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A Variant of Hop Domination in Graphs

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Abstract. Let G be a connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A set $S \subseteq V(G)$ is a hop dominating set of G if for each $v \in V(G) \setminus S$, there exists $w \in S$ such that $d_G(v, w) = 2$. A set $S \subseteq V(G)$ is a super hop dominating set if $ehpn_G(v, V(G) \setminus S) \neq \varnothing$ for each $v \in V(G) \setminus S$, where $ehpn_G(v, V(G) \setminus S)$ is the set containing all the external hop private neighbors of v with respect to $V(G) \setminus S$. The minimum cardinality of a super hop dominating set of G, denoted by $\gamma_h^s(G)$, is called the super hop domination number of G. In this paper, we investigate the concept and study it for graphs resulting from some binary operations. Specifically, we characterize the super hop dominating sets in the join, and lexicographic products of graphs, and determine bounds of the super hop domination number of each of these graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: hop domination, super hop domination, complement-super domination, join, lexicographic product

1. Introduction

Super domination in a graph was first introduced and studied by Lemanska et al. in [8]. This concept uses the concept of external private neighbor of a vertex in some subset of the vertex set of a graph. Dettlaff et al. in [3] determined the super domination number of lexicographic products of graphs. Also, Dettlaff in [2] determined some values and bounds for the super domination number of some Cartesian products of graphs. Paraico and Canoy in [13] characterized the super dominating sets in the lexicographic and the Cartesian products of graphs and obtained bounds for the super domination numbers of these graphs.

Recently, Natarajan and Ayyaswamy [11] introduced and studied the concept of hop domination in a graph. Ayyaswamy et al. in [1] also investigated the concept and obtained

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bounds of the hop domination number of some graphs. The concept and some of its variations are also studied in [5], [6], [7], [10], [9], [12], and [14]. Motivated by these previous studies and, in particular, [3], [8], and [13], we introduce herein the concept of super hop domination and investigate it for some graphs and graphs resulting from the join, and lexicographic product of two graphs.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a connected graph and let $v \in V(G)$. The *open neighborhood* of v is the set $N_G(v) = \{z \in V(G) : vz \in E(G)\}$ and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. The *open hop neighborhood* of v is the set $N_G^2(v) = \{u \in V(G) : d_G(u, v) = 2\}$, where $d_G(u, v)$ denotes the distance between vertices u and v in G. The *closed hop neighborhood* of v is $N_G^2[v] = N_G^2(v) \cup \{v\}$. The *open hop neighborhood* of $A \subseteq V(G)$ is the set $N_G^2(A) = \cup_{v \in A} N_G^2(v)$ and its *closed hop neighborhood* is $N_G^2[A] = A \cup N_G^2(A)$.

A set $S \subseteq V(G)$ is a *dominating set* (*hop dominating set*) of a graph G if for each $v \in V(G) \setminus S$, there exists $w \in S$ such that $d_G(v, w) = 1$ (resp. $d_G(v, w) = 2$). The smallest cardinality of a dominating (resp. hop dominating) set of G, denoted by $\gamma(G)$ (resp. $\gamma_h(G)$), is called the *domination number* (resp. *hop domination number*) of G.

Let S be a subset of $V(G)$ and let $v \in S$. A vertex $w \in V(G) \setminus S$ is an *external private neighbor* (*external hop private neighbor*) of v with respect to S if $N_G(w) \cap S = \{v\}$ (resp. $N_G^2(w) \cap S = \{v\}$). The set containing all the external private neighbors (resp. hop private) neighbors of v with respect to S is denoted by $epn_G(v, S)$ (resp. $ehpn_G(v, S)$). A set $S \subseteq V(G)$ is called a *super dominating set* (resp. *super hop dominating set*) if $epn_G(v, V(G) \setminus S) \neq \varnothing$ (resp. $ehpn_G(v, V(G) \setminus S) \neq \varnothing$) for each $v \in V(G) \setminus S$. The smallest cardinality of a super dominating (resp. super hop dominating) set of G, denoted by $\gamma_s(G)$ (resp. $\gamma_h^s(G)$), is called the *super domination number* (resp. *super hop domination number*) of G. Any super dominating (resp. super hop dominating) set of G with cardinality $\gamma_s(G)$ (resp. $\gamma_h^s(G)$) is called a γ_s-set (resp. γ_h^s-set) of G.

A subset D of $V(G)$ is a *complement-super dominating set* of G (super dominating set of $\overline{G}$) if for each $v \in V(G) \setminus D$, there exists $w \in D \setminus N_G(v) = D \cap N_{\overline{G}}(v)$ such that $[V(G) \setminus N_G(w)] \cap [V(G) \setminus D] = N_{\overline{G}}(w) \cap [V(\overline{G}) \setminus D] = \{v\}$. The smallest cardinality of a complement-super dominating set of G, denoted by $\gamma_{cs}(G) = \gamma_s(\overline{G})$, is called the *complement-super domination number* of G (super domination number of $\overline{G}$). Any complement-super dominating set of G with cardinality equal to $\gamma_{cs}(G)$ is called a γ_{cs}-set.

3. Results

Remark 1. $\gamma_h^s(K_n) = n$ *for all* $n \geq 1$.

Proposition 1. *Let G be a graph of order $n \geq 1$. Then* $\max\{\gamma_h(G), \lceil \frac{n}{2} \rceil\} \leq \gamma_h^s(G) \leq n$.

Proof. Since every super hop dominating set is hop dominating, it follows that $\gamma_h(G) \leq \gamma_h^s(G)$. Now let S be a γ_h^s-set of G. Then, by definition of super hop dominating set,

$|S| \geq |V(G) \setminus S|$. This implies that $\gamma_h^s(G) = |S| \geq \lceil \frac{n}{2} \rceil$. Moreover, since $V(G)$ is a super hop dominating set, we have $\lceil \frac{n}{2} \rceil \leq \gamma_h^s(G) \leq n$. Therefore, the assertion holds. □

Theorem 1. *Let G be a graph of order $n \geq 1$. Then $\gamma_h^s(G) = n$ if and only if each component C of G is a complete graph.*

Proof. Suppose that $\gamma_h^s(G) = n$. Suppose further that there exists a component C of G such that C is not a complete graph. Then there exist $x, y \in V(C)$ such that $d_C(x, y) = d_G(x, y) = 2$. This implies that $S = V(G) \setminus \{x\}$ is a super hop dominating set of G. Hence, $\gamma_h^s(G) \leq |S| = n - 1$, contrary to the assumption that $\gamma_h^s(G) = n$. Thus, each component of G is a complete graph.

For the converse, suppose that each component of G is a complete graph. Let S be a γ_h^s-set of G and suppose that $S \neq V(G)$. Let $v \in V(G) \setminus S$ and let C be the component of G with $v \in V(C)$. Let $w \in ehpn_G(v, V(G) \setminus S)$. Then $w \in V(C)$ and $d_G(v, w) = 2$, contrary to the assumption that C is a complete graph. Thus, $S = V(G)$, showing that $\gamma_h^s(G) = n$. □

The next results are consequences of Theorem 1.

Corollary 1. *Let G be a connected graph of order n. Then $\gamma_h^s(G) = n$ if and only if $G = K_n$.*

Corollary 2. *Let G be a connected non-complete graph of order n. Then $\gamma_h^s(G) \leq n - 1$.*

Corollary 3. *If G is the complete graph of order n, then $\gamma_h^s(G) + \gamma_h^s(\overline{G}) = 2n$ and $\gamma_h^s(G).\gamma_h^s(\overline{G}) = n^2$.*

Theorem 2. *Let G be a connected non-complete graph of oder n. Then*

(i) $n \leq \gamma_h^s(G) + \gamma_h^s(\overline{G}) \leq 2n - 1$ *and*

(ii) $\frac{n^2}{4} \leq \gamma_h^s(G).\gamma_h^s(\overline{G}) \leq n^2 - n$.

Proof. By Corollary 2, $\gamma_h^s(G) \leq n - 1$. Also, by Proposition 1, $\gamma_h^s(\overline{G}) \leq n$. These imply that $\gamma_h^s(G) + \gamma_h^s(\overline{G}) \leq (n-1) + n = 2n - 1$ and $\gamma_h^s(G).\gamma_h^s(\overline{G}) \leq (n-1)n = n^2 - n$. The left inequalities follow from Proposition 1. □

Note that the upper bounds in Theorem 2 are tight. Indeed, if $G = K_{1,n-1}$, then $\overline{G} = K_1 \cup K_{n-1}$. It is easy to show that $\gamma_h^s(G) = n - 1$. By Theorem 1, $\gamma_h^s(\overline{G}) = n$. Hence, $\gamma_h^s(G) + \gamma_h^s(\overline{G}) = 2n - 1$ and $\gamma_h^s(G).\gamma_h^s(\overline{G}) = n^2 - n$. The lower bounds are also attainable. It can be verified that $\gamma_h^s(P_4) + \gamma_h^s(\overline{P}_4) = 4$ and $\gamma_h^s(P_4).\gamma_h^s(\overline{P}_4) = \frac{16}{4} = 4$.

The *join* of graphs G and H is the graph $G + H$ with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Theorem 3. *Let G and H be any two graphs. A subset S of $V(G + H)$ is a super hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$ where S_G and S_H are complement-super dominating sets of G and H, respectively.*

Proof. Suppose that S is a super hop dominating set of $G+H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Let $v \in V(G) \setminus S_G$. Since S is a super hop dominating set of $G+H$, there exists $w \in ehpn(v, V(G+H) \setminus S)$. Since $V(H) \subseteq N_{G+H}(v)$, it follows that $w \in S_G \setminus N_G(v)$ and

$$N^2_{G+H}(w) \cap (V(G+H) \setminus S) = [V(G) \setminus N_G(w)] \cap [V(G) \setminus S_G] = \{v\}.$$

This shows that S_G is a complement-super dominating set of G. Similarly, S_H is a complement-super dominating set of H.

For the converse, suppose that $S = S_G \cup S_H$, where S_G and S_H are complement-super dominating sets of G and H, respectively. Let $v \in V(G+H) \setminus S$. If $v \in V(G)$, then $v \in V(G) \setminus S_G$. Since S_G is a complement-super dominating set of G, there exists $w \in S_G \setminus N_G(v)$ such that $[V(G) \setminus N_G(w)] \cap [V(G) \setminus S_G] = N^2_{G+H}(w) \cap (V(G+H) \setminus S) = \{v\}$. This implies that $w \in ehpn(v, V(G+H) \setminus S)$. Similarly, if $v \in V(G) \setminus S_H$, then there exists $z \in ehpn(v, V(G+H) \setminus S)$. This shows that S is a super hop dominating set of $G+H$. □

Theorem 4. *Let G be a graph of order n. Then $\lceil \frac{n}{2} \rceil \leq \gamma_{cs}(G) \leq n$. Moreover,*

(i) $\gamma_{cs}(G) = n$ if and only if $G = K_n$; and

(ii) for $n \geq 4$ and even, we have $\gamma_{cs}(G) = \frac{n}{2}$ if and only if G has an $(\frac{n}{2} - 1)$-regular bipartite subgraph H with partite sets A and B such that $V(G) = A \cup B = V(H)$, $|A| = |B| = \frac{n}{2}$ and $E(G) = E(\langle A \rangle) \cup E(\langle B \rangle) \cup E(H)$, where $\langle A \rangle$ is the graph induced by A.

Proof. Let S be a complement-super dominating set of G. By definition, $|S| \geq |V(G) \setminus S| = n - |S|$. Hence, $|S| \geq \frac{n}{2}$, showing that $\lceil \frac{n}{2} \rceil \leq \gamma_{cs}(G) \leq n$.

For (i), suppose that $\gamma_{cs}(G) = n$. Since $\gamma_{cs}(G) = \gamma_s(\overline{G})$, it follows that $\overline{G} = \overline{K}_n$. Hence, $G = K_n$.

The converse is clear.

To show (ii), suppose first that $\gamma_{cs}(G) = \frac{n}{2}$. Let S be an γ_{cs}-set of G. Then $|S| = \frac{n}{2}$. Let $v \in V(G) \setminus S$. Then there exists $x_v \in S \setminus N_G(v)$ such that $[V(G) \setminus N_G(x_v)] \cap (V(G) \setminus S) = \{v\}$. Since $|V(G) \setminus S| = \frac{n}{2}$ and S is a complement-super dominating set of G, $vy \in E(G)$ for all $y \in S \setminus \{x_v\}$. Thus, $N_G(v) \cap S = S \setminus \{x_v\}$. Let $u, v \in V(G) \setminus S$ with $u \neq v$. Since $N_G(x_v) \cap (V(G) \setminus S) = V(G) \setminus (S \cup \{v\})$, it follows that $u \in N_G(x_v)$. Since $u \notin N_G(x_u)$, $x_u \neq x_v$. Hence, $S = \{x_v : v \in V(G) \setminus S\}$. Let $A = S$ and $B = V(G) \setminus S$. Consider the bipartite graph H with partite sets A and B. Then $|A| = |B| = \frac{n}{2}$. Since $N_G(v) \cap S = S \setminus \{x_v\}$ for each $v \in B$, it follows that $deg_H(v) = \frac{n}{2} - 1$ for each $v \in B$. Also, since $[V(G) \setminus N_G(x_v)] \cap (V(G) \setminus S) = \{v\}$ for each $v \in B$, $deg_H(x_v) = \frac{n}{2} - 1$ for each $x_v \in A$. Thus, H is an $(\frac{n}{2} - 1)$-regular graph. Moreover, $E(G) = E(\langle A \rangle) \cup E(\langle B \rangle) \cup E(H)$.

For the converse, suppose that G has the given property. Let $v \in B = V(G) \setminus A$. Then by assumption, $deg_H(v) = \frac{n}{2} - 1$. This implies that there exists $w \in A$ such $[V(H) \setminus N_H(w)] \cap B = \{v\}$. Since $E(G) = E(\langle A \rangle) \cup E(\langle B \rangle) \cup E(H)$, it follows that

$[V(G) \setminus N_G(w)] \cap B = [V(H) \setminus N_H(w)] \cap B = \{v\}$. This shows that A is a complement-super dominating set of G. Therefore $\frac{n}{2} \leq \gamma_{cs}(G) \leq |A| = \frac{n}{2}$, i.e., $\gamma_{cs}(G) = \frac{n}{2}$. □

Theorem 5. *Let G be a graph of order n. Then*

(i) $\gamma_{cs}(G) = 1$ *if and only if* $G = K_1$ *or* $G = \overline{K}_2$.

(ii) $\gamma_{cs}(G) = 2$ *if and only if* $G \in \{K_2, P_3, K_2 \cup K_1, \overline{K}_3, P_4, C_4, K_2 \cup K_2\}$.

Proof. (i) Suppose $\gamma_{cs}(G) = 1$ and let $S = \{v\}$ be a γ_{cs}-set of G. By Theorem 4, $1 \leq n \leq 2$. If $n = 1$, then $G = K_1$. Suppose $n = 2$ and let $w \in VG) \setminus \{v\}$. Since S is a complement-super dominating set of G, $vw \notin E(G)$. It follows that $G = \overline{K}_2$.

The converse is clear.

(ii) Suppose $\gamma_{cs}(G) = 2$. Then $2 \leq n \leq 4$ by Theorem 4. Let $S = \{a, b\}$ be a γ_{cs}-set of G. If $n = 2$, then $G = K_2$ since $\gamma_{cs}(\overline{K}_2) = 1$. Suppose $n = 3$ and let $c \in V(G) \setminus S$. We may assume that $[V(G) \setminus N_G(a)] \cap [V(G) \setminus S] = \{c\}$. If $ab \in E(G)$, then $G = P_3$ or $G = K_2 \cup K_1$. If $ab \notin E(G)$, then $G = \overline{K}_3$ or $G = K_2 \cup K_1$. Finally, let $n = 4$ and let $V(G) = \{a, b, c, d\}$. Since S is a γ_{cs}-set, we may assume that $[V(G) \setminus N_G(a)] \cap [V(G) \setminus S] = \{c\}$ and $[V(G) \setminus N_G(b)] \cap [V(G) \setminus S] = \{d\}$. Then $ad, bc \in E(G)$. Suppose $ab \in E(G)$. If $cd \in E(G)$, then $G = C_4$. If $cd \notin E(G)$, then $G = P_4$. Next, suppose that $ab \notin E(G)$. If $cd \in E(G)$, then $G = P_4$. Otherwise, $G = K_2 \cup K_2$. Therefore, $G \in \{K_2, P_3, K_2 \cup K_1, \overline{K}_3, P_4, C_4, K_2 \cup K_2\}$.

The converse is clear. □

Theorem 6. *Let n be a positive integer. Then*

$$\gamma_{cs}(P_n) = \begin{cases} n & \text{if } n = 1, 2 \\ 2 & \text{if } n = 3 \\ n - 2 & \text{if } n \geq 4. \end{cases}$$

Proof. Let $P_n = [v_1, v_2, \cdots, v_n]$. Clearly, $\gamma_{cs}(P_1) = 1$, $\gamma_{cs}(P_2) = 2$, and $\gamma_{cs}(P_3) = 2$. Suppose $n \geq 4$. Since $S_0 = V(P_n) \setminus \{v_1, v_n\}$ is a complement-super dominating set of P_n, it follows that $\gamma_{cs}(P_n) \leq n - 2$. Suppose $\gamma_{cs}(P_n) < n - 2$. Let S be a γ_{cs}-set of P_n and let $v, w,$ and x be distinct elements of $V(P_n) \setminus S$. We may assume that $v = v_j, w = v_r,$ and $x = v_s$ where $1 \leq j < r < s \leq n$. Since S is a complement-super dominating set of P_n and $w \in V(P_n) \setminus S$, there exists $p \in S$ such that $[V(P_n) \setminus N_{P_n}(p)] \cap [V(P_n) \setminus S] = \{w\}$. This implies that $pv, px \in E(P_n)$. Since $j < s$, it follows that $p = v_{j+1}$ and $s = j+2$. This, however, would imply that $r = j+1$ (i.e., $w = v_r = p$ because $j < r < s$), a contradiction. Therefore, $\gamma_{cs}(P_n) = n - 2$. □

Theorem 7. *Let n be a positive integer and $n \geq 3$. Then*

$$\gamma_{cs}(C_n) = \begin{cases} 3 & \text{if } n = 3, 6 \\ n - 2 & \text{if } n = 4, 5 \text{ and } n \geq 7. \end{cases}$$

Proof. Let $C_n = [v_1, v_2, \cdots, v_n, v_1]$. Now $\gamma_{cs}(C_3) = 3$ by Theorem 4. Suppose $4 \leq n \leq 5$. It can be verified easily that $S_0 = V(C_n) \setminus \{v_1, v_2\}$ is a γ_{cs}-set of C_n. Thus, $\gamma_{cs}(C_n) = n - 2$. Next, suppose that $n = 6$. Since $S_1 = V(C_6) \setminus \{v_1, v_3, v_5\} = \{v_2, v_4, v_6\}$ is a complement-super dominating set of C_n, $\gamma_{cs}(C_6) = 3$ by Theorem 4. Lastly, suppose that $n \geq 7$. The set $S^* = V(C_n) \setminus \{v_1, v_2\} = \{v_3, v_4, \cdots, v_n\}$ is a complement-super dominating set of C_n. Consequently, $\gamma_{cs}(C_n) \leq n-2$. Let S be a γ_{cs}-set of C_n and suppose $|S| < n-2$. Let $a, b,$ and c be distinct elements of $V(C_n) \setminus S$. We may assume that $a = v_1$, $b = v_m$, and $c = v_r$ where $1 < m < r \leq n$. Since S is a complement-super dominating set of C_n and $c \in V(C_n) \setminus S$, there exists $y \in S$ such that $[V(C_n) \setminus N_{C_n}(y)] \cap [V(C_n) \setminus S] = \{c\}$. This implies that $ay, yb \in E(C_n)$. With the assumption that $a = v_1$ and $m < r$, we find that $y = v_2$ and $b = v_3$. Applying the same argument to $a \in V(C_n) \setminus S$, we find that there exists $z \in S$ such that $[V(C_n) \setminus N_{C_n}(z)] \cap [V(C_n) \setminus S] = \{a\}$. Hence, $bz, zc \in E(C_n)$, implying that $z = v_4$ and $c = v_5$. Finally, for $b \in V(C_n) \setminus S$, there also exists $q \in S$ such that $[V(C_n) \setminus N_{C_n}(q)] \cap [V(C_n) \setminus S] = \{b\}$. It follows that $qv_1, qv_5 \in E(C_n)$. Since $n \geq 7$, no such vertex q of C_n exists. Consequently, S is not a complement-super dominating set of C_n, a contradiction. Therefore, $\gamma_{cs}(C_n) = n - 2$. □

Theorem 8. *Let G be a graph of order $n \geq 2$. If S is a complement-super dominating set of G, then $V(G) \setminus S$ contains at most a single isolated vertex of G. In particular, if $G = \overline{K}_n$, then $S = V(G) \setminus \{v\}$ is a γ_{cs}-set of G for each $v \in V(G)$, that is, $\gamma_{cs}(G) = n-1$.*

Proof. If $S = V(G)$, then we are done. Suppose $S \neq V(G)$ and let $v \in V(G) \setminus S$. Then there exists $w \in S$ such that $[V(G) \setminus N_G(w)] \cap [V(G) \setminus S] = \{v\}$. Hence, if v is an isolated vertex of G, then there can be no other isolated vertex in $V(G) \setminus S$. In other words, $V(G) \setminus S$ contains at most a single isolated vertex of G.

Suppose now that $G = \overline{K}_n$ and let $v \in V(G)$. Then clearly, $S = V(G) \setminus \{v\}$ is a complement-super dominating set of G. Since it is not possible for $V(G) \setminus S$ to contain more than one isolated vertices, it follows that S is a γ_{cs}-set of G. Thus, $\gamma_{cs}(G) = n-1$. □

Theorem 9. *Let $G_1, G_2, \cdots, G_n$ be the components of a graph G, where $n \geq 2$. Then $S \subseteq V(G)$ is a complement-super dominating set of G if and only if one of the following holds:*

(i) *$S = D \cup [\cup_{j \neq k} V(G_j)]$ for some $k \leq n$ and for some complement-super dominating set D of G_k.*

(ii) *$S = V(G) \setminus \{v\}$ for some isolated vertex v or $S = [V(G_k) \setminus \{v\}] \cup [\cup_{j \neq k} V(G_j)]$ for some non-trivial component G_k and for some $v \in V(G_k)$.*

(iii) *$S = [V(G_k) \setminus \{v\}] \cup [V(G_r) \setminus \{w\}] \cup [\bigcup_{j \neq k,r} V(G_j)]$ for distinct non-trivial components G_k and G_r and for some $v \in V(G_k)$ and $w \in V(G_r)$.*

Proof. Suppose S is a complement-super dominating set of G. If $S = V(G)$, then we may take $D = V(G_1)$ which is a complement-super dominating set of G_1. Hence, S of type (i). Suppose $S \neq V(G)$. Let $v \in V(G) \setminus S$ and let $k \in \{1, 2, \cdots, n\}$ such that $v \in V(G_k)$.

Since S is a complement-super dominating set of G, there exists $z \in S$ such that $[V(G) \setminus N_G(z)] \cap [V(G) \setminus S] = \{v\}$. Suppose first that $z \in V(G_k)$. Then clearly, $\cup_{j\neq k} V(G_j) \subseteq S$. Let $D = S \cap V(G_k)$. Then $[V(G_k) \setminus N_{G_k}(z)] \cap [V(G_k) \setminus D] = \{v\}$. Thus, if $D = V(G_k) \setminus \{v\}$, then D is a complement-super dominating set of G_k. Suppose $D \neq V(G_k) \setminus \{v\}$ and let $x \in V(G_k) \setminus (D \cup \{v\})$. Then there exists $y \in S$ such that $[V(G) \setminus N_G(y)] \cap [V(G) \setminus S] = \{x\}$. Since $x \neq v$, it follows that $y \in D$ and $[V(G_k) \setminus N_{G_k}(y)] \cap [V(G_k) \setminus D] = \{x\}$. Thus, D is a complement-super dominating set of G_k, showing that (i) holds. Next, suppose that $z \in V(G_r)$ for some $r \leq n$ with $r \neq k$ and let $D^* = S \cap V(G_r)$. Note that if $G_k \neq \langle v \rangle$, then $[V(G_k) \setminus \{v\}] \cup [\cup_{j\neq k,r} V(G_j)] \subseteq S$. Suppose first that $D^* = V(G_r)$. If $G_k = \langle v \rangle$, then $S = V(G) \setminus \{v\}$. Otherwise, $S = [V(G_k) \setminus \{v\}] \cup [\cup_{j\neq k} V(G_j)]$. Hence, S is of type (ii). Next, suppose that $D^* \neq V(G_r)$. Let $w \in V(G_r) \setminus D^*$. Then $z \in N_G(w)$. Let $p \in S$ such that $[V(G) \setminus N_G(p)] \cap [V(G) \setminus S] = \{w\}$. Since $v \in [V(G) \setminus S] \cap V(G_k))]$, $p \notin \cup_{j\neq k} V(G_j)$. Hence, $p \in V(G_k) \cap N_{G_k}(v)$. Moreover, $|V(G_r) \setminus D^*| = 1$, that is, $D^* = V(G_r) \setminus \{w\}$. Consequently, $S = [V(G_k) \setminus \{v\}] \cup [V(G_r) \setminus \{w\}] \cup [\cup_{j\neq r,k} V(G_j)]$, showing that (iii) holds.

For the converse, suppose first that S is of type (i), that is, $S = D \cup [\cup_{j\neq k} V(G_j)]$ for some $k \leq n$ and for some complement-super dominating set D of G_k. Let $v \in V(G) \setminus S$. Then $v \in V(G_k) \setminus D$. Since D is a complement-super dominating set of G_k, there exists $z \in D$ such that $[V(G_k) \setminus N_{G_k}(z)] \cap [V(G_k) \setminus D] = \{v\}$. It follows that $[V(G) \setminus N_G(z)] \cap [V(G) \setminus S] = \{v\}$, showing that S is a complement-super dominating set of G. Clearly, if S is of type (ii), then S is a complement-super dominating set of G. Finally, suppose that $S = [V(G_k) \setminus \{v\}] \cup [V(G_r) \setminus \{w\}] \cup [\bigcup_{j\neq k,r} V(G_j)]$ for some $k, r \leq n$, where $v \in V(G_k)$, $w \in V(G_r)$, and both $[V(G_k) \setminus \{v\}]$ and $[V(G_r) \setminus \{w\}]$ are non-empty sets. Pick $p \in V(G_k) \cap N_{G_k}(v)$ and $q \in V(G_r) \cap N_{G_r}(w)$. Then $[V(G) \setminus N_G(p)] \cap [V(G) \setminus S] = \{w\}$ and $[V(G) \setminus N_G(q)] \cap [V(G) \setminus S] = \{v\}$, showing that S is a complement-super dominating set of G. □

Theorem 10. *Let $G_1, G_2, \cdots, G_k$, where $k \geq 2$, be the components of a graph G of order n. Then each of the following statements holds.*

- (i) *$\gamma_{cs}(G) = n-1$ if and only if $G = \overline{K}_n$ or G has exactly a single non-trivial component G_s and $|V(G_s)| - 1 \leq \gamma_{cs}(G_s) \leq |V(G_s)|$.*
- (ii) *If G has at least two non-trivial components, then*

$$\gamma_{cs}(G) = \min\{n-2, n-\eta_G\},$$

where $\eta_G = \max\{|V(G_j)| - \gamma_{cs}(G_j) : j = 1, 2, \cdots, k\}$.

Proof. (i) Suppose $\gamma_{cs}(G) = n-1$. Suppose $G \neq \overline{K}_n$ and assume that G has two non-trivial components, say G_m and G_r. Pick $v \in V(G_m)$ and $w \in V(G_r)$. Then $S^* = [V(G_k) \setminus \{v\}] \cup [V(G_r) \setminus \{w\}] \cup [\cup_{j\neq k,r} V(G_j)]$ is a complement-super dominating set of G by Theorem 9(iii). Hence, $\gamma_{cs}(G) \leq |S^*| = n-2$, a contradiction. Since $G \neq \overline{K}_n$, G has exactly a single non-trivial component, say G_s. Let D be a γ_{cs}-set of G_s. Then $S = D \cup [\cup_{j\neq k} V(G_j)]$ is a complement-super dominating set of G by Theorem 9(i). Since $\gamma_{cs}(G) = n-1 \leq |S|$, $|V(G_s)| - 1 \leq |D| = \gamma_{cs}(G_s) \leq |V(G_s)|$.

For the converse, suppose first that $G = \overline{K}_n$. Then $\gamma_{cs}(G) = n-1$ by Theorem 8. Next, suppose G has a single non-trivial component G_s and $|V(G_s)| - 1 \leq \gamma_{cs}(G_s) \leq |V(G_s)|$. Since $V(G) \setminus \{x\}$ is a complement-super dominating set of G for each isolated vertex x, it follows that $\gamma_{cs}(G) \leq n-1$. Let S be a γ_{cs}-set of G. If $\gamma_{cs}(G_s) = |V(G_s)|$, then S of type (ii) by Theorem 9. Hence, $\gamma_{cs}(G) = n-1$. If $\gamma_{cs}(G_s) = |V(G_s)| - 1$, then $S = D \cup [\cup_{j \neq s} V(G_j)]$ where D is γ_{cs}-set of G_s or S is of type (ii) by Theorem 9. In either case, $\gamma_{cs}(G) = n-1$.

(ii) Let G_m and G_r be non-trivial components and pick $v \in V(G_m)$ and $w \in V(G_r)$. Then $S' = [V(G_m) \setminus \{v\}] \cup [V(G_r) \setminus \{w\}] \cup [\bigcup_{j \neq k,r} V(G_j)]$ is a complement-super dominating set of G by Theorem 9(iii). It follows that $\gamma_{cs}(G) \leq n-2$. Let S be a γ_{cs}-set of G. Since every complement-super dominating set of type (ii) (see Theorem 9) is of cardinality $n-1$, it follows that S is of type (i) or type (iii). If S is of type (iii), then $\gamma_{cs}(G) = n-2$. If S is of type (i), then there exists a (non-trivial component) G_s such that $S = D_s \cup [\cup_{j \neq s} V(G_j)]$, where D_s is a γ_{cs}-set of G_s. In this case, $\gamma_{cs}(G) = \gamma_{cs}(G_s) + \sum_{j \neq s} |V(G_j)|$. Hence, $\gamma_{cs}(G) + |V(G_s)| - \gamma_{cs}(G_s) = n$, that is, $\gamma_{cs}(G) = n - [|V(G_s)| - \gamma_{cs}(G_s)]$. Clearly, $\eta_G = |V(G_s)| - \gamma_{cs}(G_s)$. Accordingly, $\gamma_{cs}(G) = \min\{n-2, n-\eta_G\}$. □

The next result is a consequence of Theorem 3, Theorem 4, and Theorem 10.

Corollary 4. *Let G and H be any two graphs of orders m and n, respectively. Then*

$$\gamma_h^s(G+H) = \gamma_{cs}(G) + \gamma_{cs}(H).$$

In particular,

- (i) $\gamma_h^s(G+H) = m+n$ *if G and H are complete;*
- (ii) $\gamma_h^s(G+H) = m + \gamma_{cs}(H)$ *if $G = K_m$; and*
- (iii) $\gamma_h^s(G+H) = m+n-2$ *if $G = \overline{K}_m$ and $H = \overline{K}_n$ for $m, n \geq 2$.*

The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ such that $(v,a)(u,b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$.

Note that every non-empty subset C of $V(G) \times V(H)$ can be expressed as $C = \cup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$.

Theorem 11. *Let G and H be any connected non-trivial graphs. Then $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a super hop dominating set of $G[H]$ if and only if the following statements hold.*

- (i) $S = V(G)$.
- (ii) *For each $x \in S$ with $|V(H) \setminus T_x| \geq 2$, the following conditions hold:*
  - (a) *T_x is a complement-super dominating set of H, and*

(b) $T_y = V(H)$ *for all* $y \in N_G^2(x)$.

(iii) *For each* $x \in V(G)$ *with* $|V(H) \setminus T_x| = 1$, *at least one of the following conditions holds:*

(a) T_x *is a complement-super dominating set of* H *and* $T_y = V(H)$ *for all* $y \in N_G^2(x)$.

(b) *There exist* $z \in N_G^2(x)$ *and* $t \in T_z$ *such that* $T_w = V(H)$ *for all* $w \in N_G^2(z) \setminus \{x\}$ *and* $V(H) \setminus T_z \subseteq N_H(t)$, *where* $|V(H) \setminus T_z| \leq 1$.

Proof. Suppose $C = \cup_{x \in S}(\{x\} \times T_x)$ is a super hop dominating set of $G[H]$. Suppose there exists $x \in V(G) \setminus S$ and let $a \in V(H)$. Since H is non-trivial, it follows that $ehpn_{G[H]}((x,a), V(G[H]) \setminus C) = \varnothing$, contrary to our assumption that C is a super hop dominating set. Thus, $S = V(G)$, showing that (i) holds.

Now let $x \in S = V(G)$ with $|V(H) \setminus T_x| \geq 2$ and let $p \in V(H) \setminus T_x$. Since C is a super hop dominating set of $G[H]$, $ehpn_{G[H]}((x,p), V(G[H]) \setminus C) \neq \varnothing$. Let $(y,q) \in ehpn_{G[H]}((x,p), V(G[H]) \setminus C)$. By assumption, $y \notin N_G^2(x)$. Hence, $y = x$ and $q \in T_x \setminus N_H(p)$. Moreover, since $(x,q) \in ehpn_{G[H]}((x,p), V(G[H]) \setminus C)$, $[V(H) \setminus N_H(q)] \cap [V(H) \setminus T_x] = \{p\}$ and $T_y = V(H)$ for all $y \in S \cap N_G^2(x)$, showing that (a) and (b) of (ii) hold.

Finally, let $x \in S$ with $|V(H) \setminus T_x| = 1$ and let $p \in V(H) \setminus T_x$. Suppose that (a) of (iii) does not hold. Since C is a super hop dominating set of $G[H]$ and $(x,p) \in V(G[H]) \setminus C$, let $(z,t) \in ehpn_{G[H]}((x,p), V(G[H]) \setminus C)$. This, together with the assumption, implies that $z \in N_G^2(x)$, $t \in T_z$, $T_w = V(H)$ for all $w \in N_G^2(z) \setminus \{x\}$ and $V(H) \setminus T_z \subseteq N_H(t)$. Suppose $|V(H) \setminus T_z| > 1$, say $c, d \in V(H) \setminus T_z$ where $c \neq d$. Then $ehpn_{G[H]}((z,c), V(G[H]) \setminus C) = \varnothing$, contrary to the assumption that C is a super hop dominating set of $G[H]$. Therefore, $|V(H) \setminus T_z| \leq 1$, showing that (b) of (iii) holds.

For the converse, suppose that $C = \cup_{x \in S}(\{x\} \times T_x)$ satisfies conditions (i), (ii), and (iii). Let $(x,p) \in V(G[H]) \setminus C$. Since $S = V(G)$, $p \notin T_x$. If $|V(H) \setminus T_x| \geq 2$, then by condition (ii), there exists $q \in ehpn_H(p, V(H) \setminus T_x)$. Clearly, $(x,q) \in ehpn_{G[H]}((x,p), V(G[H]) \setminus C)$. Suppose that $|V(H) \setminus T_x| = 1$. Then by (iii), $ehpn_{G[H]}((x,p), V(G[H]) \setminus C) \neq \varnothing$. Accordingly, C is a super hop dominating set of $G[H]$. □

Let G be a graph. A set $S \subseteq V(G)$ is a *hop independent set* of G if $d_G(x,y) \neq 2$ for any two vertices $x, y \in S$. The *hop independence number of* G, denoted by $\alpha_h(G)$, is the largest cardinality of a hop independent set of G. Any hop independent set of G with cardinality $\alpha_h(G)$ is called an α_h-set of G. The concept has been introduced and studied in [4].

Corollary 5. *Let* G *and* H *be non-trivial connected graphs of orders* m *and* n, *respectively. Then* $\gamma_h^s(G[H]) \leq (\gamma_{cs}(H) - n)\alpha_h(G) + mn$. *Moreover, if* $\gamma_{cs}(H) \leq n-2$, *then* $\gamma_h^s(G[H]) = (\gamma_{cs}(H) - n)\alpha_h(G) + mn$.

Proof. Let A be a α_h-set of G and let D be a γ_{cs}-set of H. Set $T_x = D$ for each $x \in A$ and $T_x = V(H)$ for each $x \in V(G) \setminus A$. Then, by Theorem 11, $C_0 = \cup_{x\in V(G)}(\{x\} \times T_x) = (A \times D) \cup [(V(G) \setminus A) \times V(H)]$ is a super hop dominating set of $G[H]$. Hence,

$$\begin{aligned}\gamma_h^s(G[H]) &\leq |C_0| \\ &= |A||D| + (m - |A|)n \\ &= \alpha_h(G)\gamma_{cs}(H) + [m - \alpha_h(G)]n \\ &= (\gamma_{cs}(H) - n)\alpha_h(G) + mn.\end{aligned}$$

Next, let $C = \cup_{x\in V(G)}(\{x\} \times T_x)$ be a γ_h^s-set of $G[H]$. Let $R = \{v \in V(G) : |V(H) \setminus T_v| \geq 2\}$. Since C is a super hop dominating set, R must be a hop independent set of G by Theorem 11(ii). Also, since C is a γ_h^s-set of $G[H]$, T_v is a γ_{cs}-set of H for each $v \in R$. Now let $u \in V(G) \setminus R$ and suppose that $|V(H) \setminus T_u| = 1$. By Theorem 11(ii), $d_G(u, w) \neq 2$ for all $w \in R$ (otherwise, $T_u = V(H)$, a contradiction). Suppose that $T_x = V(H)$ for all $x \in N_G^2(u)$. Replace T_u by a γ_{cs}-set T'_u of H. Then the set $C^* = \cup_{x\in V(G)\setminus\{u\}}(\{x\} \times T_x) \cup (\{u\} \times T'_u)$ is a super hop dominating set of $G[H]$ by Theorem 11. Moreover, $|C^*| < |C|$, contrary to the assumption that C is a γ_h^s-set of $G[H]$. Hence, $T_x \neq V(H)$ for some $x \in N_G^2(u)$. It follows from Theorem 11(iii) that there exists a $z \in N_G^2(u)$ and $t \in T_z$ such that $T_w = V(H)$ for all $w \in N_G^2(z) \setminus \{u\}$ and $V(H) \setminus T_z \subseteq N_H(t)$, where $|V(H) \setminus T_z| \leq 1$. Replace T_u by $T'_u = V(H)$ and T_z by a γ_{cs}-set T'_z of H. Then the set $C_1 = \cup_{x\in V(G)\setminus\{z,u\}}(\{x\} \times T_x) \cup (\{u\} \times T'_u) \cup (\{z\} \times T'_z)$ is a super hop dominating set of $G[H]$ by Theorem 11. Clearly, $|T'_u| = |T_u| + 1$. If $\gamma_{cs}(H) \leq n - 3$, then $|T'_z| < |T_z| - 1$ and a straightforward computation shows that $|C_1| < |C|$, a contradiction. Thus, $T_u = V(H)$ for all $u \in V(G) \setminus R$. Since C is a γ_h^s-set of $G[H]$, it follows that R is a α_h-set of G. This implies that $\gamma_h^s(G[H]) = |C| = (\gamma_{cs}(H) - n)\alpha_h(G) + mn$. Suppose now that $\gamma_{cs}(H) = n - 2$. Suppose further that $T_z = V(H)$. Then $|T'_z| < |T_z| - 1$ and $|C_1| < |C|$, a contradiction. Hence, $|T_z| = n - 1$. It follows that $|T'_z| = |T_z| - 1$ and $|C_1| = |C|$. This means that in the γ_h^s-set C we may assume further that $T_u = V(H)$ for each $u \in V(G) \setminus R$. Again, as C is a γ_h^s-set of $G[H]$, R would be a α_h-set of G, establishing the desired equality. □

We point out that the equality in Corollary 5 does not necessarily hold if $\gamma_{cs}(H) = n-1$, where $n = |V(H)|$. To see this, consider $G = P_4$ and $H = P_3$. Then $\alpha_h(G) = 2$ and $\gamma_{cs}(H) = 2$. It can be verified easily that $\gamma_h^s(G[H]) = 8 < 10 = (2 - 3)(2) + (4)(3) = [\gamma_{cs}(H) - 3]\alpha_h(G) + 12$.

Corollary 6. *Let H be a connected graph and let m be a positive integer. Then*

$$\gamma_h^s(K_m[H]) = \begin{cases}\gamma_h^s(H)\ , & m = 1 \\ m.\gamma_{cs}(H)\ , & m \geq 2.\end{cases}$$

Proof. The result is clear if $m = 1$. Suppose $m \geq 2$ and let let $C = \cup_{x\in V(K_m)}(\{x\} \times T_x)$ be a γ_h^s-set of $K_m[H]$. Then each T_x is a complement-super dominating set of H

by Theorem 11. In particular, T_x is a γ_{cs}-set of H for all $x \in V(K_m)$. Accordingly, $\gamma_h^s(K_m[H]) = |C| = m.\gamma_{cs}(H)$. □

Conclusion: Super hop domination, a variant of hop domination, has been introduced and studied for some graphs and graphs resulting from the join and lexicographic product of two graphs. In the case of the join of graphs, the concept of complement-super domination plays a vital role. Finding the complement-super domination number of a graph is the same as determining the super domination number of the complement of the graph. It is conjectured that the problem of finding a super hop dominating set is not easy, that is, NP-hard (NP-complete). It is recommended that some bounds on the super hop domination be determined and that the parameter be studied for other graphs.

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