Computer Virus Fractional Order Model with Effects of Internal and External Storage Media

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Abstract. In this work, we focus on the implementation of epidemic techniques on computer virus and study the dynamic transmission of several viruses to minimize the destruction of computers. We aim to make and analyze computer viruses through the Atangana-Baleanu sense and the Atangan-Taufik scheme, which is used for the fractional derivative model for the computer virus epidemic. It contained infected external computer effects and removable storage media on the computer viruses. For the validation of the model, we also discussed its positivity and boundedness. Fixed point theory and the iterative methods helped a lot to find out the existence and uniqueness of the model. In the case of numerical simulation, we used Atanagana-Taufik technique to illustrate the effects of varying the fractional order. The graphical results support our theoretical results from which, we analyze the infected external computer effects and removable storage media on the computer viruses.

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Key Words and Phrases: Computer virus, Atangana-Baleanu sense, Atangan-Taufik, Fixed point theory, positivity and boundedness, Existence and uniqueness.

1. Introduction

The malicious and destructive results that can be obtained from program codes are known as computer viruses. They are automated programs that, against the user’s wishes, replicate themselves to spread to new targets and infect computers. A lot of time and effort has gone into researching how to avoid harmful actions. To effectively control the spread of computer viruses, it is critical to understand how malicious codes spread over the Internet. To reduce the threat of virus several techniques can be proposed with the help of epidemic

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models. Computer viruses have different types ranging from host dependent viruses and network worms, which cause a significant threat to our daily life and work [35]. The interconnected networks are the crucial channel of the fast spread of computer viruses. The network security community provides long and continuous attention to the virus diffusion. In Cohen [13] and Murray [26] noticed the analogy between computer viruses, their biological counterparts, and different techniques that investigate the dynamics of computer sciences. The first dynamical model for computer viruses was proposed by Kephart and White [20]. After that, different models for viruses on the computer were proposed [14, 18, 24, 25, 31, 33, 38, 39, 42]. There was no infectivity in the previous models. The original SLBS (susceptible-latent-breaking-susceptible) model for the computer virus was organized by Yang et al [40].

In recent years the fractional calculus has fascinated the attention of researchers and the various features of that study under investigation. Genetic mutations is dominant tool for defining the dynamic function of various body systems. The power of these component operators is their non-local features that are not in the integer separator operator. The real-world problems include characterization of memory and hereditary properties, while fractional order problems include integration and transects differentiation that can be understandable through fractional calculus [6, 10]. Riemann Liouville proposed idea of fractional derivative. Scientist used the latest fractional order derivative in the exponential kernel [7, 17]. The approaches of epidemic model shows different problems of non-singular kernel, which includes trigonometric and exponential functions [15, 16, 34]. The COVID-19 disease conceptual model which effectively catches the proposed outbreak of this virus [4, 37, 41]. The nonlinear fractional differential equation containing Atanagana-Baleanu fractional derivative is solved by Toufik and Atangana [36]. The most repetitive feature of these models is their global (non-local) features that include fractional application are also discuss in [1–3, 5, 29]. Khan and co-authors investigated several fractional models to study the dynamics of the zika virus, pine with disease, covid-19, and gonorrhea with optimal control [9, 21–23]. A new fractional derivative with a nonsingular kernel involving exponential, Mittag-leffler, power functions, and some advanced approaches for epidemic models have been elaborated in [8, 11, 27, 27, 28, 30]. The Lie group method is used in [32] to obtain the Lie symmetry algebra admitted by the time fractional Black-Scholes equation. In [12], authors build a proper extension of the classical prolongation formula of conformable derivative point transformations. This method was demonstrated and used to build a symmetry group admitting conformable ordinary and partial differential equations. A new dynamical model was developed in [18]. The authors thoroughly examined the model, and discovered that the unique (viral) equilibrium is globally asymptotically stable. The writers [19] presented an approximation algorithm that transforms the given system of equations into a nonlinear matrix equation by representing the unknown solutions and their derivatives in matrix forms along with the collocation points. A combination of the idea of quasi-linearization and the Bessel/Legendre-collocation method was applied to the original nonlinear system in addition to the direct Bessel or Legendre-collocation method.

In this work, we apply the Atangana-Baleanu fractional derivative with sumudu transform and Atangan-Tufik scheme with Mittag-Leffler kernel to a non-integer order for the
computer virus model. We also discussed the positivity and boundedness of the fractional order system. The existence and uniqueness of the solutions of the proposed fractional scheme are reputable using fixed-point theory and an iterative method. Lastly, simulation are made to see actual behaviour of this physical phenomena.

2. Preliminaries

**Definition 1.** For any function $\Psi(t)$ over a set, the **Sumudu transform**

$$Z = \left\{ \Psi(t) : \exists \phi, such \ that \phi_1, \phi_2 > 0, |\Psi(t)| < \phi e^{\frac{|t|}{\phi}}, \ if \ t \in (-1)^j \times [0, \infty) \right\}$$

is defined by

$$A(r) = ST[\phi] = \int_0^\infty \exp(-\phi) \Psi(rt)d\phi \quad r \in (-\phi_1, \phi_2)$$

**Definition 2.** The **Atangana Baleanu in Caputo sense** in [18, 25] for function $\Psi(t)$ is given as

$$ABC_{\sigma} D^\sigma_t \Psi(t) = \frac{AB(\sigma)}{m-\sigma} \int_0^t \frac{d^m}{dw^m} h(w) E_\alpha \left\{ -\frac{\sigma(t-w)^\sigma}{m-\sigma} \right\} dw \quad m-1 < \sigma < m$$

we have

$$\left[ ABC_{\sigma} D^\sigma_t \Psi(t) \right](s) = \frac{AB(\sigma) s^\sigma L[\Psi(t)](s) - s^{\sigma-1}\Psi(0)}{s^{\sigma} + \frac{\sigma}{1-\sigma}}$$

The sumudu transform for (1), we have

$$ST \left[ ABC_{\sigma} D^\sigma_t \Psi(t) \right](s) = \frac{B(\sigma)}{1-\sigma} \left\{ \sigma \Gamma(\alpha + 1) E_\sigma \left( -\frac{w^\sigma}{1-\sigma} \right) \right\} \times [ST(\Psi(t)) - \Psi(0)]$$

**Definition 3.** A function $\Psi(t)$ for ABC fractional order $\sigma$ is given by

$$ABC_{\sigma} I^\sigma_t (\Psi(t)) = \frac{(1-\sigma)\Psi(t)}{B-\sigma} + \frac{\alpha}{B(\sigma)\Gamma(\sigma)} \int_0^t \Psi(s)(t-s)^{\sigma-1}ds$$

3. Fractional Order Computer Virus Model

The classical computer virus model given in [41], having four compartments $S(t), L(t), B(t)$ and $R(t)$ which represent the susceptible, latent computers, breaking-out computers and recovered computer after infection with internal and external sources with time $t$ respectively. Besides the subsequent basis assumptions are created as: the exploit rate of each compartment is positive constant $\mu$; the coming into rates of the four compartments are $b_1, b_2, b_3, b_4$ are positive constants respectively; each computer in the S-compartment is diseased with L (or B) computers with a chance of $\beta_1 L$ (or $\beta_2 B$), where $\beta_1, \beta_2$ are favorable times; each computer in S-room is diseased with changeable storage media that may have
We can express the above system in ABC fractional order form as:

\[ \frac{dS}{dt} = b_1 + \gamma_2 L + \gamma_3 B + \eta R - \mu S - \gamma_1 S - \beta_1 LS - \beta_2 BS - \theta S, \]
\[ \frac{dL}{dt} = b_2 + \beta_1 LS + \beta_2 BS + \theta S - \gamma_1 L - \gamma_2 L - \mu L - \alpha L, \]
\[ \frac{dB}{dt} = b_3 + \alpha L - \mu B - \gamma_1 B - \gamma_3 B, \]
\[ \frac{dR}{dt} = b_4 + \gamma_1 S + \gamma_1 L + \gamma_1 B - \eta R - \mu R. \]

We can express the above system in ABC fractional order form as:

\[
\begin{align*}
ABC_{0}D^\alpha_t S(t) &= b_1 + \gamma_2 L + \gamma_3 B + \eta R - \mu S - \gamma_1 S - \beta_1 LS - \beta_2 BS - \theta S, \\
ABC_{0}D^\alpha_t L(t) &= b_2 + \beta_1 LS + \beta_2 BS + \theta S - \gamma_1 L - \gamma_2 L - \mu L - \alpha L, \\
ABC_{0}D^\alpha_t B(t) &= b_3 + \alpha L - \mu B - \gamma_1 B - \gamma_3 B, \\
ABC_{0}D^\alpha_t R(t) &= b_4 + \gamma_1 S + \gamma_1 L + \gamma_1 B - \eta R - \mu R.
\end{align*}
\]

We have the following system of differential equations

\[
\begin{align*}
\frac{dS}{dt} &= b_1 + \gamma_2 L + \gamma_3 B + \eta R - \mu S - \gamma_1 S - \beta_1 LS - \beta_2 BS - \theta S, \\
\frac{dL}{dt} &= b_2 + \beta_1 LS + \beta_2 BS + \theta S - \gamma_1 L - \gamma_2 L - \mu L - \alpha L, \\
\frac{dB}{dt} &= b_3 + \alpha L - \mu B - \gamma_1 B - \gamma_3 B, \\
\frac{dR}{dt} &= b_4 + \gamma_1 S + \gamma_1 L + \gamma_1 B - \eta R - \mu R.
\end{align*}
\]

with given initial conditions

\[ S(0), L(0), B(0), R(0) \geq 0 \]

3.1. Analysis of the Model:

**Theorem 1.** The solution of the proposed system (6) is bounded and unique in \( R^4_+ \) according to initial conditions.

**Proof.** The existence and uniqueness of the solution of system of the equations (6) on the time interval \((0, \infty)\) can be obtained in the region \( R^4_+ \) is positively invariant. From model of system of the equations (6), we find

\[
\begin{align*}
ABC_{0}D^\alpha_t S(t)|_{S=0} &= b_1 + \gamma_2 L + \gamma_3 B + \eta R \geq 0, \\
ABC_{0}D^\alpha_t L(t)|_{L=0} &= b_2 + \beta_2 BS + \theta S \geq 0, \\
ABC_{0}D^\alpha_t B(t)|_{B=0} &= b_3 + \alpha L \geq 0, \\
ABC_{0}D^\alpha_t R(t)|_{R=0} &= b_4 + \gamma_1 L + \gamma_1 B \geq 0.
\end{align*}
\]

If \( \{S(0), L(0), B(0), R(0)\} \in R^4_+ \), then according to above equations, the solution \( \{S(t), L(t), B(t), R(t)\} \) cannot escape from the hyperplanes \( S = 0, L = 0, B = 0 \) and \( R = 0 \). Solution’s lie in the domain \( R^4_+ \) is a positively invariant set.

**Theorem 2.** The region

\[ A = \{ABC_{0}D^\alpha_t S(t), ABC_{0}D^\alpha_t L(t), ABC_{0}D^\alpha_t B(t), ABC_{0}D^\alpha_t R(t) \in R^4_+ \} \]

and

\[ ABC_{0}D^\alpha_t S(t) + ABC_{0}D^\alpha_t L(t) + ABC_{0}D^\alpha_t B(t) + ABC_{0}D^\alpha_t R(t) \leq \frac{\Lambda}{\mu} \]

is a positive invariant set for the system (6).
Proof. For the verification of the results used the system (6), we have

\[ A^{BC}_0 D^\alpha_t N(t) = (b_1 + b_2 + b_3 + b_4) - \mu N(t) \]

let \( \Lambda = (b_1 + b_2 + b_3 + b_4) \), then

\[ A^{BC}_0 D^\alpha_t N(t) = \Lambda - \mu N(t) \]

Using the Laplace transformation, we get

\[ s^\alpha N(t) - s^{\alpha - 1} N(0) = \frac{\Lambda}{s} - \mu N(s) \]

which further gives

\[ N(s) = \frac{s^{-1} \Lambda}{s^\alpha + \mu} - \frac{s^{\alpha - 1} N(0)}{s^\alpha + \mu} \]

From equations (6), we infer that if \((S_0, L_0, B_0, R_0) \in R^4_+\), then

\[ N(t) = \Lambda t^\alpha E_{\alpha, \alpha+1}(-\mu t^\alpha) + E_{\alpha, 1}(-\mu t^\alpha) \]

\[ = \left(\Omega - \delta\right) \left(\frac{\mu t^\alpha E_{\alpha, \alpha+1}(-\mu t^\alpha)}{\mu} + E_{\alpha+1}(-\mu t^\alpha)\right) \leq \frac{\Lambda}{\mu F(1)} \]

\[ \leq \frac{\Lambda}{\mu} \]

and

\[ \leq \frac{(b_1 + b_2 + b_3 + b_4)}{\mu} \]

Hence the solution is bounded in the given domain for sub-compartments of the system. This proved the results.

3.2. Equilibrium Point:

By substituting the left hand side of the system equal to zero, we get equilibrium point of the system. Let \( A^{BC}_0 D^\alpha_t N(t) = A^{BC}_0 D^\alpha_t S(t) + A^{BC}_0 D^\alpha_t L(t) + A^{BC}_0 D^\alpha_t B(t) + A^{BC}_0 D^\alpha_t R(t) \) and \( b = b_1 + b_2 + b_3 + b_4 \). By solving the system (6), we have \( \lim_{t \to \infty} N = N^* \) and \( \lim_{t \to \infty} R = R^* \), where

\[ N^* = \frac{b}{\mu}, \quad R^* = \frac{\mu b_4 + b \gamma_1}{\mu (\gamma_1 + \eta + \mu)} \]

We get

\[ \begin{cases} \dot{L} = b_2 + (\beta_1 L + \beta_3 B + \theta)(N^* - R^* - L - B) - (\gamma_1 + \gamma_2 + \mu + \alpha)L \\ \dot{B} = b_3 + \alpha L - (\mu + \gamma_1 + \gamma_3) B \end{cases} \]

with initial conditions \((L(0), B(0)) \in \zeta\), where

\[ \zeta = \{(L, B) | L \geq 0, B \geq 0, 0 \leq L + B \leq N^* - R^*\} \]

Hence it is invariant.
4. Computer Virus Model with ABC Derivative

In this section, consider the system (6) with ABC derivative and definition of sumudu transform, we get

\[ B(\alpha)\alpha\Gamma(\alpha+1)E_\alpha \left( \frac{-w^\alpha}{1 - \alpha} \right) ST[S(t)-S(0)] = ST \left[ b_1 + \gamma_2 L + \gamma_3 B + \eta R - \mu S - \gamma_1 S - \beta_1 L S - \beta_2 B S - \theta S \right] \]

(9)

\[ B(\alpha)\alpha\Gamma(\alpha+1)E_\alpha \left( \frac{-w^\alpha}{1 - \alpha} \right) ST[L(t)-L(0)] = ST \left[ b_2 + \beta_1 L S + \beta_2 B S + \theta S - \gamma_1 L - \gamma_2 L - \mu L - \alpha L \right] \]

(10)

By reorganizing the system of equations (9)-(12), we have

\[ ST[S(t)] = S(0) + \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha+1)E_\alpha \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST \left[ b_1 + \gamma_2 L + \gamma_3 B + \eta R - \mu S - \gamma_1 S - \beta_1 L S - \beta_2 B S - \theta S \right] \]

(13)

\[ ST[L(t)] = L(0) + \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha+1)E_\alpha \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST \left[ b_2 + \beta_1 L S + \beta_2 B S + \theta S - \gamma_1 L - \gamma_2 L - \mu L - \alpha L \right] \]

(14)

\[ ST[B(t)] = B(0) + \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha+1)E_\alpha \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST \left[ b_3 + \alpha L - \mu B - \gamma_1 B - \gamma_3 B \right] \]

(15)

\[ ST[R(t)] = R(0) + \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha+1)E_\alpha \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST \left[ b_4 + \gamma_1 S + \gamma_1 L + \gamma_1 B - \eta R - \mu R \right] \]

(16)

we get

\[ S(t) = S(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha+1)E_\alpha \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST \left[ b_1 + \gamma_2 L + \gamma_3 B + \eta R - \mu S - \gamma_1 S - \beta_1 L S - \beta_2 B S - \theta S \right] \right\} \]

(17)

\[ L(t) = L(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha+1)E_\alpha \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST \left[ b_2 + \beta_1 L S + \beta_2 B S + \theta S - \gamma_1 L - \gamma_2 L - \mu L - \alpha L \right] \right\} \]

(18)

\[ B(t) = B(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha+1)E_\alpha \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST \left[ b_3 + \alpha L - \mu B - \gamma_1 B - \gamma_3 B \right] \right\} \]

(19)
\[
R(t) = R(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha) \alpha \Gamma(\alpha + 1) E_{\alpha} \left( \frac{w^{\alpha}}{1 - \alpha} \right)} \times ST[b_1 + \gamma_1 S + \gamma_1 L + \gamma_1 B - \eta R - \mu R] \right\}
\]  

Therefore, the following is obtained

\[
S_{(n+1)}(t) = S(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha) \alpha \Gamma(\alpha + 1) E_{\alpha} \left( \frac{w^{\alpha}}{1 - \alpha} \right)} \times ST[b_1 + \gamma_2 L_n + \gamma_3 B_n + \eta R_n - (\mu + \gamma_1 + \theta) S_n - \beta_1 L_n S_n - \beta_2 B_n S_n] \right\}
\]  

\[
L_{(n+1)}(t) = L(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha) \alpha \Gamma(\alpha + 1) E_{\alpha} \left( \frac{w^{\alpha}}{1 - \alpha} \right)} \times ST[b_2 + \beta_1 L_n S_n + \beta_2 B_n S_n + \theta S_n - (\gamma_1 + \gamma_2 + \mu + \alpha) L_n] \right\}
\]  

\[
B_{(n+1)}(t) = B(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha) \alpha \Gamma(\alpha + 1) E_{\alpha} \left( \frac{w^{\alpha}}{1 - \alpha} \right)} \times ST[b_3 + \alpha L_n - (\mu + \gamma_1 + \gamma_3) B_n] \right\}
\]  

\[
R_{(n+1)}(t) = R(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha) \alpha \Gamma(\alpha + 1) E_{\alpha} \left( \frac{w^{\alpha}}{1 - \alpha} \right)} \times ST[b_4 + \gamma_1 S_n + \gamma_1 L_n + \gamma_1 B_n - (\eta + \mu) R_n] \right\}
\]  

And obtained solution of the system of equations (21)-(24) is presented as \( S(t) = \lim_{n \to \infty} S_n(t) \); \( L(t) = \lim_{n \to \infty} L_n(t) \); \( B(t) = \lim_{n \to \infty} B_n(t) \) and \( R(t) = \lim_{n \to \infty} R_n(t) \)

**Theorem 3.** Let \((X, |.|)\) be a Banach space and \(H\) a self-map of \(X\) satisfying

\[
\|Hx - Hr\| \leq \theta \|X - Hx\| + \theta \|x - r\|
\]

\( \forall x, r \in X, \text{ where } 0 \leq \theta < 1. \) Assume that \(H\) is a Picard \(H\)-stable. Let us consider the system of equations (21)-(24) and we obtained

\[
\frac{1 - \alpha}{B(\alpha) \alpha \Gamma(\alpha + 1) E_{\alpha} \left( \frac{w^{\alpha}}{1 - \alpha} \right)}
\]

It is also known as fractional Lagrange multiplier.

**Theorem 4.** Express \(K\) be a self-map is specified by

\[
K[S_{(n+1)}(t)] = S_{(n+1)}(t) = S(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha) \alpha \Gamma(\alpha + 1) E_{\alpha} \left( \frac{w^{\alpha}}{1 - \alpha} \right)} \times ST[b_1 + \gamma_2 L_n + \gamma_3 B_n + \eta R_n - (\mu + \gamma_1 + \theta) S_n - \beta_1 L_n S_n - \beta_2 B_n S_n] \right\}
\]
\[
K[L_{(n+1)}(t)] = L_{(n+1)}(t) = L(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha + 1)E_{\alpha} \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST[b_2 + \beta_1 L_n S_n + \beta_2 B_n S_n + \theta S_n - (\gamma_1 + \gamma_2 + \mu + \alpha)L_n] \right\}
\]

\[
K[B_{(n+1)}(t)] = B_{(n+1)}(t) = B(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha + 1)E_{\alpha} \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST[b_3 + \alpha L_n - (\mu + \gamma_1 + \gamma_3)B_n] \right\}
\]

\[
K[R_{(n+1)}(t)] = R_{(n+1)}(t) = R(0) + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha + 1)E_{\alpha} \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST[b_4 + \gamma_1 S_n + \gamma_1 L_n + \gamma_1 B_n - (\eta + \mu)R_n] \right\}
\]

**Proof.** Considering the norm properties and triangular inequalities, we get

\[
\|K[S_n(t)] - K[S_m(t)]\| \leq \|S_n(t) - S_m(t)\| + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha + 1)E_{\alpha} \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST[b_1 + \gamma_2(L_n(t) - L_m(t)) + \gamma_3(B_n(t) - B_m(t)) + \eta(R_n(t) - R_m(t)) - (\theta + \mu + \gamma_1)(S_n(t) - S_m(t)) - \beta_2(B_n(t) - B_m(t))(S_n(t) - S_m(t))]\right\}
\]

\[
\|K[L_n(t)] - K[L_m(t)]\| \leq \|L_n(t) - L_m(t)\| + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha + 1)E_{\alpha} \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST[b_2 + \beta_1(L_n(t) - L_m(t))(S_n(t) - S_m(t)) + \beta_2(B_n(t) - B_m(t))(S_n(t) - S_m(t)) + \theta(S_n(t) - S_m(t)) - (\gamma_1 + \gamma_2 + \mu + \alpha)(L_n(t) - L_m(t)))]\right\}
\]

\[
\|K[B_n(t)] - K[B_m(t)]\| \leq \|B_n(t) - B_m(t)\| + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha + 1)E_{\alpha} \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST[b_3 + \alpha (L_n(t) - L_m(t)) - (\mu + \gamma_1 + \gamma_3)(B_n(t) - B_m(t)))]\right\}
\]

\[
\|K[R_n(t)] - K[R_m(t)]\| \leq \|R_n(t) - R_m(t)\| + ST^{-1} \left\{ \frac{1 - \alpha}{B(\alpha)\alpha\Gamma(\alpha + 1)E_{\alpha} \left( \frac{w^\alpha}{1 - \alpha} \right)} \times ST[b_4 + \gamma_1(S_n(t) - S_m(t)) + \gamma_1(L_n(t) - L_m(t)) + \gamma_1(B_n(t) - B_m(t)) - (\eta + \mu)(R_n(t) - R_m(t))]\right\}
\]

K fulfills the condition associated with theorem (3) when

\[
\theta = (0, 0, 0, 0)
\]
Theorem 5. Uniqueness of the system (6) solution obtained with iteration method. Concern the following Hilbert space \( H = L^2((p, q) \times (0, r)) \)

\[ h : (p, q) \times [0, T] \to R, \quad \int \int ghdgdh < \infty \]

In this regard, the following operations are considered

\[ \theta = \theta_0 = 0, 0, 0, 0 \]

\[ \theta = b_1 + \gamma_2 L(t) + \gamma_3 B(t) + \eta R(t) - \mu S(t) - \gamma_1 S(t) - \beta_1 L(t) S(t) - \beta_2 B(t) S(t) - \theta S(t) \]

\[ + b_2 + \beta_1 LS + \beta_2 BS + \theta S - \gamma_1 L - \gamma_2 L - \mu L - \alpha L + b_3 + \alpha L - \mu B - \gamma_1 B - \beta_3 B \]

\[ + b_4 + \gamma_1 S + \gamma_1 L + \gamma_1 B - \eta R - \mu R \]

(30)

We establish that the inner product of

\[ T(S_{11}(t) - S_{12}(t), L_{11}(t) - L_{12}(t), B_{11}(t) - B_{12}(t), R_{11}(t) - R_{12}(t), (v_1, v_2, v_3, v_4)) \]

where \( \{S_{11}(t) - S_{12}(t), L_{11}(t) - L_{12}(t), B_{11}(t) - B_{12}(t), R_{11}(t) - R_{12}(t)\} \) are the special solutions of the system.

\[ [b_1 + \gamma_2 (L_{21}(t) - L_{22}) + \gamma_3 (B_{31}(t) - B_{32}(t)) + \eta (R_{41}(t) - R_{42}(t)) - (\theta + \mu + \gamma_1) (S_{11}(t) - S_{12}(t)) \]

\[ - \beta_1 (L_{21}(t) - L_{22}(t)) (S_{11}(t) - S_{12}(t)) - \beta_2 (B_{31}(t) - B_{32}(t)) (S_{11}(t) - S_{12}(t)), v_1] \]

\[ \leq [b_1 + \gamma_2 (L_{21}(t) - L_{22}(t)) ||v_1|| + \gamma_3 (B_{31}(t) - B_{32}(t)) ||v_1|| + \eta ||R_{41}(t) - R_{42}(t)|| ||v_1|| \]

\[ - (\theta + \mu + \gamma_1) ||S_{11}(t) - S_{12}(t)|| ||v_1|| - \beta_1 ||L_{21}(t) - L_{22}(t)|| ||S_{11}(t) - S_{12}(t)|| ||v_1|| \]

\[ - \beta_2 ||B_{31}(t) - B_{32}(t)|| ||S_{11}(t) - S_{12}(t)|| ||v_1|| \]

\[ \leq [b_2 + \beta_1 ||L_{21}(t) - L_{22}(t)|| ||S_{11}(t) - S_{12}(t)|| + \beta_2 (B_{31}(t) - B_{32}(t)) (S_{11}(t) - S_{12}(t)) \]

\[ + \theta (S_{11}(t) - S_{12}(t)) - (\gamma_1 + \gamma_2 + \mu + \alpha) (L_{21}(t) - L_{22}(t)), v_2] \]

\[ \leq [b_2 + \beta_1 ||L_{21}(t) - L_{22}(t)|| ||S_{11}(t) - S_{12}(t)|| ||v_2|| + \beta_2 (B_{31}(t) - B_{32}(t)) ||S_{11}(t) - S_{12}(t)|| ||v_2|| \]

\[ + \theta ||S_{11}(t) - S_{12}(t)|| ||v_2|| - (\gamma_1 + \gamma_2 + \mu + \alpha) ||L_{21}(t) - L_{22}(t)|| ||v_2|| \]

Hence proved its stable according to defied condition in theorem (3)
\[ b_3 + \alpha(L_1(t) - L_2(t)) - (\mu + \gamma_1 + \gamma_3)(B_{31}(t) - B_{32}(t)), v_3 \]
\[ \leq [b_3 + \alpha\|(L_1(t) - L_2(t))\|_v^3 - (\mu + \gamma_1 + \gamma_3)\|(B_{31}(t) - B_{32}(t))\|_v^4] \]
\[ b_4 + \gamma_1(S_{11}(t) - S_{12}(t)) + \gamma_1(L_1(t) - L_2(t)) + \gamma_1(B_{31}(t) - B_{32}(t)) - (\eta + \mu)(R_{41}(t) - R_{42}(t)), v_4 \]
\[ \leq [b_4 + \gamma_1\|(S_{11}(t) - S_{12}(t))\|_v^4 + \gamma_1\|(L_1(t) - L_2(t))\|_v^4 + \gamma_1\|(B_{31}(t) - B_{32}(t))\|_v^4 - (\eta + \mu)\|(R_{41}(t) - R_{42}(t))\|_v^4] \]

Hence it is converge with topological concepts in parameters \((\chi_{e_1}, \chi_{e_2}, \chi_{e_3}, \chi_{e_4})\)

\[ \|S(t) - S_{11}(t)\|, \|S(t) - S_{12}(t)\| \leq \frac{\chi_{e_1}}{w} \]
\[ \|L(t) - L_{21}(t)\|, \|L(t) - L_{22}(t)\| \leq \frac{\chi_{e_2}}{\delta} \]
\[ \|B(t) - B_{31}(t)\|, \|B(t) - B_{32}(t)\| \leq \frac{\chi_{e_3}}{\zeta} \]
\[ \|R(t) - R_{41}(t)\|, \|R(t) - R_{42}(t)\| \leq \frac{\chi_{e_4}}{h} \]

where

\[ w = 4(b_1 + \gamma_2\|L_1(t) - L_2(t)\| + \gamma_3\|B_{31}(t) - B_{32}(t)\| + \eta\|R_{41}(t) - R_{42}(t)\| - (\theta + \mu + \gamma_1)\|S_{11}(t) - S_{22}(t)\| - \beta_1\|L_1(t) - L_2(t)\||\|S_{11}(t) - S_{12}(t)\| - \beta_2\|B_{31}(t) - B_{32}(t)\||\|S_{11}(t) - S_{12}(t)\|)\|v_1\| \]
\[ \delta = 4(b_2 + \beta_1\|(L_1(t) - L_2(t))\|\|\|S_{11}(t) - S_{12}(t)\| + \beta_2\|(B_{31}(t) - B_{32}(t))\|\|\|S_{11}(t) - S_{12}(t)\| + \theta\|(S_{11}(t) - S_{12}(t))\| - (\gamma_1 + \gamma_2 + \mu + \alpha)\|(L_1(t) - L_2(t))\|\|v_2\| \]
\[ \zeta = 4(b_3 + \alpha\|(L_1(t) - L_2(t))\| - (\mu + \gamma_1 + \gamma_3)\|(B_{31}(t) - B_{32}(t))\|\|v_3\| \]
\[ h = 4(b_4 + \gamma_1\|(S_{11}(t) - S_{12}(t))\| + \gamma_1\|(L_1(t) - L_2(t))\| + \gamma_1\|(B_{31}(t) - B_{32}(t))\| - (\eta + \mu)\||(R_{41}(t) - R_{42}(t))\|\|v_4\| \]

where

\[ \leq (b_1 + \gamma_2\|L_1(t) - L_2(t)\| + \gamma_3\|B_{31}(t) - B_{32}(t)\| + \eta\|R_{41}(t) - R_{42}(t)\| - (\theta + \mu + \gamma_1)\|S_{11}(t) - S_{22}(t)\| - \beta_1\|L_1(t) - L_2(t)\||\|S_{11}(t) - S_{12}(t)\| - \beta_2\|B_{31}(t) - B_{32}(t)\||\|S_{11}(t) - S_{12}(t)\| \neq 0 \]
\[ \leq (b_2 + \beta_1\|(L_1(t) - L_2(t))\|\|\|S_{11}(t) - S_{12}(t)\| + \beta_2\|(B_{31}(t) - B_{32}(t))\|\|\|S_{11}(t) - S_{12}(t)\| + \theta\|S_{11}(t) - S_{12}(t)\| - (\gamma_1 + \gamma_2 + \mu + \alpha)\|(L_1(t) - L_2(t))\| \neq 0 \]
\[ \leq (b_3 + \alpha\|(L_1(t) - L_2(t))\| - (\mu + \gamma_1 + \gamma_3)\|(B_{31}(t) - B_{32}(t))\| \neq 0 \]
\[ \leq (b_4 + \gamma_1\|(S_{11}(t) - S_{12}(t))\| + \gamma_1\|(L_1(t) - L_2(t))\| + \gamma_1\|(B_{31}(t) - B_{32}(t))\| - (\eta + \mu)\|(R_{41}(t) - R_{42}(t))\| \neq 0 \]

Where \(\{\|v_1\|, \|v_2\|, \|v_3\|, \|v_4\|\} \neq 0\) \((S_{11}(t) - S_{12}(t)) = 0, (L_1(t) - L_2(t)) = 0, (B_{31}(t) - B_{32}(t)) = 0, (R_{41}(t) - R_{42}(t)) = 0\) so \(S_{11}(t) = S_{12}(t), L_1(t) = L_2(t), B_{31}(t) = B_{32}(t), R_{41}(t) = R_{42}(t)\) This complete proof is uniqueness.
5. Advanced Numerical Scheme

Here in this section considering the numerical scheme is defined in [36], we have

\[
\begin{cases}
ABC_0 Dy(t) = h(t, w(t)) \\
w(0) = w_0
\end{cases}
\]  

(31)

We have

\[
w(t) - w(0) = \frac{1 - \sigma}{ABC(\sigma)} h(t, w(t)) + \frac{\alpha}{\Gamma(\sigma) \times ABC(\sigma)} \int_0^t h(\tau, w(\tau))(t - \tau)^{\sigma - 1} d\tau
\]

(32)

At a given point \(t_{(m+1)}\), \(m = 0, 1, 2, 3, 4, 5, \ldots\), we can write above equation

\[
w(t_{m+1}) - w(0) = \frac{1 - \sigma}{ABC(\sigma)} h(t_m, w(t_m)) + \frac{\alpha}{\Gamma(\sigma) \times ABC(\sigma)} \int_0^{t_{m+1}} h(\tau, w(\tau))(t_{m+1} - \tau)^{\sigma - 1} d\tau
\]

\[
= \frac{1 - \sigma}{ABC(\sigma)} h(t_m, w(t_m)) + \frac{\alpha}{\Gamma(\sigma) \times ABC(\sigma)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} h(\tau, w(\tau))(t_{m+1} - \tau)^{\sigma - 1} d\tau
\]

(33)

(34)

By using interval \([t_k, t_{k+1}]\), the function \(h(\tau, y(\tau))\), with the help of two-steps Lagrange polynomial interpolation, we have

\[
P_k(\tau) = \frac{\tau - t_{k-1}}{t_k - t_{k-1}} h(t_k, w(t_k)) - \frac{\tau - t_k}{t_k - t_{k-1}} h(t_{k-1}, w(t_{k-1}))
\]

\[
= \frac{h(t_k, w(t_k))}{h}(\tau, t_{k-1}) - \frac{h(t_{k-1}, w(t_{k-1}))}{h}(\tau, t_k)
\]

\[
\cong \frac{h(t_k, w(t_k))}{h}(\tau, t_{k-1}) - \frac{h(t_{k-1}, w(t_{k-1}))}{h}(\tau, t_k)
\]

(35)

By using (34), we get

\[
w_{m+1} = w_0 + \frac{1 - \sigma}{ABC(\sigma)} h(t_m, w(t_m)) + \frac{\alpha}{\Gamma(\sigma) \times ABC(\sigma)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{h(t_k, y_k)}{h}(\tau - t_{k-1})(t_{n+1} - \tau)^{\sigma - 1} d\tau - \frac{h(t_{k-1}, y_{k-1})}{h} \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{m+1} - \tau)^{\sigma - 1} d\tau
\]

(36)

For simplification, we will consider

\[
B_{\sigma, k, 1} = \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{m+1} - \tau)^{\sigma - 1} d\tau
\]

(37)

similarly

\[
B_{\sigma, k, 2} = \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{m+1} - \tau)^{\sigma - 1} d\tau
\]

(38)
By putting (37) and (38), we get

\[
B_{\sigma,k,1} = h^{\alpha+1} \frac{(m + 1 - k)^\sigma(m - k + 2 + \sigma) - (m - k)^\sigma(m - k + 2 + 2)}{\alpha(\alpha + 1)} \\
B_{\sigma,k,2} = h^{\alpha+1} \frac{(m + 1 - k)^{\sigma+1} - (m - k)^{\sigma}(m - k + 1 + \sigma)}{\alpha(\alpha + 1)}
\]

By putting (37) and (38), we get

\[
w_{m+1} = w_0 + \frac{1 - \sigma}{ABC(\sigma)} h(t_m, y(t_m)) + \frac{\sigma}{ABC(\sigma)} \sum_{k=0}^{n} \frac{h^{\sigma} h(t_k, y_k)}{\Gamma(\alpha + 2)} ((q_3)^\sigma q_2 - (q_4)^\sigma q_1) \\
- \frac{h^{\sigma} h(t_{k-1}, y_{k-1})}{\Gamma(\alpha + 2)} ((q_3)^{\sigma+1} - (q_4)^\sigma q_3) \tag{39}
\]

We obtain the following for the system of equations (6).

\[
S_{m+1} = S_0 + \frac{1 - \sigma}{ABC(\sigma)} f(t_m, S(t_m)) + \frac{\sigma}{ABC(\sigma)} \sum_{k=0}^{n} \frac{h^{\sigma} f(t_k, S_k)}{\Gamma(\alpha + 2)} ((q_3)^\sigma q_2 - (q_4)^\sigma q_1) \\
- \frac{h^{\sigma} f(t_{k-1}, S_{k-1})}{\Gamma(\alpha + 2)} ((q_3)^{\sigma+1} - (q_4)^\sigma q_3) \tag{40}
\]

\[
L_{m+1} = L_0 + \frac{1 - \sigma}{ABC(\sigma)} f(t_m, L(t_m)) + \frac{\sigma}{ABC(\sigma)} \sum_{k=0}^{n} \frac{h^{\sigma} f(t_k, L_k)}{\Gamma(\alpha + 2)} ((q_3)^\sigma q_2 - (q_4)^\sigma q_1) \\
- \frac{h^{\sigma} f(t_{k-1}, L_{k-1})}{\Gamma(\alpha + 2)} ((q_3)^{\sigma+1} - (q_4)^\sigma q_3) \tag{41}
\]

\[
B_{m+1} = B_0 + \frac{1 - \sigma}{ABC(\sigma)} f(t_m, B(t_m)) + \frac{\sigma}{ABC(\sigma)} \sum_{k=0}^{n} \frac{h^{\sigma} f(t_k, B_k)}{\Gamma(\alpha + 2)} ((q_3)^\sigma q_2 - (q_4)^\sigma q_1) \\
- \frac{h^{\sigma} f(t_{k-1}, B_{k-1})}{\Gamma(\alpha + 2)} ((q_3)^{\sigma+1} - (q_4)^\sigma q_3) \tag{42}
\]

\[
R_{m+1} = R_0 + \frac{1 - \sigma}{ABC(\sigma)} f(t_m, R(t_m)) + \frac{\sigma}{ABC(\sigma)} \sum_{k=0}^{n} \frac{h^{\sigma} f(t_k, R_k)}{\Gamma(\alpha + 2)} ((q_3)^\sigma q_2 - (q_4)^\sigma q_1) \\
- \frac{h^{\sigma} f(t_{k-1}, R_{k-1})}{\Gamma(\alpha + 2)} ((q_3)^{\sigma+1} - (q_4)^\sigma q_3) \tag{43}
\]

Where \( q_1 = m - k + 2 + 2\sigma, q_2 = m - k + 2 + \sigma, q_3 = m + 1 - k, q_4 = m - k \) and \( q_5 = m - k + 1 + \sigma \)

6. Results and Discussion

The mathematical analysis of the epidemic computer virus model with the effect of external and internal storage media is proposed with the new fractional operator with given parameters details in [41]. Numerical simulation carried out with ABC fractional derivative for computer virus model. The graphs of the approximate solutions in different
fractional order are provided in Figures 1-8, which show the memory effect. It can be observed that internal and external storage media are the rich source of the spread of the virus. Also, we found that external storage media has significant effect for the viral infection. From figures (1) and (5), we can see that susceptible computers decreases due to the rapid increase in latent, breaking-out, and recovered computers. In fig (2), the simulation show the dynamics of latent computers increase with time while decreasing with time. In fig (3), we can see that the break out of computers also growing fast and causing rapid and severe damage to computers. In fig (4), the recovery of computers is smoothly increasing due to the effect of fractional operator. From figures 6-8, the simulation shows the quick increase in latent and break out computers with time $t$ and causes the disturbance rapidly and slow the recovery rate of computers.

Figure 1: Simulation of $S(t)$ Compartment with proposed fractional order scheme

Figure 2: Simulation of $L(t)$ Compartment with proposed fractional order scheme
Figure 3: Simulation of B(t) Compartment with proposed fractional order scheme

Figure 4: Simulation of R(t) Compartment with proposed fractional order scheme

Figure 5: Simulation of S(t) Compartment with proposed fractional order scheme
Figure 6: Simulation of L(t) Compartment with proposed fractional order scheme

Figure 7: Simulation of B(t) Compartment with proposed fractional order scheme

Figure 8: Simulation of R(t) Compartment with proposed fractional order scheme
7. Conclusion

In this work, we presented the new result for the fractional order computer virus model with ABC fractional derivative and the Atangana-Toufik scheme. Qualitative analysis with positivity and boundedness of the proposed system was also discussed. Additionally, the uniqueness and stability of the proposed scheme of iterative results are proved by using the fixed point theorem. Finally, numerical simulations at different fractional orders are obtained, which support the theoretical results of the computer virus model. It is observed that internal and external storage media are the cause of the spread of the virus. Also, we found that all external storage media has a greater effect on viral infection with increasing the time t. These results are very helpful to control the virus infection in computers to overcome the threats on results. In future work, we intend to expand the modeling of a computer virus in the stochastic fractional-order derivatives as well as partial differential equation.

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References


