \emptyset$, there exists a pre-$\mathscr{I}$-closed set F such that $A \cap F \neq \emptyset$ and $F \subseteq V$.

(3) $F = \Lambda_{p(\star)}(F)$ for every pre-$\mathscr{I}$-closed set F.

(4) $p\imath Cl(\{x\}) = \Lambda_{p(\star)}(\{x\})$ for each $x \in X$.

(5) $p\imath Cl(\{x\}) \subseteq \Lambda_{p(\star)}(\{x\})$ for each $x \in X$.

Proof. (1) $\Rightarrow$ (2): Let A be any nonempty subset of X and let V be any pre-$\mathscr{I}$-open set such that $A \cap V \neq \emptyset$. Then, there exists $x \in A \cap V$ and hence $p\imath\text{Cl}(\{x\}) \subseteq V$. Put $F = p\imath\text{Cl}(\{x\})$, then F is pre-$\mathscr{I}$-closed, $A \cap F \neq \emptyset$ and $F \subseteq V$.

(2) $\Rightarrow$ (3): Let F be any pre-$\mathscr{I}$-closed set and $x \notin F$. Then, $x \in X - F$ and by (2), there exists a pre-$\mathscr{I}$-closed set K such that $x \in K$ and $K \subseteq X - F$. Now, put $V = X - K$. Then, V is a pre-$\mathscr{I}$-open set such that $F \subseteq V$ and $x \notin V$. Thus, $x \notin \Lambda_{p(\star)}(F)$ and hence $F \supseteq \Lambda_{p(\star)}(F)$. On the other hand, we have $F \subseteq \Lambda_{p(\star)}(F)$. Consequently, we obtain $F = \Lambda_{p(\star)}(F)$.

(3) $\Rightarrow$ (4): Let $x \in X$ and $y \notin \Lambda_{p(\star)}(\{x\})$. Then, there exists a pre-$\mathscr{I}$-open set U such that $x \in U$ and $y \notin U$. Therefore, $p\imath\text{Cl}(\{y\}) \cap U = \emptyset$ and by (3), $\Lambda_{p(\star)}(p\imath\text{Cl}(\{y\})) \cap U = \emptyset$. Since $x \notin \Lambda_{p(\star)}(p\imath\text{Cl}(\{y\}))$, there exists a pre-$\mathscr{I}$-open set V such that $p\imath\text{Cl}(\{y\}) \subseteq V$ and $x \notin V$. Thus, $p\imath\text{Cl}(\{x\}) \cap V = \emptyset$. Since $y \in V$, $y \notin p\imath\text{Cl}(\{x\})$. Therefore,

$$p\imath\text{Cl}(\{x\}) \subseteq \Lambda_{p(\star)}(\{x\}).$$

Moreover, $p\imath\text{Cl}(\{x\}) \subseteq \Lambda_{p(\star)}(\{x\}) \subseteq \Lambda_{p(\star)}(p\imath\text{Cl}(\{x\})) = p\imath\text{Cl}(\{x\})$. This shows that $p\imath\text{Cl}(\{x\}) = \Lambda_{p(\star)}(\{x\})$.

(4) $\Rightarrow$ (5): The proof is obvious.

(5) $\Rightarrow$ (1): Let V be any pre-$\mathscr{I}$-open set and $x \in V$. Suppose that $y \notin V$. Then, $p\imath\text{Cl}(\{y\}) \cap V = \emptyset$ and $x \notin p\imath\text{Cl}(\{y\})$. By Lemma 8, $y \notin \Lambda_{p(\star)}(\{x\})$ and by (5), we have $y \notin p\imath\text{Cl}(\{x\})$. Thus, $p\imath\text{Cl}(\{x\}) \subseteq V$ and hence $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0.

Corollary 3. *An ideal topological space $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0 if and only if $\Lambda_{p(\star)}(\{x\}) \subseteq p\imath Cl(\{x\})$ for each $x \in X$.*

Proof. This is obvious by Theorem 16.

Conversely, suppose that $\Lambda_{p(\star)}(\{x\}) \subseteq p\imath\mathrm{Cl}(\{x\})$ for each $x \in X$. Let $x \in p\imath\mathrm{Cl}(\{y\})$. By Lemma 8, we have $y \in \Lambda_{p(\star)}(\{x\})$ and hence $y \in p\imath\mathrm{Cl}(\{x\})$. Similarly, if $y \in p\imath\mathrm{Cl}(\{x\})$, then $x \in p\imath\mathrm{Cl}(\{y\})$. It follows from Theorem 15 that $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0.

Corollary 4. *An ideal topological space $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0 if and only if $\prec x \succ_{p(\star)} = p\imath Cl(\{x\})$ for each $x \in X$.*

Proof. Let $x \in X$. By Theorem 16, we have $p\imath\mathrm{Cl}(\{x\}) = \Lambda_{p(\star)}(\{x\})$ and hence $p\imath\mathrm{Cl}(\{x\}) = \Lambda_{p(\star)}(\{x\}) \cap p\imath\mathrm{Cl}(\{x\}) = \prec x \succ_{p(\star)}$.

Conversely, suppose that $\prec x \succ_{p(\star)} = p\imath\mathrm{Cl}(\{x\})$ for each $x \in X$. Let $x \in X$. By the hypothesis, we have $p\imath\mathrm{Cl}(\{x\}) = \prec x \succ_{p(\star)} (\{x\}) = p\imath\mathrm{Cl}(\{x\}) \cap \Lambda_{p(\star)}(\{x\}) \subseteq \Lambda_{p(\star)}(\{x\})$. It follows from Theorem 16 that $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-R_0.

7. Conclusion

Topology plays an important role in both pure and applied sciences such as quantum physics, high energy physics, data mining, computational topology, digital topology and mathematical sciences. The notions of closed sets and low separation axioms are fundamental with respect to the investigation of topological spaces. Various types of generalizations of closed sets and some new separation axioms have been researched by many mathematicians. This paper is concerned with the concepts of $\Lambda_{p(\star)}$-sets and $(\Lambda, p(\star))$-closed sets which are defined by utilizing the notions of pre-$\mathscr{I}$-open sets and pre-$\mathscr{I}$-open sets. Furthermore, some properties of $(\Lambda, p(\star))$-closed sets and $(\Lambda, p(\star))$-open sets are considered. Several characterizations of $(\Lambda, p(\star))$-continuous functions are obtained. Additionally, some characterizations of $(\Lambda, p(\star))$-extremally disconnected and pre-$\mathscr{I}$-R_0 ideal topological spaces are explored. The ideas and results of this paper may motivate further research.

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