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E-Bayesian Estimation under Loss Functions in Competing Risks

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Abstract. Using gamma prior distribution of which shape hyperparameter has beta distribution and rate parameter has three different distributions over a finite interval, we studied the E-Bayesian estimation of one scale parameter of Gompertz distribution based on progressively type I censored sample from the competing risks model subject to K independent causes. The estimators obtained generalize those issued from the quadratic loss, entropy loss and DeGroot loss functions.

2020 Mathematics Subject Classifications: 62C10, 91G70, 33B15, 62P20

Key Words and Phrases: Bayesian estimation, E-Bayesian estimation, Competing risks, Type-I progressively censoring, Loss function

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1. Introduction

Several estimations of the parameters of the Gompertz distribution in a competing risks context have been studied in the literature. The maximum likelihood estimation has been studied by [17], while the Bayesian estimation and the hierarchical Bayesian estimation have been investigated by [19], [3, 27, 28], [30], [33] and [23]. All these methods involve integrals whose computation is not easy and may require numerical methods.

The progress in computational mathematics and statistics during the past two decades has contributed to the development of a new method called E-Bayesian estimation introduced by [10]. By considering the quadratic loss function, [10] proved by means of simulations, that the E-Bayesian estimator is more efficient and easier to implement than others

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authors. Since then, there has been a growing interest in studying E-Bayesian estimation. The studies are done under different distributions such as the binomial distribution [19], the exponential distribution ([30]), the Pareto distribution [26] and the distribution of Lomax ([6]). All these papers came to the same conclusion that the E-Bayesian estimate is better than the Bayesian estimate. [32] studied E-Bayesian estimation of a parameter in the context of competing risks model under the quadratic and Linex loss functions. They also concluded that the new method is more efficient under these loss functions.

In this article, we assume that the survival time X is a positive and absolutely continuous random variable. Instead of observing independent and identically distributed realizations (i.i.d.) of duration X, we observe the realization of the variable X subjected to various perturbations independent or not of the phenomenon studied. In the presence of right random censorship, the lifetimes are not all observed. For some of them, one only knows that they are greater than a certain known value.

There are several types of censorship: Type I, II, and III censorship. The reader interested in the notion of censorship can refer to [1] or [7]. Type I censorship describes the situation where a test ends at a certain period and one knows that the remaining individuals have not yet been observed. In this case, the censorship time is fixed in advance and the number of individuals not observed is a random variable. Let C be a fixed value, instead of observing the complete life time variables $X_1, \cdots, X_n$, one observes X_i when $X_i \leq C_i$ if not, one knows that $X_i > C_i$. We use the following notation $T_i = X_i \wedge C_i = \min(X_i, C_i)$, with $i = 1, \cdots, n$.

In the case of simple type I censorship, all the individuals are censored after the same length of time while in the case of progressive type I censorship which is used in this article, all the individuals are censored at the same date whatever the time span they were followed. Progressively censored type I data were first proposed by [15]. Indeed, in the context of our article, progressively censored type I data are described as follows : assume that n units are wagered in a progressive life-test censorship scheme: $(R_1, R_2, \ldots, R_r)$, $1 \leq r \leq n$. The experiment is over on the date $\tau \in (0, \infty)$, R_i $(i = 1, 2, \ldots, r)$ and r is fixed in advance. At the time of the first failure t_1, R_1 of the remaining units are randomly removed, at the time of the second failure t_2, R_2 of the remaining units are randomly removed and so on. If the rth failure time t_r occurs before time τ, all the remaining units $R_r = n - r - (R_1 + \ldots + R_{r-1})$ are removed and the terminal time of the experiment is t_r. On the other hand, if the rth failure time t_r does not occur before time τ and only J failures occur before time τ, where $0 \leq J \leq r$, then at the time τ, all the remaining $R^*_\tau = n - J - (R_1 + \ldots + R_J)$ units are removed, and the terminal time of the experiment is τ. We denote the two cases as:

Case 1

$$t_1 < t_2 < \ldots < t_r, \quad t_r < \tau;$$

Case 2

$$t_1 < t_2 < \ldots < t_J < \tau < t_{J+1} < \ldots < t_r, \quad t_r > \tau.$$

In the reminder of this article, we briefly present the notion of competing risks for the

Gompertz modeling, and the E-Bayesian estimation of the scale parameter of the Gompertz distribution under several loss functions.

2. Competing risks

2.1. Background

In survival analysis, a competing risks situation is that where the event of interest is subject to several causes. In this context, the event of interest has been modelled by various distributions such as Gompertz distribution ([29], [32]), exponential distribution ([20]) and Lindley distribution ([21]), stochastic process ([22]).

In this paper, we adopt the same concepts as Wu et al. [32] and Njamen et al. [23]. An examination of the literature has shown that for a given subject, at most one event denoted by δ_k ($k \in \{1, \cdots, K\}$) among K events will be observed. If no event occurs, then the subject is censored at the end of its tracking ($\delta_k = 0$). In practice, one fixes a single event of interest ($\delta_k = 1$) among the possible K. We assumed that :

- there are K competing independent failure modes;
- the system failure only occurs in one of the competing K failure modes with durations $T_1, \cdots, T_K$;
- the system failure time is $T = \min\{T_1, \cdots, T_K\}$ which is a latent time ;
- the lifetime of the concurrent failure mode k ($k = 1, \cdots, K$) denoted by T_k, follows a Gompertz distribution with parameters α_k and β_k: Gompertz(α_k, β_k).

2.2. Gompertz distribution in competing risks

[14] used the [9] distribution to model the cumulative incidence function (CIF) associated with an event. This CIF associated with an event of type k is denoted here by $F_k(t, \Psi_k)$, defined by

$$F_k(t; \Psi_k) = 1 - \exp\left\{-\frac{\beta_k}{\alpha_k}\left[\exp(\alpha_k t) - 1\right]\right\}, \tag{1}$$

with $\Psi_k = (\alpha_k; \beta_k) \in \mathbb{R}^* \times \mathbb{R}$; α_k is the shape parameter and β_k the scale parameter.

The curve below is that of CIF. It is obtained by the R software version R i386 3.1.3 downloadable online. The CIF is a function defined on the interval [0; 1]. The curves of the distribution function of the Gompertz distribution inform us that, for any values of the parameters α_k and β_k, the curves take their origin in 0 and increase until they reach the value 1. The graph illustrates this perfectly. The CIF is used in survival data in aging biology where α_k is called the coefficient of the age-dependent mortality rate, and β_k is called the coefficient of the death-age-independent rate (see [31]). These models are

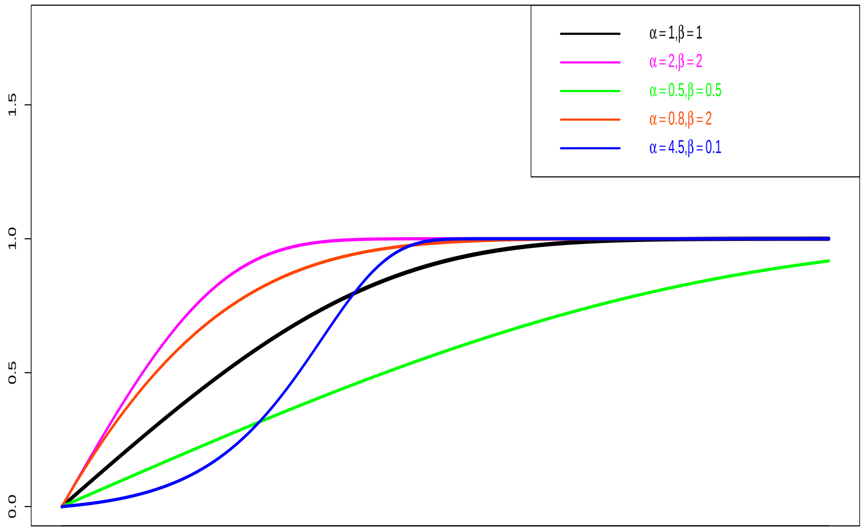

Figure 1: Curve of the CIF of the Gompertz distribution

also widely used in demography where they make it possible to estimate the lifespans of populations ([2]).

The associated density function f_k is obtained by deriving the CIF with respect to time. One has :

$$f_k(t;\Psi_k) = \frac{\partial F_k(t;\Psi_k)}{\partial t} = \beta_k \exp(\alpha_k t)\exp\left\{-\frac{\beta_k}{\alpha_k}\left[\exp(\alpha_k t) - 1\right]\right\}. \tag{2}$$

Figure 2 below is the curve of the density function of the Gompertz distribution. It is obtained by the R software. We notice that when β_k tends to 0, we obtain the curve of the Exponential distribution, which is a particular case of the Gompertz distribution (see the curve in blue). Thus, when $\beta_k \to 0$, the curve presents exponential distribution. Actually, through limit concept,

$$\lim_{\alpha_k \to 0} f_k(t,\Psi_k) = \beta_k \exp(-t\beta_k), \quad \text{for } t > 0.$$

If we fix the parameter β_k, we obtain a family of distributions indexed by the parameter α_k (> 0) which in fact constitutes a family of distributions at risk proportional. Thus, the other curves (green, black, red and pink) give us the basic Gompertz density under different parameters.

Under progressive type I censorship, we consider a population of K competing risks. Let k be a fixed constant, $k \in \{1, 2, \ldots, K\}$. Let $\tau^* = \min\{t_r, \tau\}$ and $R^* = r$, $t_r \leq \tau$; $R^* = J$, $t_r > \tau$ where τ^* is the final time of the experiment, R^* the number of failures before time τ^*. The couples $(t_1,\ \alpha_1), \cdots, (t_{R^*},\ \alpha_{R^*})$ are the observed failure data, where $t_1, t_2, \ldots, t_{R^*}$ are the failure times in statistical order and α_i takes any integer in the set

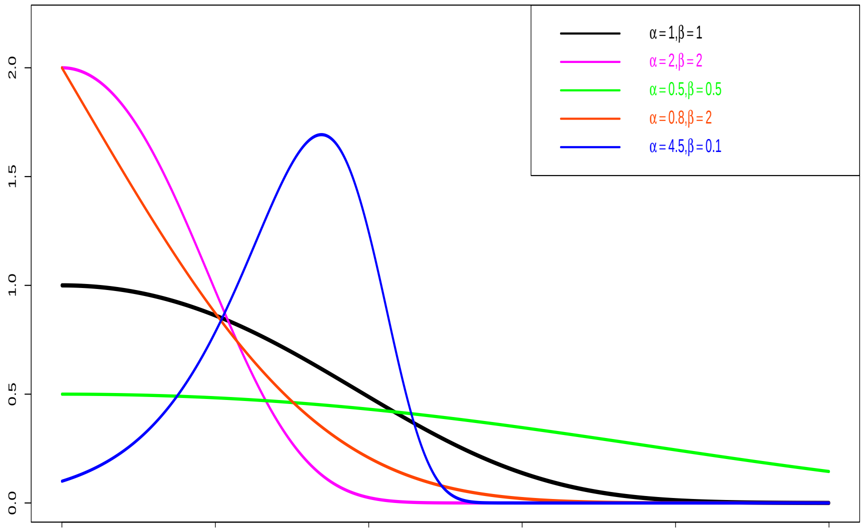

Figure 2: Curve of the density function of the Gompertz distribution

$\{1, \cdots, K\}$. Note that $\alpha_i = k$ $(k = 1, 2, \ldots, K)$ indicates the failure mode caused by the kth event.

Let

$$\delta_k(\alpha_i) = \begin{cases} 1 & \text{if } \alpha_i = k \\ 0 & \text{if } \alpha_i \neq k, \end{cases}$$

and $n_k = \sum_{i=1}^{R^*} \delta_k(\alpha_i) \geq 0$ the total number of failures caused by the kth event. Under type I progressive censorship as defined in the introduction, the likelihood function is given for all $t = (t_1, t_2, \ldots, t_{R*})$ by :

$$L_k(t|\alpha_k, \beta_k) \propto \prod_{k=1}^{K} \left[\prod_{i=1}^{R^*} f_k(t_i)^{\delta_k(\alpha_i)} [1 - F_k(t_i)]^{1-\delta_k(\alpha_i)} [1 - F_k(t_i)]^{R_i} [1 - F_k(\tau^*)]^{n - R^* - \sum_{i=1}^{R^*} R_i} \right], \quad (3)$$

where R_i is the ith remainder in the random variables.
After calculation (see [23]), the likelihood function is obtained from (2) as :

$$L_k(t|\alpha_k, \beta_k) = \prod_{k=1}^{K} \left[\beta_k^{n_k} \exp \left\{ \alpha_k \sum_{i=1}^{R^*} \delta_k(\alpha_i) t_i - \left(\frac{\beta_k}{\alpha_k} \right) \times A_k \right\} \right], \quad (4)$$

with

$$A_k = \sum_{i=1}^{R^*} (R_i + 1) \left(e^{\alpha_k t_i} - 1 \right) + \left(n - R^* - \sum_{i=1}^{R^*} R_i \right) \left(e^{\alpha_k r^*} - 1 \right) \quad \text{and} \quad \alpha_k > 0.$$

In addition to the distribution of [9], we consider the gamma distribution with parameters $a > 0$ and $b > 0$, whose density function π is defined for all $x \in \mathbb{R}$ by :

$$\pi(x; a, b) = \frac{b^a}{\Gamma(k)} x^{a-1} \exp(-bx)\mathbb{I}(x > 0), \tag{5}$$

where Γ is the Gamma function defined for all $a > 0$ by :

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx. \tag{6}$$

It is easy to check that for all $a > 0$, $\Gamma(a + 1) = a\Gamma(a)$, and in particular for an integer a, $\Gamma(a) = (a - 1)!$. By convention, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

We also consider the Beta density function of parameters $u > 0$ and $v > 0$, defined for $x \in \mathbb{R}$ by:

$$\frac{x^{u-1}(1 - x)^{v-1}}{B(u, v)} \mathbb{I}(0 \leq x \leq 1),$$

where for $u > 0$ and $v > 0$,

$$B(u, v) = \int_0^1 t^{u-1}(1 - t)^{v-1} dt \tag{i}$$

One easily shows that for all $x > 0$ and $y > 0$,

$$B(x, y + 1) = \frac{y}{x + y} B(x, y) \tag{ii}$$

and

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x, y)} = B(y, x). \tag{iii}$$

Bayesian estimation has received great attention by the researchers who have that Bayes estimators perform better than classical estimators. Expected Bayesian or E-Bayesian method as an extension to Bayesian estimation has been introduced by [11]. He obtained the E- Bayes estimate of failure probability by considering quadratic loss function and discussed the properties of E-Bayes estimate and showed that E-Bayes estimate is efficient and easy to operate.

In this article, we examine E-Bayesian estimation of the parameter of the reliability function for the competing risk model from the Gompertz distribution developed in [23]. Under the type I progressive censorship, the new estimators obtained generalize not only the estimators of the generalized quadratic loss function proposed by [32], but also those of the DeGroot and Entropy loss functions.

3. The E-Bayesian notion

Bayesian methods in the context of competing risks are studied for instance by [10–13], [32], [24], [33], [18] and [25]. E-Bayesian estimation is a new method for estimating the probability of failure in the case of two hyper-parameters. Introduced by [10–12], in the context of a single risk, it is based on the calculation of the posterior mean of the Bayes estimators.

Here we consider E-Bayesian estimation of the shape parameter λ of the Gompertz distribution in a competing risks context. Under the Entropy loss function, we assume that λ follows a Gamma prior distribution $\pi(\lambda|a,b)$ where $a>0$ and $b>0$ are the hyper-parameters.

Definition 1. *([10]) Then the E-Bayesian estimate of λ is given by :*

$$\widehat{\lambda}_{EB}=\int\int_{\mathcal{D}}\widehat{\lambda}_B\pi(\lambda|(a,b)dadb=\mathbb{E}_\pi[\widehat{\lambda}_B(a,b)], \tag{7}$$

where $\mathcal{D}$ is the domain of space of the parameters a and b.

Definition 2. *([10]) An E-Bayesian estimate of λ is $\mathbb{E}_\pi[\widehat{\lambda}_B(a,b)]$ the expectation of $\widehat{\lambda}_B(a,b)$ obtained with respect to any joint distribution $\pi(a,b)$ of (a,b).*

In the context of competitive risks, [32] considered the Bayesian estimation of the parameter β_k and assumed that β_k follows an a-priori Gamma distribution $\pi(\beta_k|a_k,b_k)$, with hyper-parameters $a_k>0$ and $b_k>0$. These must be selected to guarantee that $\pi(\beta_k)$ be a decreasing function of β_k. For this reason, they must be choosen such that

$$\frac{\partial\pi(\beta_k|a_k,b_k)}{\partial\beta_k}<0.$$

Since

$$\frac{\partial\pi(\beta_k|a_k,b_k)}{\partial\beta_k}=\frac{b_k^{a_k}}{\Gamma(a_k)}\left(a_k-1-b_k\beta_k\right)\beta_k^{a_k-2}\exp(-b_k\beta_k), \tag{8}$$

from (8), a_k and b_k must achieve $0<a_k<1$ and $b_k>0$.

We assume subsequently that a_k and b_k are independent random variables, with joint (a-priori) distribution of the form $\widetilde{\pi}(a_k,b_k)=\widetilde{\pi}(a_k)\widetilde{\pi}(b_k)$, where $\widetilde{\pi}$ is a density function. In the sequel, we use three different a priori distributions for a_k and b_k as defined by [32]. The influence of each of these on the E-Bayesian estimation of β_k is investigated. The three a priori joint distributions of the hyperparameters a_k and b_k are defined as Beta distributions by :

$$\begin{cases}\widetilde{\pi_1}(a_k,b_k)=\frac{1}{c_kB(u_k,v_k)}a_k^{u_k-1}(1-a_k)^{v_k-1}\\ \widetilde{\pi_2}(a_k,b_k)=\frac{2}{c_k^2B(u_k,v_k)}(c_k-b_k)a_k^{u_k-1}(1-a_k)^{v_k-1}\\ \widetilde{\pi_3}(a_k,b_k)=\frac{2b_k}{c_k^2B(u_k,v_k)}a_k^{u_k-1}(1-a_k)^{v_k-1},\end{cases} \tag{9}$$

where a_k and b_k are such that

$$0<a_k<1,\quad 0<b_k<c_k,\quad c_k>0.$$

4. E-Bayesian estimation under the generalized quadratic loss function

4.1. Generalized quadratic loss function

The quadratic loss function proposed by [16] and [8] is defined by :

$$L(\theta, d) = (\theta - d)^2,$$

where $\theta \in \Theta$ the parameters space, $d \in \mathcal{D}$ the decision space.

A variant of this loss function is the weighted squared loss function of the form

$$L(\theta, d) = \omega(\theta)(\theta - d)^2,$$

where ω is a weight function.

A cost function is any real-valued function L defined on $\Theta \times \mathcal{D}$. In general, a loss function is a measurable positive function defined on $\Theta \times \mathcal{D}$.

The parameters space Θ is endowed with a probability π such that $(\Theta, \mathcal{A}, \pi)$ is a probabilized space. We write $\theta \sim \pi$ to mean that θ has distribution π called a priori law. This distribution determines what we know and what we don't know before observing the the event under consideration.

Under the assumption of a quadratic cost, the Bayes estimator $\delta^{\pi}(\underline{x})$ of θ associated with the prior distribution π is the conditional mean a posteriori of θ defined for any observation $\underline{x} = (x_1, x_2, \cdots, x_n)$ by :

$$\delta^{\pi}(\underline{x}) = \mathbb{E}_{\pi}(\cdot|\underline{x})(\theta) = \int_{\theta\in\Theta} L(\theta, \delta(x))\pi(\theta|\underline{x})d\theta.$$

The E-Bayesian estimator of β_k with hyper-parameters a_k and b_k is given by :

$$\widehat{\beta_{k(EBQG_i)}} = \int\int_{\mathcal{D}} \widehat{\beta_{k(BQG)}}(a_k, b_k)\pi_i(a_k, b_k)db_k da_k, \quad i = 1, 2, 3,$$

where $\mathcal{D}$ is the decision space, and $\widehat{\beta}_{k(BQG)}$ is the Bayesian estimator of β_k defined in Theorem 4.1 of [23] and recalled below :

$$\widehat{\beta}_{k(BQG)}(\alpha_k, \beta_k) = \frac{n_k + a_k + \alpha - 1}{b_k + \frac{A_k}{\alpha_k}}, \quad \text{with } \alpha_k > 0.$$

The a priori distributions defined above will allow us in the following subsection to determine the estimators of the Bayesian expectation for the different loss functions considered.

4.2. The E-Bayesian estimators

Theorem 1. *Under the generalized quadratic loss function, the E-Bayesian estimators of β_k obtained with the priors $\pi_i(a_k, b_k)$, $i \in \{1,2,3\}$ are given respectively by :*

$$\left\{\begin{array}{lcl} \widehat{\beta}_{k(EBQG1)} & = & c_k^{-1}\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\left(n_k+\alpha-1+\frac{u_k}{u_k+v_k}\right) \\ \widehat{\beta}_{k(EBQG2)} & = & 2c_k^{-2}\left[-c_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\left(n_k+\alpha-1+\frac{u_k}{u_k+v_k}\right) \\ \widehat{\beta}_{k(EBQG3)} & = & 2c_k^{-2}\left[c_k-\left(\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\left(n_k+\alpha-1+\frac{u_k}{u_k+v_k}\right), \end{array}\right. \tag{10}$$

with $\alpha_k > 0,\ \ 0 < a_k < 1$ *and* $0 < b_k < c_k$.

Proof. For $i = 1$, we have, for the generalized quadratic loss function, and the prior $\pi_1(a_k, b_k)$, the E-Bayesian estimator of β_k given by :

$$\begin{aligned} \widehat{\beta}_{k(EBQG1)} &= \int_0^1\int_0^{c_k}\frac{n_k+a_k+\alpha-1}{b_k+\frac{A_k}{\alpha_k}}\times\frac{1}{c_kB(u_k,v_k)}\times a_k^{u_k-1}(1-a_k)^{v_k-1}db_kda_k \\ &= \int_0^1\frac{1}{c_kB(u_k,v_k)}\times a_k^{u_k-1}(1-a_k)^{v_k-1}\times\left(\int_0^{c_k}\frac{n_k+a_k+\alpha-1}{b_k+\frac{A_k}{\alpha_k}}db_k\right)da_k \\ &= \int_0^1\frac{n_k+a_k+\alpha-1}{c_kB(u_k,v_k)}\times a_k^{u_k-1}(1-a_k)^{v_k-1}\times\left(\int_0^{c_k}\frac{1}{b_k+\frac{A_k}{\alpha_k}}db_k\right)da_k \\ &= \int_0^1\frac{n_k+a_k+\alpha-1}{c_kB(u_k,v_k)}\times a_k^{u_k-1}(1-a_k)^{v_k-1}\times\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)da_k \\ &= \frac{1}{c_kB(u_k,v_k)}\times\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\int_0^1(n_k+a_k+\alpha-1)\times a_k^{u_k-1}(1-a_k)^{v_k-1}da_k; \\ &= \frac{1}{c_kB(u_k,v_k)}\times\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\times I, \end{aligned}$$

with

$$I=\int_0^1(n_k+a_k+\alpha-1)\times a_k^{u_k-1}(1-a_k)^{v_k-1}da_k.$$

One has the following equalities :

$$\begin{aligned} I &= \int_0^1(n_k+a_k+\alpha-1)\times a_k^{u_k-1}(1-a_k)^{v_k-1}da_k+\int_0^1 a_k^{u_k}(1-a_k)^{v_k-1}da_k \\ &= \int_0^1[(n_k+\alpha-1)+a_k]a_k^{u_k-1}(1-a_k)^{v_k-1} \end{aligned}$$

$$
\begin{aligned}
&= (n_k+\alpha-1)\int_0^1 (a_k^{u_k-1})da_k + \int_0^1 a_k a_k^{u_k-1}(1-a_k)^{v_k-1}da_k \\
&= (n_k+\alpha-1)\int_0^1 (a_k^{u_k-1})da_k + \int_0^1 a_k^{(u_k-1)+1}(1-a_k)^{v_k-1}da_k \\
&= (n_k+\alpha-1)\int_0^1 (a_k^{u_k-1})da_k + \int_0^1 a_k^{(u_k+1)-1}(1-a_k)^{v_k-1}da_k \\
&= (n_k+\alpha-1)B(u_k,v_k) + B(u_k+1,v_k) \qquad \text{by } (i) \\
&= (n_k+\alpha-1)B(u_k,v_k) + B(v_k,u_k+1) \qquad \text{since } B(x,y)=B(y,x) \\
&= (n_k+\alpha-1)B(u_k,v_k) + \frac{u_k}{u_k+v_k}B(u_k,v_k) \qquad \text{by } (ii) \\
&= \left(n_k+\alpha-1+\frac{u_k}{u_k+v_k}\right)B(u_k,v_k). \qquad (11)
\end{aligned}
$$

It results from above that

$$
\begin{aligned}
\widehat{\beta}_{k(EBQG1)} &= \frac{1}{c_k B(u_k,v_k)} \times \ln\left(1+\frac{c_k}{\frac{A_k}{c_k}}\right) \times \left[n_k+\alpha-1+\frac{u_k}{u_k+v_k}\right]B(u_k,v_k) \\
&= \frac{1}{c_k}\ln\left(1+\frac{c_k}{\frac{A_k}{c_k}}\right) \times \left[n_k+\alpha-1+\frac{u_k}{u_k+v_k}\right].
\end{aligned}
$$

For $i=2$, we have for the generalized quadratic loss function, and for the prior $\pi_2(a_k,b_k)$, the E-Bayesian estimator of β_k given by :

$$
\begin{aligned}
\widehat{\beta}_{k(EBQG2)} &= \int_0^1\int_0^{c_k} \frac{n_k+a_k+\alpha-1}{b_k+\frac{A_k}{\alpha_k}} \times \frac{2}{c_k^2 B(u_k,v_k)}(c_k-b_k)a_k^{u_k-1}(1-a_k)^{v_k-1}db_k da_k \\
&= \int_0^1 \frac{2(n_k+a_k+\alpha-1)}{c_k^2 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1}\left(\int_0^{c_k}\frac{c_k-b_k}{b_k+\frac{A_k}{\alpha_k}}db_k\right)da_k \\
&= \int_0^1 \frac{2(n_k+a_k+\alpha-1)}{c_k^2 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1}\left[-b_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(b_k+\frac{A_k}{\alpha_k}\right)\right]_0^{c_k} da_k \\
&= \int_0^1 \frac{2(n_k+a_k+\alpha-1)}{c_k^2 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1}\left[-c_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right] da_k \\
&= \frac{2\left[-c_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]}{c_k^2 B(u_k,v_k)}\int_0^1 (n_k+a_k+\alpha-1)a_k^{u_k-1}(1-a_k)^{v_k-1}da_k \\
&= \frac{2\left[-c_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]}{c_k^2 B(u_k,v_k)} \times I \\
&= \frac{2\left[-c_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]}{c_k^2 B(u_k,v_k)} \times B(u_k,v_k)\left[(n_k+\alpha-1)+\frac{u_k}{u_k+v_k}\right] \qquad \text{by } (14)
\end{aligned}
$$

$$= 2c_k^{-2}\left[-c_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\times\left[(n_k+\alpha-1)+\frac{u_k}{u_k+v_k}\right].$$

For $i=3$, we have for the generalized quadratic loss function, and for the prior $\pi_3(a_k,b_k)$, the E-Bayesian estimator of β_k given by :

$$\begin{aligned}
\widehat{\beta}_{k(EBQG3)} &= \int_0^1\int_0^{c_k}\frac{n_k+a_k+\alpha-1}{b_k+\frac{A_k}{\alpha_k}}\times\frac{2}{c_k^2B(u_k,v_k)}\times(b_k)a_k^{u_k-1}(1-a_k)^{v_k-1}db_kda_k\\
&= \int_0^1\frac{2(n_k+a_k+\alpha-1)}{c_k^2B(u_k,v_k)}\times a_k^{u_k-1}(1-a_k)^{v_k-1}\times\left(\int_0^{c_k}\frac{b_k}{b_k+\frac{A_k}{\alpha_k}}db_k\right)da_k\\
&= \int_0^1\frac{2(n_k+a_k+\alpha-1)}{c_k^2B(u_k,v_k)}\times a_k^{u_k-1}(1-a_k)^{v_k-1}\times\left[b_k-\left(\frac{A_k}{\alpha_k}\right)\ln\left(b_k+\frac{A_k}{\alpha_k}\right)\right]_0^{c_k}da_k\\
&= \int_0^1\frac{2(n_k+a_k+\alpha-1)}{c_k^2B(u_k,v_k)}\times a_k^{u_k-1}(1-a_k)^{v_k-1}\times\left[c_k-\left(\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]da_k\\
&= \frac{2}{c_k^2B(u_k,v_k)}\left[c_k-\left(\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\int_0^1(n_k+a_k+\alpha-1)a_k^{u_k-1}(1-a_k)^{v_k-1}da_k\\
&= \frac{2}{c_k^2B(u_k,v_k)}\left[c_k-\left(\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\times I\\
&= \frac{2}{c_k^2B(u_k,v_k)}\left[c_k-\left(\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\times B(u_k,v_k)\left[(n_k+\alpha-1)+\frac{u_k}{u_k+v_k}\right]\text{ by (14)}\\
&= 2c_k^{-2}\left[c_k-\left(\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\times\left[(n_k+\alpha-1)+\frac{u_k}{u_k+v_k}\right].
\end{aligned}$$

Remark 1. *From the decompositions resulting from the above theorem, one observes that :*

- *For $\alpha=1$, one obtains the estimator of β_k under the quadratic loss function as in [32];*
- *For $\alpha=2$, one gets the estimator of β_k under the DeGroot loss function;*
- *For $\alpha=0$, one gets the estimator of β_k under the Entropy loss function.*

Thus, our estimators generalize not only the one associated with the quadratic loss function of [32], but also those associated with the DeGroot and the Entropy loss functions.

5. E-Bayesian Estimation for the DeGroot loss function

5.1. The DeGroot loss function

[5] introduced several types of loss functions and then obtained the Bayes estimators under them. An example of a symmetric loss function is defined by :

$$L(\theta, \delta(\underline{x})) = \left(\frac{\theta - \delta(\underline{x})}{\delta(\underline{x})}\right)^2.$$

Under this loss function, the Bayes estimator is defined by :

$$\delta_\pi(\underline{x}) = \frac{E_\pi(\theta^2|\underline{x})}{E_\pi(\theta|\underline{x})}.$$

The E-Bayesian estimator of β_k with hyper-parameters a_k and b_k is given by the formula:

$$\widehat{\beta_{k(EBD_i)}} = \int\int_{\mathcal{D}} \widehat{\beta_{k(BD)}}(a_k, b_k)\pi_i(a_k, b_k)db_k da_k, \quad i = 1, 2, 3,$$

where $\mathcal{D}$ is the decision space and where $\widehat{\beta}_{k(BD)}$ is the Bayesian estimator of β_k defined in Theorem 4.2 of [23] and given below:

$$\widehat{\beta_{k(BD)}}(\alpha_k, \beta_k) = \frac{n_k + a_k + 1}{b_k + \frac{A_k}{\alpha_k}}, \quad \text{with } \alpha_k > 0.$$

The a priori distributions defined above will allow us in the following subsection to determine the E-Bayesian estimators for the different loss functions of DeGroot.

5.2. The E-Bayesian Estimators

Theorem 2. *Under DeGroot's loss function, the E- Bayesian estimators of β_k with the priors $\pi_i(a_k, b_k)$, $i \in \{1, 2, 3\}$ are given by :*

$$\begin{cases} \widehat{\beta_{k(EBD1)}} &= c_k^{-1}\ln\left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}}\right)\left(n_k + 1 + \frac{u_k}{u_k + v_k}\right) \\ \widehat{\beta_{k(EBD2)}} &= 2c_k^{-2}\left[-c_k + \left(c_k + \frac{A_k}{\alpha_k}\right)\ln\left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\left(n_k + 1 + \frac{u_k}{u_k + v_k}\right) \\ \widehat{\beta_{k(EBD3)}} &= 2c_k^{-2}\left[c_k - \left(\frac{A_k}{\alpha_k}\right)\ln\left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\left(n_k + 1 + \frac{u_k}{u_k + v_k}\right), \end{cases} \tag{12}$$

where $\alpha_k > 0$, $0 < a_k < 1$ and $0 < b_k < c_k$.

Proof. For $i = 1$, under DeGroot's loss function, and for the prior $\pi_1(a_k, b_k)$, the E-Bayesian estimator of β_k is given by :

$$\begin{aligned} \widehat{\beta_{k(EBD1)}} &= \int_0^1\int_0^{c_k} \frac{n_k + a_k + 1}{b_k + \frac{A_k}{\alpha_k}} \times \frac{1}{c_k B(u_k, v_k)} \times a_k^{u_k - 1}(1 - a_k)^{v_k - 1} db_k da_k \\ &= \int_0^1 \frac{1}{c_k B(u_k, v_k)} \times a_k^{u_k - 1}(1 - a_k)^{v_k - 1} \times \left(\int_0^{c_k} \frac{n_k + a_k + 1}{b_k + \frac{A_k}{\alpha_k}} db_k\right) da_k \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{n_k+a_k+1}{c_k B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1} \times \left(\int_0^{c_k} \frac{1}{b_k+\frac{A_k}{\alpha_k}} db_k \right) da_k \\
&= \int_0^1 \frac{n_k+a_k+1}{c_k B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1} \times \ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right) da_k \\
&= \frac{1}{c_k B(u_k,v_k)} \times \ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right) \int_0^1 (n_k+a_k+1) \times a_k^{u_k-1}(1-a_k)^{v_k-1} da_k; \\
&= \frac{1}{c_k B(u_k,v_k)} \times \ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right) \times I_1,
\end{aligned}
$$

with

$$I_1 = \int_0^1 (n_k+a_k+1) \times a_k^{u_k-1}(1-a_k)^{v_k-1} da_k.$$

One has the following developments:

$$
\begin{aligned}
I_1 &= (n_k+1)\int_0^1 a_k^{u_k-1}(1-a_k^{v_k-1}) da_k + \int_0^1 a_k a_k^{u_k-1}(1-a_k^{v_k-1}) da_k \\
&= (n_k+1)B(u_k,v_k) + \int_0^1 a_k^{(u_k-1)-1}(1-a_k) da_k \\
&= (n_k+1)B(u_k,v_k) + B(u_k+1,v_k) \\
&= (n_k+1)B(u_k,v_k) + B(v_k,u_k+1) \quad \text{because } B(x,y)=B(y,x) \\
&= (n_k+1)B(u_k,v_k) + \frac{u_k}{u_k+v_k} B(u_k,v_k) \\
&= \left[(n_k+1) + \frac{u_k}{u_k+v_k}\right] B(u_k,v_k), \qquad (13)
\end{aligned}
$$

From which one has:

$$
\begin{aligned}
\widehat{\beta}_{k(EBD1)} &= \frac{1}{c_k B(u_k,v_k)} \times \ln\left(1+\frac{c_k}{\frac{A_k}{c_k}}\right) \times \left(n_k+1+\frac{u_k}{u_k+v_k}\right) B(u_k,v_k) \\
&= \frac{1}{c_k} \ln\left(1+\frac{c_k}{\frac{A_k}{c_k}}\right) \times \left(n_k+1+\frac{u_k}{u_k+v_k}\right).
\end{aligned}
$$

For $i=2$, under DeGroot's loss function, and for the a priori $\pi_2(a_k,b_k)$, the E-Bayesian estimator of β_k is given by :

$$
\begin{aligned}
\widehat{\beta}_{k(EBD2)} &= \int_0^1 \int_0^{c_k} \frac{n_k+a_k+1}{b_k+\frac{A_k}{\alpha_k}} \times \frac{2}{c_k^2 B(u_k,v_k)} \times (c_k-b_k) a_k^{u_k-1}(1-a_k)^{v_k-1} db_k da_k \\
&= \int_0^1 \frac{2(n_k+a_k+1)}{c_k^1 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1} \times \left(\int_0^{c_k} \frac{c_k-b_k}{b_k+\frac{A_k}{\alpha_k}} db_k \right) da_k
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 \frac{2(n_k+a_k+1)}{c_k^1 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1} \times \left[-b_k + \left(c_k + \frac{A_k}{\alpha_k}\right)\ln(b_k+\frac{A_k}{\alpha_k})\right]_0^{c_k} da_k \\
&= \int_0^1 \frac{2(n_k+a_k+1)}{c_k^1 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1} \times \left[-c_k + \left(c_k + \frac{A_k}{\alpha_k}\right)\ln(1+\frac{c_k}{\frac{A_k}{\alpha_k}})\right] da_k \\
&= \frac{2\left[-c_k + \left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]}{c^2 B(u_k,v_k)} \times \int_0^1 (n_k+a_k+1)a_k^{u_k+1}(1-a_k)^{v_k-1}da_k \\
&= \frac{2\left[-c_k + \left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]}{c^2 B(u_k,v_k)} \times I_1,
\end{aligned}
$$

where I_1 is defined above.

Thus,

$$
\begin{aligned}
\widehat{\beta_{k(EBD2)}} &= \frac{2\left[-c_k + \left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]}{c^2 B(u_k,v_k)} \times \left[(n_k+1)+\frac{u_k}{u_k+v_k}\right] B(u_k,v_k) \\
&= \frac{2\left[-c_k + \left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]}{c^2} \times \left[(n_k+1)+\frac{u_k}{u_k+v_k}\right] \\
&= 2c_k^{-2}\left[-c_k + \left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right] \times \left(n_k+1+\frac{u_k}{u_k+v_k}\right).
\end{aligned}
$$

For $i=3$, under DeGroot's loss function, and for the a priori distribution $\pi_3(a_k,b_k)$, the E-Bayesian estimator of β_k is given by :

$$
\begin{aligned}
\widehat{\beta_{k(EBD3)}} &= \int_0^1\int_0^{c_k} \widehat{\beta_{k(BD)}}(a_k,b_k)\pi_3(a_k,b_k)db_k da_k \\
&= \int_0^1\int_0^{c_k} \frac{n_k+a_k+1}{b_k+\frac{A_k}{\alpha_k}} \times \frac{2b_k}{c_k^2 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1}db_k da_k \\
&= \int_0^1 \frac{2(n_k+a_k+1)}{c_k^2 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1} \times \left(\int_0^{c_k} \frac{b_k}{b_k+\frac{A_k}{\alpha_k}}db_k\right) da_k \\
&= \int_0^1 \frac{2(n_k+a_k+1)}{c_k^2 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1} \times \left[b_k - \left(\frac{A_k}{\alpha_k}\right)\ln\left(b_k+\frac{A_k}{\alpha_k}\right)\right]_0^{c_k} da_k \\
&= \int_0^1 \frac{2(n_k+a_k+1)}{c_k^2 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1} \times \left[c_k - \left(\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right] da_k \\
&= \frac{2}{c_k^2 B(u_k,v_k)}\left[c_k - \left(\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\int_0^1 (n_k+a_k+1)a_k^{u_k-1}(1-a_k)^{v_k-1}da_k \\
&= \frac{2}{c_k^2 B(u_k,v_k)}\left[c_k - \left(\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right] \times I_1
\end{aligned}
$$

$$= \frac{2}{c_k^2 B(u_k, v_k)}\left[c_k - \left(\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right] \times B(u_k, v_k)\left[(n_k+1)+\frac{u_k}{u_k+v_k}\right] \text{ from (18)}$$

$$= \frac{2}{c_k^2}\left[c_k - \left(\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right] \times \left[(n_k+1)+\frac{u_k}{u_k+v_k}\right].$$

This expression gives the E-Bayesian estimator of β_k for the a priori $\pi_3(a_k, b_k)$.

6. E-Bayesian estimation for the entropy loss function

6.1. Entropy loss function

[4] proposed a loss function which results from the Linex loss function called the entropy loss function, defined by :

$$L_E(\theta, d) \propto \left(\frac{d}{\theta}\right)^p - p\ln\left(\frac{d}{\theta}\right) - 1.$$

$L_E(\theta, d)$ is minimal at $d = \theta$.

The Bayes estimator of the parameter θ under this loss function is defined for all $p \in \mathbb{R}$ by :

$$\delta(\underline{x}) = \left(E_\theta(\theta)^{-p}\right)^{\frac{-1}{p}}.$$

- When $p = 1$, the Bayes estimator coincides with the Bayes estimator under the weighted squared loss function :
$$\frac{(d-\theta)^2}{\theta}.$$
- When $p = -1$, the Bayes estimator coincides with the Bayes estimator under the quadratic loss function.

The E-Bayesian estimator of β_k for the hyper-parameters a_k and b_k is given by the formula :

$$\widehat{\beta_{k(EBE_i)}} = \int\int_{\mathcal{D}} \widehat{\beta_{k(BE)}}(a_k, b_k)\pi_i(a_k, b_k)db_k da_k, \quad i = 1, 2, 3,$$

where $\mathcal{D}$ is the decision space and $\widehat{\beta_{k(BE)}}$ is the Bayesian estimator of β_k defined by (see [23]) :

$$\widehat{\beta_{k(BE)}}(\alpha_k, \beta_k) = \left[\frac{1}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{-1}} \times \frac{\Gamma(n_k + a_k - p)}{\Gamma(n_k + a_k)}\right]^{-1/p}, \quad \text{with } \alpha_k > 0.$$

6.2. E-Bayesian estimators of β_k

Theorem 3. *Under the Entropy loss function for $p = 1$, the E-Bayesian estimators of β_k for the priors $\pi_i(a_k, b_k)$, $i \in \{1, 2, 3\}$ are given by :*

$$\begin{cases} \widehat{\beta}_{k(EBE1)} &= c_k^{-1} \ln\left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}}\right)\left(n_k - 1 + \frac{u_k}{u_k + v_k}\right) \\ \widehat{\beta}_{k(EBE2)} &= 2c_k^{-2}\left[-c_k + \left(c_k + \frac{A_k}{\alpha_k}\right)\ln\left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\left(n_k - 1 + \frac{u_k}{u_k + v_k}\right) \\ \widehat{\beta}_{k(EBE3)} &= 2c_k^{-2}\left[c_k - \left(\frac{A_k}{\alpha_k}\right)\ln\left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]\left(n_k - 1 + \frac{u_k}{u_k + v_k}\right) \end{cases} \tag{14}$$

where $\alpha_k > 0$, $0 < a_k < 1$ and $0 < b_k < c_k$.

Proof. For $i = 1$, under the entropy loss function for $p = 1$ and for the prior $\pi_1(a_k, b_k)$, the E-Bayesian estimator of β_k is given by :

$$\begin{aligned} \widehat{\beta}_{k(EBE1)} &= \int_0^1 \int_0^{c_k} \left[\frac{1}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{-1}} \times \frac{\Gamma(n_k + a_k - p)}{\Gamma(n_k + a_k)}\right]^{-1/1} \times \frac{1}{c_k B(u_k, v_k)} a_k^{u_k - 1}(1 - a_k)^{v_k - 1} db_k da_k \\ &= \int_0^1 \int_0^{c_k} \left[\frac{1}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{-1}} \times \frac{\Gamma(n_k + a_k - p)}{(n_k + a_k - 1)\Gamma(n_k + a_k - 1)}\right]^{-1} \\ &\times \frac{1}{c_k B(u_k, v_k)} a_k^{u_k - 1}(1 - a_k)^{v_k - 1} db_k da_k \\ &= \int_0^1 \int_0^{c_k} \frac{n_k + a_k - 1}{b_k + \frac{A_k}{\alpha_k}} \times \frac{1}{c_k B(u_k, v_k)} a_k^{u_k - 1}(1 - a_k)^{v_k - 1} db_k da_k \\ &= \int_0^1 \frac{n_k + a_k - 1}{c_k B(u_k, v_k)} \times a_k^{u_k - 1}(1 - a_k)^{v_k - 1} \times \left(\int_0^{c_k} \frac{1}{b_k + \frac{A_k}{\alpha_k}} db_k\right) da_k \\ &= \int_0^1 \frac{n_k + a_k - 1}{c_k B(u_k, v_k)} \times a_k^{u_k - 1}(1 - a_k)^{v_k - 1} \times \ln\left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}}\right) da_k \\ &= \frac{1}{c_k B(u_k, v_k)} \times \ln\left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}}\right) \int_0^1 (n_k + a_k - 1) \times a_k^{u_k - 1}(1 - a_k)^{v_k - 1} da_k \\ &= \frac{1}{c_k B(u_k, v_k)} \times \ln\left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}}\right) \times I_2, \end{aligned}$$

with

$$I_2 = \int_0^1 (n_k + a_k - 1) \times a_k^{u_k - 1}(1 - a_k)^{v_k - 1} da_k.$$

As for I and I_1, one has:

$$\begin{aligned} I_2 &= \int_0^1 [(n_k-1)+a_k] \times a_k^{u_k-1}(1-a_k)^{v_k-1} \\ &= \left[(n_k-1)+\frac{u_k}{u_k+v_k}\right] B(u_k,v_k). \end{aligned} \tag{15}$$

For $i=2$, under the Entropy loss function for $p=1$ and for the prior $\pi_2(a_k,b_k)$, the E-Bayesian estimator of β_k is given by :

$$\begin{aligned} \widehat{\beta_{k(EBE2)}} &= \int_0^1\int_0^{c_k} \widehat{\beta_{k(BE)}}(a_k,b_k)\pi_2(a_k,b_k)db_k da_k \\ &= \int_0^1\int_0^{c_k} \frac{n_k+a_k-1}{b_k+\frac{A_k}{\alpha_k}} \times \frac{2}{c_k^2 B(u_k,v_k)}(c_k-b_k)a_k^{u_k-1}(1-a_k)^{v_k-1}db_k da_k \\ &= \int_0^1 \frac{2(n_k+a_k-1)}{c_k^2 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1}\left(\int_0^{c_k}\frac{c_k-b_k}{b_k+\frac{A_k}{\alpha_k}}db_k\right)da_k \\ &= \int_0^1 \frac{2(n_k+a_k-1)}{c_k^2 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1}\left[-b_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(b_k+\frac{A_k}{\alpha_k}\right)\right]_0^{c_k} da_k \\ &= \int_0^1 \frac{2(n_k+a_k-1)}{c_k^2 B(u_k,v_k)} \times a_k^{u_k-1}(1-a_k)^{v_k-1}\left[-c_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right] da_k \\ &= \frac{2\left[-c_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]}{c_k^2 B(u_k,v_k)}\int_0^1 (n_k+a_k-1)a_k^{u_k-1}(1-a_k)^{v_k-1}da_k \\ &= \frac{2\left[-c_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]}{c_k^2 B(u_k,v_k)} \times I_2 \end{aligned}$$

By replacing I_2 in the expression for $\widehat{\beta_{k(EBE2)}}$ above, we have :

$$\begin{aligned} \widehat{\beta_{k(EBE2)}} &= \frac{2\left[-c_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]}{c_k^2 B(u_k,v_k)} \times \left[(n_k-1)+\frac{u_k}{u_k+v_k}\right] B(u_k,v_k) \\ &= \frac{2\left[-c_k+\left(c_k+\frac{A_k}{\alpha_k}\right)\ln\left(1+\frac{c_k}{\frac{A_k}{\alpha_k}}\right)\right]}{c_k^2} \times \left[(n_k-1)+\frac{u_k}{u_k+v_k}\right]. \end{aligned}$$

For $i=3$, under the entropy loss function for $i=3$ and for the prior $\pi_3(a_k,b_k)$, the E-Bayesian estimator of β_k is given by:

$$\widehat{\beta_{k(EBE3)}} = \int_0^1\int_0^{c_k} \widehat{\beta_{k(BE)}}(a_k,b_k)\pi_3(a_k,b_k)db_k da_k$$

$$
\begin{aligned}
&= \int_0^1 \int_0^{c_k} \frac{n_k + a_k - 1}{b_k + \frac{A_k}{\alpha_k}} \times \frac{2}{c_k^2 B(u_k, v_k)} \times (b_k) a_k^{u_k - 1}(1 - a_k)^{v_k - 1} db_k da_k \\
&= \int_0^1 \frac{2(n_k + a_k - 1)}{c_k^2 B(u_k, v_k)} \times a_k^{u_k - 1}(1 - a_k)^{v_k - 1} \times \left(\int_0^{c_k} \frac{b_k}{b_k + \frac{A_k}{\alpha_k}} db_k \right) da_k \\
&= \int_0^1 \frac{2(n_k + a_k - 1)}{c_k^2 B(u_k, v_k)} \times a_k^{u_k - 1}(1 - a_k)^{v_k - 1} \times \left[b_k - \left(\frac{A_k}{\alpha_k} \right) \ln \left(b_k + \frac{A_k}{\alpha_k} \right) \right]_0^{c_k} da_k \\
&= \int_0^1 \frac{2(n_k + a_k - 1)}{c_k^2 B(u_k, v_k)} \times a_k^{u_k - 1}(1 - a_k)^{v_k - 1} \times \left[c_k - \left(\frac{A_k}{\alpha_k} \right) \ln \left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}} \right) \right] da_k \\
&= \frac{2}{c_k^2 B(u_k, v_k)} \left[c_k - \left(\frac{A_k}{\alpha_k} \right) \ln \left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}} \right) \right] \int_0^1 (n_k + a_k - 1) a_k^{u_k - 1}(1 - a_k)^{v_k - 1} da_k \\
&= \frac{2}{c_k^2 B(u_k, v_k)} \left[c_k - \left(\frac{A_k}{\alpha_k} \right) \ln \left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}} \right) \right] \times I_2 \\
&= \frac{2}{c_k^2 B(u_k, v_k)} \left[c_k - \left(\frac{A_k}{\alpha_k} \right) \ln \left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}} \right) \right] \times B(u_k, v_k) \left[(n_k - 1) + \frac{v_k}{u_k + v_k} \right] \\
&= \frac{2}{c_k^{-2}} \left[c_k - \left(\frac{A_k}{\alpha_k} \right) \ln \left(1 + \frac{c_k}{\frac{A_k}{\alpha_k}} \right) \right] \times \left[(n_k - 1) + \frac{v_k}{u_k + v_k} \right].
\end{aligned}
$$

This expression gives the E-Bayesian estimator of β_k for the a-priori distribution of the density $\pi_3(a_k, b_k)$ of the hyper-parameters a_k and b_k under this loss function.

7. Conclusion and Perspectives

In this paper, we have studied the E-Bayesian estimation of the scaling parameter in the context of competing risks of the Gompertz distribution under several loss functions. Under progressive type I censoring, we have determined the new estimators which generalize not only the generalized quadratic loss function estimators proposed by [32] but also those of the DeGroot and Entropy loss functions. We also determine the E-Bayesian estimators of the scale parameter for the prior distributions of the hyper-parameters under the entropy loss function for $p = 1$ and under the DeGroot loss function.

As perspectives, we plan to work on the relations existing among the estimators $\widehat{\beta}_{k(EBQGi)}$ $(i = 1, 2, 3)$, $\widehat{\beta}_{k(EBDi)}$ $(i = 1, 2, 3)$ and the relations among $\widehat{\beta}_{k(EBEi)}$ $(i = 1, 2, 3)$. In a second step, we will do simulation experiments to evaluate the performance of Bayesian and E-Bayesian estimation of the reliability functions based on the generalized quadratic loss, DeGroot and Entropy functions in terms of estimated risks by Monte Carlo methods. We will also make applications in survival data in biology of aging where α_k is called the coefficient of the age-dependent mortality rate, and β_k is called the coefficient of the death-age-independent rate. The curves that we will obtain will be compared with those obtained by [31] and we will allow us to judge the robustness and/or efficiency of our estimators obtained. Finally, we will consider the case of the 0-1 loss function.

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