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British call option on stocks under stochastic interest rate

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Abstract. The closed form expression for the price of the British put and call options have long been established where both interest rate and volatility are assumed to be constant. In reality, these assumptions do not fully reflect the variable nature of the financial markets. In this paper, we derived a closed form expression for the arbitrage-free price of the British call option by assuming stochastic interest rate which follows the Cox-Ingersoll-Ross model and constant volatility as

$$V(t, r_t, x) = p(t, r_t; T) + \int_t^T J(t, r_t, x, v, b_D(v, r_v))dv,$$

where the first term is the arbitrage-free price of the European call option under stochastic interest rate and the second term is the early-exercise premmium. We have also shown that the price function of the British call option satisfies the partial differential equation given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X_t^2 \frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}\frac{\partial^2 V}{\partial x \partial r} + \frac{1}{2}\sigma_2^2 r_t \frac{\partial^2 V}{\partial r^2} + \frac{\partial V}{\partial x} r_t X_t + [a\theta - (a + \lambda\sigma)r]\frac{\partial V}{\partial r} - r_t V = 0.$$

Moreover, we have shown that the contract drift satisfies $\mu_c < r_t + \rho\sigma_1\sigma_2\sqrt{r_t}\lambda(0, t+u)$ for $u \in [0, \tau]$ and $t \in [0, T]$.

2020 Mathematics Subject Classifications: 35A01, 35A02, 35C05, 35C15, 35R35, 60H15, 60H35, 60J65

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1. Introduction

Over the years, derivatives have become increasingly important in the global financial market, with great impact on national economics. They are embedded in capital investment opportunities, added to bond issues[5], used as price discovery and price stabilizer [14] and so on. The most common forms of derivatives is an option. An option is defined to be a contract between two parties granting one party the opportunity to buy or sell a security from or to the other party at a specified price also known as strike price on or before a specified maturity date [7]. The two parties involved are called the buyer and seller of the option. The buyer has the right but not the obligation to exercise the option. In order to acquire the option, the buyer should pay the option price, which is also known as premium to the seller [7].

In 2013, G.Peskir and F. Samee introduced a new type of call option called British call options where the holder enjoys the early exercise feature of the American option whereupon his payoff is the best prediction of the European payoff under the hypothesis that the true drift of the stock price equals the contract drift [11]. British option provides its holder with a protection mechanism against unfavorable stock price movements as well as securing higher returns when the movements are favourable. The motivation for the British call option starts from the difference between the paid premium and the expected payoff when the true drift deviates from the risk-free rate. An added feature is built into this instrument which aim at both providing protection against unfavourable price movements as well as securing higher returns when these movements are favourable [11]. Accordingly, the value function of the British call option is similar to American call option but they differ on their respective boundary functions.

The closed form expression for the price of the British call option has been derived by Peskir and Samee (2013) by assuming constant interest rate and constant volatility. However, the assumptions of constant interest rate and constant volatility fail to reflect the fact that these market rates are stochastic in the real world. This paper extends the results in [11] to address the mentioned shortcomings by considering British call option under stochastic interest rate and constant volatility. In particular, it will be assumed that the short rate follows the Cox-Ingersoll-Ross (CIR) model. Furthermore, this paper focuses on the theoretical framework of British option pricing. Actual implementation through simulation and numerical approximation will not be included.

2. Setting of the Problem

Let us consider the financial market consisting of a risky stock with price process $X = (X_t : t \in [0, T])$ and a zero-coupon bond with price process $P = (P_t : t \in [0, T])$ where the prices respectively evolve as

$$dX_t = \mu X_t dt + \sigma_1 X_t dW_t, \tag{1}$$

$$dP_t = r_t P_t dt - \lambda(t, T)\sigma_2 P_t d\bar{W}_t, \tag{2}$$

where $\mu \in \mathbb{R}$ is the appreciation rate, σ_1 and σ_2 are volatility coefficients, $W = (W_t)_{t\geq 0}$ and $\bar{W} = (\bar{W}_t)_{t\geq 0}$ are standard Wiener processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$cov(dW_t \cdot d\bar{W}_t) = \rho dt \text{ , } (|\rho| < 1), \text{ (see [1])}$$

r_t follows the Cox-Ingersoll-Ross (CIR) model

$$dr_t = a(\theta - r_t)dt + \sigma_2 \sqrt{r_t} d\bar{W}_t \tag{3}$$

where a, θ, σ_2 are positive constants and the market price of risk is given by $\lambda(t,T) = \lambda\sqrt{r_t}$. In this model, the standard deviation of the stochastic term $\sigma_2\sqrt{r_t}d\bar{W}_t$ is proportional to the square root of the interest rate, that is, as the rate increases, the standard deviation also increases and as the interest rate approaches zero, $\sigma_2\sqrt{r_t}d\bar{W}_t$ also approaches zero. Moreover this model is a mean reverting process, that is, when $r_t > \theta$ the drift is negative and when $r_t < \theta$, the drift is positive where a is the speed of the mean reversion and θ is the equilibrium level. The deterministic part of the solution of (3) is given by

$$r_t = \theta + (r_0 - \theta)e^{-at}. \tag{4}$$

Note that the price $P_t = P(t, r; T)$ of a zero coupon bond satisfies the partial differential equation (see [4])

$$\frac{\partial P}{\partial t} + [a\theta - (a + \lambda\sigma)r]\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_2^2 r \frac{\partial^2 P}{\partial r^2} - rP = 0 \tag{5}$$

such that $P(T, r; T) = 1$ for all $r \in \mathbb{R}$ and $t \in [0, T]$. The price of a zero-coupon bond at time t with maturity time T using the risk neutral valuation framework is given by

$$P(t, r_t, T) = E[e^{-\int_t^T r_u du}|\mathcal{F}_t], \qquad \text{(see [8])} \tag{6}$$

where $\mathcal{F}_t$ denotes the natural filtration generated by the price process X. The natural filtration $\mathcal{F}_t$ represents the information generated by the process X as time progresses.

Note that the interest rate r_u is a Markovian process (see [8]). This implies that, r_u is dependent on r_t for $u > t$, i.e., r_u is a function of r_t. We then have,

$$\begin{aligned} P(t, r_t, T) &= E[e^{-\int_t^T r_u du}|\mathcal{F}_t]. \\ &= E[e^{-\int_t^T r_u du}|r_t] \end{aligned}$$

where E is taken with respect to the probability measure $\mathbb{P}$. The solution to the partial differential equation (5) is given by

$$P(t, r, T) = e^{-\lambda(t,T)r - A(t,T)} \qquad \text{see [4]} \tag{7}$$

where

$$A(t,T) = \frac{2a\theta}{\Gamma\sigma^2}\left[-\Gamma\left(\frac{T-t}{h_1}\right) - \frac{h_1 - h_2}{h_1 h_2}\ln\left(\frac{h_1 - h_2 e^{\Gamma(T-t)}}{h_1 - h_2}\right)\right] \tag{8}$$

$$\lambda(t,T) = \frac{2}{\sigma^2} \frac{\left[1 - e^{\Gamma(T-t)}\right]}{\left[h_1 e^{-\Gamma(T-t)} - h_2\right]} \tag{9}$$

$$\Gamma = \sqrt{(a+\gamma)^2 + 2\sigma^2} \tag{10}$$

$$h_1 = \frac{-a+\gamma}{\sigma^2} + \frac{\Gamma}{\sigma^2} \tag{11}$$

$$h_2 = \frac{-a+\gamma}{\sigma^2} - \frac{\Gamma}{\sigma^2} \tag{12}$$

Let us now consider the British call option on stock given the financial market described above. Moreover, we assume that the stock does not pay dividends and there are no transaction costs involved in its trade. In 2013, Peskir and Samee defined the british call option with strike price $K > 0$ and maturity time $T > 0$ in years as follows:

Definition [11] The British call option is a financial contract between a seller/hedger and a buyer/holder entitling the latter to exercise at any (stopping) time τ prior to T whereupon his payoff (deliverable immediately) is the 'best prediction' of the European payoff $(X_T - K)^+$ given all the information up to time τ under the hypothesis that the true drift of the stock price equals the contract drift μ_c.

In [11], the price of the British call option is derived under the hypothesis that the risk-free rate is constant, that is, $r_t = r$ for all $t \in [0,T]$. Hence, this paper presents an extension of the results in Peskir and Samee (2013) by considering stochastic interest rate.

Let $(\Omega, \mathcal{F}, P)$ denote the probability space for which all the succeeding processes are defined. Let X_t denote the price of a risky stock at time $t \in [0,T]$ which satisfies $dX_t = \mu X_t dt + \sigma_1 X_t dW_t$. We note that the coefficients depend on the maturity time T as well as the current time t. Let $(\mathcal{F})_{0 \le t \le T}$ denote the natural filtration generated by the process X_t. All local martingales involved are with respect to the filtration $\mathcal{F}_t$. Let $\mu_c > 0$ be the risk-free rate and μ be the expected rate of return (appreciation rate) such that $\mu_c \neq \mu$.

Let

$$Z_t := \exp\left[\int_0^t \beta(u) dW_u - \frac{1}{2}\int_0^t \beta^2(u) du\right]$$

for $0 \le u \le t \le T$ where

$$\beta(t) := \frac{\mu_c - \mu}{\sigma_1}$$

is independent of T. By Itö's formula [2], we have

$$\frac{dZ_t}{Z_t} = \beta(t) dW_t.$$

This shows that Z_t is a local martingale with $E\left[Z_t\right] = 1$ for $0 \le t \le T$.

Moreover, define the Radon-Nikodym derivative of $\mathbb{P}^{\mu_c}$ with respect to $\mathbb{P}$ via the following:

$$\frac{d\mathbb{P}^{\mu_c}}{d\mathbb{P}} = Z_T. \tag{13}$$

By the generalized Girsavanov's theorem (see Lemma 1 in [15]), we have

$$W^{\mu_c} := W_t - \int_0^t \beta(u)du$$

is a $\mathbb{P}^{\mu_c}$-Brownian motion. Let E^{μ_c} be an expectation with respect to the measure $\mathbb{P}^{\mu_c}$. Under the probability $\mathbb{P}^{\mu_c}$, the stock price process in equation (1) becomes

$$dX_t = \mu_c X_t dt + \sigma_1 X_t dW_t^{\mu_c} \tag{14}$$

where $0 \leq t \leq T$ with $X_0 = x \in (0, \infty)$. Thus,

$$Law(X(\mu)|\mathbb{P}) = Law(X(\mu_c)|\mathbb{P}^{\mu_c})$$

where Law means "distribution". Furthermore, making use of the property of Z_T, we have

$$E^{\mu_c}(X) = E(Z_T X) = E(Z_T)E(X) = E(X)$$

for any random variable X.

Thus, the payoff of the British call option at a given stopping time $t = \tau$ can be written as

$$E^{\mu_c}[(X_T - K)^+|\mathcal{F}_\tau] \tag{15}$$

where the conditional expectation is taken with respect to a new (equivalent) probability measure $\mathbb{P}^{\mu_c}$. Clearly, when we exercise the British call option, we just substitute the contract drift μ_c to the true (unknown) drift μ of the stock price for the remaining term of the contract.

3. Payoff, Premium and the Price Process of the British Call Option

Using the properties of the Wiener process W^{μ_c} on X (stationary and independent increments of W^{μ_c} on X), implies that

$$E^{\mu_c}[(X_T - K)^+|\mathcal{F}_t] = G^{\mu_c}(t, X_t) \tag{16}$$

where G^{μ_c} is the payoff function given by

$$G^{\mu_c}(t, x) = E(xZ_{T-t}^{\mu_c} - K)^+ \qquad \text{(see [11])} \tag{17}$$

for $t \in [0, T]$ and $x > 0$ where $G^{\mu_c}(t, x)$ represents the value of the investment and $E(xZ_{T-t}^{\mu_c} - K)^+$ represents the expected value of its payoff, and $Z_{T-t}^{\mu_c}$ is given by

$$Z_{T-t}^{\mu_c} = exp\left[(\mu_c - \frac{\sigma_1^2}{2})(T - t) + \sigma_1 W_{T-t}\right] \tag{18}$$

for $t \in [0, T]$ and $x \in (0, \infty)$. It can be verified that (17) can be expressed as follows

$$G^{\mu_c}(t,x) = xe^{\mu_c(T-t)}\Phi(d_1) - K\Phi(d_2)$$

where

$$d_1 = \frac{\ln\frac{x}{K} + (\mu_c + \frac{1}{2}\sigma_1^2)(T-t)}{\sigma_1\sqrt{T-t}}, \quad d_2 = d_1 - \sigma_1\sqrt{T-t}$$

for $t \in [0,T]$, $x > 0$ and $\Phi(\cdot)$ is the standard normal cumulative distribution function.

By applying the standard hegding scheme based on self-financing portfolios (with consumption), the arbitrage-free price of the British call option at deal time (time 0) is given by

$$V = \sup_{0\le\tau\le T} \tilde{E}\left[e^{-\int_0^\tau r_u du} E^{\mu_c}\left[(X_T - K)^+ | \mathcal{F}_\tau\right]\right], \tag{19}$$

where the supremum is taken over all stopping time $\tau \in [0,T]$ of X and $\tilde{E}$ is taken with respect to the (unique) equivalent martingale measure $\tilde{\mathbb{P}}$ under which the stock price process evolve as

$$dX_s = r_s X_s ds + \sigma_1 X_s dW_s^{\tilde{\mathbb{P}}}, \tag{20}$$

$s \ge t$, with $X_t = x > 0$ and $W_t^{\tilde{\mathbb{P}}}$ is a Brownian motion under $\tilde{\mathbb{P}}$. Since the price is given by

$$X_s = xe^{\left(\int_t^s r_u du - \frac{1}{2}\sigma_1^2(s-t) + \sigma_1 W_{s-t}^{\tilde{\mathbb{P}}}\right)},$$

then the discounted price given by

$$e^{\left(\int_t^s r_u du\right)} X_s = xe^{\left(-\frac{1}{2}\sigma_1^2(s-t) + \sigma_1 W_{s-t}^{\tilde{\mathbb{P}}}\right)}$$

is a martingale. Note that in (19), $E\left[e^{-\int_0^\tau r_u du}\right]$ is the discounting factor which brings the payoff $[E^{\mu_c}(X_T - K)^+ | \mathcal{F}_\tau]$ from exercise date τ to time 0.

Now, we fix $t \in [0,T]$. We want to solve for a general expression for the price of the British call option at any time t with stock price $X_t = x$ and short rate $r_t = r$. We denote this by $V(t,r,x)$. Extending the argument in (19), if the exercise date is at any time $t \in [0,T]$, then using (16) and the optimal sampling theorem, we have

$$V(t,r,X_t) = \sup_{0\le\tau\le T-t} \tilde{E}_{t,x}\left[e^{-\int_t^{t+\tau} r_u du} G^{\mu_c}(t+\tau, X_{t+\tau}) | \mathcal{F}_\tau\right], \tag{21}$$

where the supremum is taken over all stopping times $\tau \in [0, T-t]$ of X and $\tilde{E}_{t,x}$ is taken with respect to the (unique) equivalent martingale measure $\tilde{\mathbb{P}}_{t,x}$ under which $X_t = x$ and $r_t \in \mathbb{R}$. Note that $V(0, r_0, x) = V$. Since the supremum in (21) is attained in the first entry time of X to the closed set where $V = G^{\mu_c}$ and $\text{Law}(X(\mu)|\mathbb{P})$ is the same as $\text{Law}(X(r_t)|\tilde{\mathbb{P}})$, then from the well-known structure of the geometric Brownian motion X,

$$V(t,r,x) = \sup_{0\le\tau\le T-t} E\left[e^{-\int_t^{t+\tau} r_u du} G^{\mu_c}(t+\tau, xX_\tau) | X_t = x, r_t = r\right], \tag{22}$$

where the process $X = (X_t(t, r_t))_{t\in[0,T]}$ under $\mathbb{P}$ evolves as

$$dX_t = r_t X_t dt + \sigma_1 X_t dW_t \quad (X_0 = 1). \tag{23}$$

Moreover, the British call option at maturity time T is given by

$$V(T, r_T, X_T) = G^{\mu_c}(T, X_T) = E(X_T - K)^+. \tag{24}$$

This implies that the price of the British call option at maturity time T coincides with the payoff of the European call option whose stock dynamics follows the stochastic differential equation (20) above. Let $G^{r_t}(t, x) = E(X_T - K)^+$, where X follows (20) with $X_t = x$. Thus

$$V(T, r_T, X_T) = G^{r_t}(t, x).$$

Note that when the interest rate is constant ($r_t = r$ for all $t \in [0, T]$), i.e., the coefficients a and σ_2 in equation (3) are all equal to zero, then the expression for $G^{r_t}(t, x)$ multiplied by $e^{-r(T-t)}$ coincides with the Black-Scholes formula for the arbitrage-free price of the European call option at time t with maturity time T.

We can directly see from (17) that $x \mapsto G^{\mu_c}(t, x)$ is convex. To clearly verify this, we present the proposition below.

Proposition 1. *For any $t \in [0, T]$ given and fixed, the mapping*

$$x \mapsto G^{\mu_c}(t, x) \tag{25}$$

is convex on $(0, \infty)$.

Proof. Let $t \in [0, T]$ be given and fixed. Moreover, let $0 \le \lambda \le 1$ and $x_2 = \lambda x_1 + (1 - \lambda)x_3$ for some $x_1, x_2, x_3 \in (0, \infty)$ with $x_1 < x_3$. Then, we have

$$\begin{aligned}
G^{\mu_c}(t, x_2) &= G^{\mu_c}(t\ ,\ \lambda x_1\ + (1-\lambda)x_3) \\
&= E^{\mu_c}\left[\left(\lambda x_1 Z^{\mu_c}_{t,T} + (1-\lambda)x_3 Z^{\mu_c}_{t,T} - K\right)^+ \mid X_t = x_2\right] \\
&= E^{\mu_c}\left[\left(\lambda x_1 Z^{\mu_c}_{t,T} + (1-\lambda)x_3 Z^{\mu_c}_{t,T} - (\lambda K + (1-\lambda)K)\right)^+ \mid X_t = x_2\right] \\
&= E^{\mu_c}\left[\left(\lambda x_1 Z^{\mu_c}_{t,T} + (1-\lambda)x_3 Z^{\mu_c}_{t,T} - \lambda K - (1-\lambda)K\right)^+ \mid X_t = x_2\right] \\
&= E^{\mu_c}\left[\left(\lambda x_1 Z^{\mu_c}_{t,T} - \lambda K + (1-\lambda)x_3 Z^{\mu_c}_{t,T} - (1-\lambda)K\right)^+ \mid X_t = x_2\right] \\
&= E^{\mu_c}\left[\left((\lambda x_1 Z^{\mu_c}_{t,T} - \lambda K) + ((1-\lambda)x_3 Z^{\mu_c}_{t,T} - (1-\lambda)K)\right)^+ |X_t = x_2\right] \\
&\le \lambda E^{\mu_c}\left[\left(x_1 Z^{\mu_c}_{t,T} K\right)^+ |X_t = x_1\right] + (1-\lambda)E^{\mu_c}\left[\left(x_3 Z^{\mu_c}_{t,T} - K\right)^+ |X_t = x_3\right] \\
&= \lambda G^{\mu_c}(t, x_1) + (1-\lambda)G^{\mu_c}(t, x_3).
\end{aligned}$$

Therefore, the mapping $x \mapsto G^{\mu_c}(t, x)$ is convex on $(0, \infty)$. $\square$

It can also be verified that the mapping in (25) is strictly increasing on $(0, \infty)$ with $G^{\mu_c}(t, 0) = 0$ and $G^{\mu_c}(t, \infty) = \infty$ for any $t \in [0, T]$ given and fixed. Define the set

$$D := \left\{ (t, r_t, X_t) \in [0, T] \times \mathbb{R} \times (0, \infty) | V(t, r_t, X_t) = G^{\mu_c(t, X_t)} \right\}. \tag{26}$$

By (20), we say that $\{T\} \times \{r_T\} \times (0, \infty) \subset D$, which is consistent with the fact that the supremum in (17) is taken over $(\mathcal{F})_{t\in[0,T]}$ - stopping times $\tau \in [t, T]$. Furthermore, by a Corollary in [12], the $(\mathcal{F})_{t\in[0,T]}$ - stopping time defined by

$$\tau_D(t, r_t, x) := inf \left\{ s \in [0, T - t] : (t + s, r_{t+s}, X_{t+s}) \in D \right\}, \tag{27}$$

with $X_{t+s} = x \in (0, \infty)$ and $r_{t+s} \in \mathbb{R}$, is an optimal stopping time for the option price in (17) since both $x \mapsto V(t, r_t, x)$ and $x \mapsto G^{\mu_c}(t, x)$ are continuous on $(0, \infty)$ and $G^{\mu_c}(t, x) \leq K$ for all $t \in [0, T]$ and $r_t \in \mathbb{R}$. Consequently, we call the set D as stopping set. Thus, we can define the continuation set $C = D^c$ as

$$C = \left\{ (t, r_t, x) \in [0, T] \times \mathbb{R} \times (0, \infty) | V(t, r_t, x) > G^{\mu_c}(t, x) \right\}. \tag{28}$$

4. Stopping Set and Boundary Function

In order to deal with the existence of an optimal stopping time for (26) above, we define

$$F(t, r_t, x) = V(t, r_t, x) - G^{\mu_c}(t, x) \geq 0,$$

which is nonnegative for $t \in [0, T], r_t \in \mathbb{R}$ and $x \in (0, \infty)$, so that we define the stopping set

$$D = \{(t, r_t, x) \in [0, T] \times \mathbb{R} \times (0, \infty) | F(t, r_t, x) = 0\}.$$

By Lemmas **2** and **3**, we say that the set D is closed. Thus, the continuation set defined by

$$C := D^c = \{(t, r_t, x) \in [0, T] \times \mathbb{R} \times (0, \infty) | F(t, r_t, x) > 0\}$$

is open. Let $(T, r_t, x) \in \{T\} \times \{r_T\} \times (0, \infty) \subset D$, which is consistent with the fact that the supremum in (21) is taken over $(\mathcal{F})_{t\in[0,T]}$ - stopping times $\tau \in [t, T]$. Furthermore, by a Corollary in [12], the $(\mathcal{F})_{t\in[0,T]}$ - stopping time defined by

$$\tau_D(t, r_t, x) := inf \left\{ s \in [0, T - t] : (t + s, r_{t+s}, X_{t+s}) \in D \right\} \tag{29}$$

with $X_{t+s} = x \in (0, \infty)$ and $r_{t+s} \in \mathbb{R}$, is an optimal stopping time for the option price in (21) since both $x \mapsto V(t, r_t, x)$ and $x \mapsto G^{\mu_c}(t, x)$ are continuous on $(0, \infty)$ and $G^{\mu_c}(t, x) \leq K$ for all $t \in [0, T]$ and $r_t \in \mathbb{R}$.

Lemma 1. *For any $(t, r_t, x) \in D$, we have*

$$\limsup_{\epsilon \searrow 0} \frac{F(t,\ r_t,\ x + \epsilon) - F(t,\ r_t,\ x)}{\epsilon} \leq 0.$$

Proof. For all $x \in (0, \infty)$ and $\epsilon > 0$, consider the $(F_s)_{s\in[t,T]}$-stopping time

$$\tau_\epsilon^+ = \tau_D(t,\ r_t,\ x+\epsilon) \in [0, T-t] \tag{30}$$

defined by $\tau_D(t,\ r_t,\ x) := \inf\{s \in [0, T-t] : (t+s,\ r_{t+s},\ X_{t+s}) \in D\}$. Note that $\tau_D(t,\ r_t,\ x)$ solves the optimal stopping problem given by

$$\begin{aligned} V(t, r_t, x+\epsilon) &= \sup_{0\le\tau\le T-t} \tilde{E}\left[e^{-\int_t^{t+\tau} r_u du} G^{\mu_c}(t+\tau, X_{t+\tau})\right] \\ &= \tilde{E}\left[e^{-\int_t^{t+\tau} r_u du} G^{\mu_c}(t+\tau_\epsilon^+, X_{t+\tau_\epsilon^+})\right] \end{aligned} \tag{31}$$

Claim: $\tau_\epsilon^+ \to 0$ as $\epsilon \to 0$. From the definition $\tau_D(t, r_t, x)$ of $\tau_D(t, r_t, x+\epsilon)$,

$$\begin{aligned} &\tau_D(t, r_t, x+\epsilon) \\ &\quad = \inf\{s \in [0, T-t] : (t+s, r_{t+s}, X_{t+s}) \in D\} \\ &\quad = \inf\Bigg\{ s \in [0, T-t] : \sup_{0\le\tau\le T-s} \tilde{E}\Bigg[e^{-\int_t^{t+\tau} r_u du} E^{\mu_c}\Bigg[\Big((x+\epsilon)x_T Z^{\mu_c}_{T-(t+\tau)} - K\Big)^+ \\ &\qquad\qquad |\mathcal{F}_{t+\tau}\Bigg]|\mathcal{F}_r\Bigg] = E^{\mu_c}\Bigg[\Big((x+\epsilon)x_T Z^{\mu_c}_{T-(t+\tau)} - K\Big)^+ |\mathcal{F}_{t+\tau}\Bigg]\Bigg\} \\ &\quad \le \inf\Bigg\{ s \in [0, T-t] : \sup_{0\le\tau\le T-s} \tilde{E}\Bigg[e^{-\int_t^{t+\tau} r_u du} E^{\mu_c}\Bigg[\Big(x x_T Z^{\mu_c}_{T-(t+\tau)} - K\Big)^+ \\ &\qquad\qquad |\mathcal{F}_{t+\tau}\Bigg]|\mathcal{F}_r\Bigg] \ge E^{\mu_c}\Bigg[\Big((x+\epsilon)x_T Z^{\mu_c}_{T-(t+\tau)} - K\Big)^+ |\mathcal{F}_{t+\tau}\Bigg]\Bigg\} \\ &\le \inf\Bigg\{ s \in [0, T-t] : \sup_{0\le\tau\le T-s} \tilde{E}\Bigg[e^{-\int_t^{t+\tau} r_u du} E^{\mu_c}\Bigg[\Big((x+\epsilon)x_T Z^{\mu_c}_{T-(t+\tau)} - K\Big)^+ \\ &\qquad\qquad |\mathcal{F}_{t+\tau}\Bigg]|\mathcal{F}_r\Bigg] \ge E^{\mu_c}\Bigg[\frac{1}{2}\Big(x Z^{\mu_c}_{T-(t+\tau)} + \epsilon Z^{\mu_c}_{T-(t+\tau)} - K \\ &\qquad\qquad + |x Z^{\mu_c}_{T-(t+\tau)} + \epsilon Z^{\mu_c}_{T-(t+\tau)} - K|\Big)^+ |\mathcal{F}_{t+\tau}\Bigg]\Bigg\}. \end{aligned}$$

This implies that

$$\begin{aligned} &\lim_{\epsilon\to 0} \tau_D(t, r_t, x+\epsilon) \\ &\quad \le \lim_{\epsilon\to 0} \inf\Bigg\{ s \in [0, T-t] : \sup_{0\le\tau\le T-s} \tilde{E}\Bigg[e^{-\int_t^{t+\tau} r_u du} E^{\mu_c}\Bigg[\Big((x+\epsilon)x_T Z^{\mu_c}_{T-(t+\tau)} - K\Big)^+ \\ &\qquad\qquad |\mathcal{F}_{t+\tau}\Bigg]|\mathcal{F}_r\Bigg] \ge E^{\mu_c}\Bigg[\frac{1}{2}\Big(x Z^{\mu_c}_{T-(t+\tau)} + \epsilon Z^{\mu_c}_{T-(t+\tau)} - K \end{aligned}$$

$$
\begin{aligned}
&+|xZ^{\mu_c}_{T-(t+\tau)}+\epsilon Z^{\mu_c}_{T-(t+\tau)}-K|\Bigg)^{+}|\mathcal{F}_{t+\tau}\Bigg]\Bigg\}\\
&=\inf\Bigg\{s\in[0,T-t]:\sup_{0\le\tau\le T-s}\tilde{E}\Bigg[e^{-\int_t^{t+\tau}r_u du}E^{\mu_c}\Bigg[\Bigg(xx_T Z^{\mu_c}_{T-(t+\tau)}-K\Bigg)^{+}\\
&\quad|\mathcal{F}_{t+\tau}\Bigg]|\mathcal{F}_r\Bigg]\ge E^{\mu_c}\Bigg[\frac{1}{2}\Bigg(xZ^{\mu_c}_{T-(t+\tau)}-K+|xZ^{\mu_c}_{T-(t+\tau)}-K|\Bigg)^{+}|\mathcal{F}_{t+\tau}\Bigg]\Bigg\}\\
&=\inf\Bigg\{s\in[0,T-t]:\sup_{0\le\tau\le T-s}\tilde{E}\Bigg[e^{-\int_t^{t+\tau}r_u du}E^{\mu_c}\Bigg[\Bigg(xx_T Z^{\mu_c}_{T-(t+\tau)}-K\Bigg)^{+}\\
&\quad|\mathcal{F}_{t+\tau}\Bigg]|\mathcal{F}_r\Bigg]\ge E^{\mu_c}\Bigg[\frac{1}{2}\Bigg((xZ^{\mu_c}_{T-(t+\tau)}-K)^{+}\Bigg]\Bigg\}\\
&=\inf\Bigg\{s\in[0,T-t]:\sup_{0\le\tau\le T-s}\tilde{E}\Bigg[e^{-\int_t^{t+\tau}r_u du}E^{\mu_c}\Bigg[\Bigg(xx_T Z^{\mu_c}_{T-(t+\tau)}-K\Bigg)^{+}\\
&=\inf\Bigg\{s\in[0,T-t]:\sup_{0\le\tau\le T-s}\tilde{E}\Bigg[e^{-\int_t^{t+\tau}r_u du}E^{\mu_c}\Bigg[\Bigg(xx_T Z^{\mu_c}_{T-(t+\tau)}-K\Bigg)^{+}\\
&\quad|\mathcal{F}_{t+\tau}\Bigg]|\mathcal{F}_r\Bigg]= E^{\mu_c}\Bigg[\frac{1}{2}\Bigg((xZ^{\mu_c}_{T-(t+\tau)}-K)^{+}\Bigg]\Bigg\}\\
&=\inf\Bigg\{s\in[0,T-t]:(t,r_t,x)\in D\Bigg\}.
\end{aligned}
$$

Now to show that $\limsup\limits_{\epsilon\searrow 0}\frac{F(t,\ r_t,\ x+\epsilon)-F(t,\ r_t,\ x)}{\epsilon}\le 0$, we use the optimal stopping problem. Hence, we have

$$
\begin{aligned}
\limsup_{\epsilon\searrow 0}\frac{V(t,\ r_t,\ x+\epsilon)-V(t,\ r_t,\ x)}{\epsilon}&=\limsup_{\epsilon\searrow 0}\frac{1}{\epsilon}\Bigg\{\tilde{E}\Bigg[e^{-\int_t^{t+\tau}r_u du}G^{\mu_c}(t+\tau_\epsilon^{+},x+\epsilon)|\mathcal{F}_t\Bigg]\\
&\quad-\sup_{0\le\tau\le T-t}\tilde{E}\Bigg[e^{-\int_t^{t+\tau}r_u du}G^{\mu_c}(t+\tau,x)\Bigg]\Bigg\}\\
&\le\limsup_{\epsilon\searrow 0}\frac{1}{\epsilon}\Bigg\{\tilde{E}\Bigg[e^{-\int_t^{t+\tau}r_u du}G^{\mu_c}(t+\tau_\epsilon^{+},x+\epsilon)|\mathcal{F}_t\Bigg]\\
&\quad-\tilde{E}\Bigg[e^{-\int_t^{t+\tau}r_u du}G^{\mu_c}(t+\tau,x)\Bigg]\Bigg\}\\
&\le\limsup_{\epsilon\searrow 0}\frac{1}{\epsilon}\Bigg\{G^{\mu_c}(t+\tau_\epsilon^{+},x+\epsilon)-G^{\mu_c}(t+\tau_\epsilon^{+},x)\Bigg\}
\end{aligned}
$$

$$= \frac{\partial G^{\mu_c}}{\partial x}(t,x).$$

Therefore, $\limsup\limits_{\epsilon \searrow 0} \frac{F(t,\ r_t,\ x+\epsilon)-F(t,\ r_t,\ x)}{\epsilon} \leq 0.$ □

Now, we characterize the stopping set defined above in terms of the boundary function $b_D(t,r_t)$.

Proposition 2. *For any* $(t,r_t,x) \in [0,T] \times \mathbb{R} \times (0,\infty)$ *such that* $(t,r_t,x) \in D$ *we have*

$$\{t\} \times \{r_t\} \times (0,\infty) \subset D \tag{32}$$

and

$$D = \left\{(t,r_t,x) \in [0,T] \times \mathbb{R} \times (0,\infty) | V(t,r_t,x) - G^{\mu_c}(t,x) \geq 0\right\}. \tag{33}$$

Proof.
Let $(t,r_t,x) \in \{t\} \times \{r_t\} \times (0,x)$. Since $(t,r_t,x) \in D$ and $V(t,r_t,x) \geq G^{\mu_c}(t,x)$ for all $x \in (0,\infty)$, we have

$$\begin{aligned}\frac{V(t,r_t,x) - V(t,r_t,y)}{x-y} &= \frac{G^{\mu_c}(t,x) - V(t,r_t,y)}{x-y} \\ &\leq \frac{G^{\mu_c}(t,x) - G^{\mu_c}(t,y)}{x-y}.\end{aligned}$$

Taking the limit on both sides as $x - y \to 0$ and by Lemma **1**, we have $(t,r_t,y) \in D$ and conclude that $\{t\} \times \{r_t\} \times (0,\infty) \subset D$. Furthermore, by the definition of the optimal stopping boundary

$$b_D(t,r_t) = \sup\left\{x \in (0,\infty) : (t,r_t,x) \in D\right\},$$

we have the equivalence

$$(t,r_t,x) \in D \iff \{t\} \times \{r_t\} \times (0,\infty) \subset D \iff x \geq b_D(t,r_t).$$

□

5. Continuity Lemmas

We next discuss the following continuity results on the payoff and option price functions to show that the stopping set D given by $D = \{(t, r_t, x) = V(t, r_t, x) - G^{\mu_c}(t, x) \geq 0\}$ is closed.

Lemma 2. *The mapping $(t, x) \mapsto G^{\mu_c}(t, x)$ is jointly continuous on $[0, T] \times (0, \infty)$.*

Proof. The continuity of the mapping $x \mapsto G^{\mu_c}(t, x)$ follows from the fact that $G^{\mu_c}(t, x)$ is convex with respect to $x \in (0, \infty)$ for any $t \in [0, T]$ given and fixed. It remains to show the uniform continuity of the mapping $t \mapsto G^{\mu_c}(t, x)$. Let $x \in (0, \infty)$ be given and fixed and $0 \leq t_1 \leq t_2 \leq T$. Thus

$$
\begin{aligned}
0 &\leq \left| G^{\mu_c}(t_2, x) - G^{\mu_c}(t_1, x) \right| \quad \text{by definition of absolute value} \\
&= \left| E^{\mu_c}\left[(xZ^{\mu_c}_{T-t_2} - K)^+ | \mathcal{F}_{t_2} \right] - E^{\mu_c}\left[(xZ^{\mu_c}_{T-t_1} - K)^+ | \mathcal{F}_{t_1} \right] \right| \quad \text{by equation (17)} \\
&\leq \left| E^{\mu_c}\left[(xZ^{\mu_c}_{T-t_2} - K)^+ - (xZ^{\mu_c}_{T-t_1} - K)^+ | \mathcal{F}_{t_2} \right] \right| \quad \text{by monotonicity} \\
&\leq E^{\mu_c}\left[\left| (xZ^{\mu_c}_{T-t_2} - K)^+ - (xZ^{\mu_c}_{T-t_1} - K)^+ \right| | \mathcal{F}_{t_2} \right] \quad \text{since } |E(X)| \leq E(|X|) \\
&= xE^{\mu_c}\left[\left| (Z^{\mu_c}_{T-t_1} - Z^{\mu_c}_{T-t_2})^+ \right| | \mathcal{F}_{t_2} \right] \quad \text{since } -K - -K = 0.
\end{aligned}
$$

Therefore, as $|t_2 - t_1| \to 0$, we have $G^{\mu_c}(t_2, x) - G^{\mu_c}(t, x) \to 0$, which completes our proof. □

Lemma 3. *The mapping $(t, r_t, x) \mapsto V(t, r_t, x)$ is jointly continuous on $[0, T] \times \mathbb{R} \times (0, \infty)$.*

Proof.
For $t \in [0, T]$ and $r_t \in \mathbb{R}$ given and fixed, the continuity of the mapping $x \mapsto V(t, r_t, x)$ on $(0, \infty)$ follows from the fact that it is convex on $(0, \infty)$. Also, the mapping $r_t \mapsto V(t, r_t, x)$ is continuous on $\mathbb{R}$ since the zero-coupon bond price $P(t, r_t; t + \tau)$ is continuous with respect to r_t on $\mathbb{R}$ for $t \in [0, T]$ and $x \in (0, \infty)$ given and fixed. It remains to show that $t \mapsto V(t, r_t, x)$ is continuous. Let $0 \leq t_1 \leq t_2 \leq T$, $\tau_1 = \tau_D(t, r_t, x)$ be the optimal stopping time for equation (19) and $\tau_2 = \tau_1 \wedge (T - t_2)$. Then

$$
\begin{aligned}
0 &\leq |V(t_1, r_{t_1}, x) - V(t_2, r_{t_2}, x)| \\
&= |\tilde{E}_{t_1,x}\left[e^{-\int_{t_1}^{t_1+\tau_1} r_u du} G^{\mu_c}(t_1 + \tau_1, X_{t_1+\tau_1}) | \mathcal{F}_{t_1} \right] \\
&- \tilde{E}_{t_2,x}\left[e^{-\int_{t_2}^{t_2+\tau_2} r_u du} G^{\mu_c}(t_2 + \tau_2, X_{t_2+\tau_2}) | \mathcal{F}_{t_2} \right] | \\
&= |\tilde{E}_{t_1,x}\left[e^{-\int_{t_1}^{t_1+\tau_2} r_u du} G^{\mu_c}(t_1 + \tau_1, X_{t_1+\tau_1}) | \mathcal{F}_{t_1} \right]
\end{aligned}
$$

$$
\begin{aligned}
&- \tilde{E}_{t_2,x}\left[e^{-\int_{t_2}^{t_2+\tau_2} r_u du} G^{\mu_c}(t_2+\tau_2, X_{t_2+\tau_2})|\mathcal{F}_{t_2}\right]| \\
&\leq \left|\tilde{E}_{t_2,x}\left[e^{-\int_{t_2}^{t_2+\tau_2} r_u du}\left\{G^{\mu_c}(t_1+\tau_1, X_{t_1+\tau_1}) - G^{\mu_c}(t_2+\tau_2, X_{t_2+\tau_2})\right\}|F_{t_2}\right]\right| \\
&\leq \tilde{E}_{t_2,x}\left[e^{-\int_{t_2}^{t_2+\tau_2} r_u du}\left|\left\{G^{\mu_c}(t_1+\tau_1, X_{t_1+\tau_1}) - G^{\mu_c}(t_2+\tau_2, X_{t_2+\tau_2})\right\}\right||F_{t_2}\right].
\end{aligned}
$$

By the continuity of the mapping $t \mapsto G^{\mu_c}(t,x)$ in Lemma 2 above, the mapping $t \mapsto V(t,r_t,x)$ is continuous on $[0,T]$, uniformly in $x \in (0,\infty)$. □

Note that from the paper of Peskir and Samee (2013) both $x \mapsto G^{\mu_c}(t,x)$ and $x \mapsto V(t,r_t,x)$ are convex. Furthermore, recall that every convex function on the open interval I is differentiable almost everywhere.

6. Boundary Value Problem

Applying the Itö 's formula on the price function $V = V(t,r_t;x)$ of the British call option, we have

$$
\begin{aligned}
dV = &\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X_t^2 \frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}X_t\frac{\partial^2 V}{\partial x\partial r} + \frac{1}{2}\sigma_2^2 r\frac{\partial^2 V}{\partial r^2}\right)dt + \frac{\partial V}{\partial x}dX_t \\
&+ \frac{\partial V}{\partial r}dr_t.
\end{aligned} \tag{34}
$$

We now derive the partial differential equation satisfied by $V(t,r_t,x)$ by creating a risk neutral portfolio. Consider the time interval $[t,t+\Delta t]$ and the following portfolio at time t: long 1 unit of derivative + short ϵ_1 units of stock + short ϵ_2 units of zero-coupon bond

Let $\prod_t$ be the value of the portfolio at time t. Then

$$\prod_t = V(t,r_t;x) - \epsilon_1 X_t - \epsilon_2 P. \tag{35}$$

The change in the value of the portfolio from time t to $t+\Delta t$ is then

$$\Delta\prod_t = \Delta V - \epsilon_1\Delta X_t - \epsilon_2\Delta P. \tag{36}$$

Using the discrete approximation of (34) for ΔV, we have

$$
\begin{aligned}
\Delta\prod_t = &\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X_t^2 \frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}X_t\frac{\partial^2 V}{\partial x\partial r} + \frac{1}{2}\sigma_2 2r_t\frac{\partial^2 V}{\partial r^2}\right)\Delta t + \frac{\partial V}{\partial x}\Delta X_t \\
&+ \frac{\partial V}{\partial r}\Delta r_t - \epsilon_1\Delta X_t - \epsilon_2\left(\frac{\partial P}{\partial t}\Delta t + \frac{\partial P}{\partial r}\Delta r_t + \frac{1}{2}\sigma_2^2 r_t\frac{\partial^2 P}{\partial r^2}\Delta t\right)
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X_t^2 \frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}X_t \frac{\partial^2 V}{\partial x \partial r} + \frac{1}{2}\sigma_2^2 r \frac{\partial^2 V}{\partial r^2} \right) \Delta t \\
&\quad + \left(\frac{\partial V}{\partial x} - \epsilon_1 \right) \Delta X_t + \left(\frac{\partial V}{\partial r} - \epsilon_2 \frac{\partial P}{\partial r} \right) \Delta r_t - \epsilon_2 \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_2^2 r \frac{\partial^2 P}{\partial r^2} \right) \Delta t
\end{aligned}
$$

To eliminate the risk, let $\epsilon_1 = \frac{\partial V}{\partial x}$ and $\epsilon_2 = \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r}$. Then equation (37) becomes

$$
\begin{aligned}
\Delta \prod_t &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X_t^2 \frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}X_t \frac{\partial^2 V}{\partial x \partial r} + \frac{1}{2}\sigma_2 r_t \frac{\partial^2 V}{\partial r^2} \right) \Delta t \\
&\quad + \left(\frac{\partial V}{\partial x} - \frac{\partial V}{\partial x} \right) \Delta X_t + \left(\frac{\partial V}{\partial r} - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} \frac{\partial P}{\partial r} \right) \Delta r_t \\
&\quad - \epsilon_2 \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_2^2 r_t \frac{\partial^2 P}{\partial r^2} \right) \Delta t \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X_t^2 \frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}X_t \frac{\partial^2 V}{\partial x \partial r} + \frac{1}{2}\sigma_2^2 r \frac{\partial^2 V}{\partial r^2} \right) \Delta t \\
&\quad - \epsilon_2 \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_2^2 r \frac{\partial^2 P}{\partial r^2} \right) \Delta t
\end{aligned} \tag{38}
$$

By equation (5),

$$
\begin{aligned}
\Delta \prod_t &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X_t^2 \frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}X_t \frac{\partial^2 V}{\partial x \partial r} + \frac{1}{2}\sigma_2 r_t \frac{\partial^2 V}{\partial r^2} \right) \Delta t \\
&\quad - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} \left([a\theta - (a+\lambda\sigma)r] \frac{\partial P}{\partial r} - \frac{1}{2}\sigma_2^2 r \frac{\partial^2 P}{\partial r^2} + r_t P + \frac{1}{2}\sigma_2^2 r_t \frac{\partial^2 P}{\partial r^2} \right) \Delta t \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X_t^2 \frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}X_t \frac{\partial^2 V}{\partial x \partial r} + \frac{1}{2}\sigma_2 r_t \frac{\partial^2 V}{\partial r^2} \right) \Delta t \\
&\quad - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} \left(r_t P - [a\theta - (a+\lambda\sigma)r] \frac{\partial P}{\partial r} \right) \Delta t \\
&= \Bigg(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X_t^2 \frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}X_t \frac{\partial^2 V}{\partial x \partial r} + \frac{1}{2}\sigma_2^2 r \frac{\partial^2 V}{\partial r^2} - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} r P \\
&\quad + [a\theta - (a+\lambda\sigma)r] \frac{\partial V}{\partial r} \Bigg) \Delta t.
\end{aligned} \tag{39}
$$

Note that the change in the value of the portfolio $\Delta \prod_t$, is deterministic. That is, the portfolio is riskless during the interval $[t, t+\Delta t]$. Hence, it must instantaneously earn the same rate of return as other short-term risk-free securities. That is,

$$
\prod_{t+\Delta t} = \prod_t e^{r_t \Delta t} \tag{40}
$$

Since $e^{r_t \Delta t} \approx 1 + r_t \Delta t$, then approximately

$$\Delta \prod_t \approx r_t \prod_t \Delta t \tag{41}$$

Substituting equations (39) and (35) to (41), we have

$$\begin{aligned}\Bigg(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X_t^2 \frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}X_t \frac{\partial^2 V}{\partial x \partial r} + \frac{1}{2}\sigma_2 r_t \frac{\partial^2 V}{\partial r^2} - r_t P \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} \\ + [a\theta - (a+\lambda\sigma)r]\frac{\partial V}{\partial r}\Bigg)\Delta t = r_t \left(V - \frac{\partial V}{\partial x}X_t - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} P\right)\Delta t.\end{aligned} \tag{42}$$

Simplifying this expression, we have

$$\begin{aligned}\Bigg(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X_t^2 \frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}X_t \frac{\partial^2 V}{\partial x \partial r} + \frac{1}{2}\sigma_2^2 r \frac{\partial^2 V}{\partial r^2} + r_t X_t \frac{\partial V}{\partial x} \\ + [a\theta - (a+\lambda\sigma)r]\frac{\partial V}{\partial r}\Bigg) - r_t V = 0.\end{aligned} \tag{43}$$

Now, we define the infinitesimal generator

$$\mathbb{L}f(s, y_1, y_2) = \left(\frac{\partial}{\partial s} + \sum_i \frac{\partial}{y_i} a_i + \frac{1}{2}\sum_i \sum_j \frac{\partial^2}{\partial y_i y_j} b_i b_j \rho_{ij}\right) f(s, y_1, y_2) \tag{44}$$

for any differentiable function $f(s, y_1, y_2)$ on $[0, T] \times \mathbb{R} \times (0, \infty)$, where the Itö's processes $Y_1 = (Y_{1,s})_{s \geq 0}$ and $Y_2 = (Y_{2,s})_{s \geq 0}$ defined on probability space $(\Omega, \mathcal{F}_s, \mathbb{P})$ satisfy the stochastic differential equations

$$dY_{1,s} = a_1 Y_{1,s} ds + b_1 Y_{1,s} dW_{1,s} \quad (Y_{1,0} = y_1)$$
$$dY_{2,s} = a_2 Y_{2,s} ds + b_2 Y_{2,s} dW_{2,s} \quad (Y_{2,0} = y_2),$$

respectively, and $\rho_{ij} = cov(dW_{i,s}, dW_{j,s})$ (see [1]).

Lemma 4. *We have*

$$D = \left\{(t, r_t, x) \in [0, T] \times \mathbb{R} \times (0, \infty) | \mathbb{L}_X(t, r_t, x) > 0\right\} \subset C \tag{45}$$

where $C = D^C$ is the continuation set.

Proof. Let $(t, r_t, x) \in [0, T] \times \mathbb{R} \times (0, \infty)$ be such that $\mathbb{L}_X V(t, r_t, x) > 0$. By Lemma 1 in [15], Dynkin's formula holds. Thus we have,

$$\begin{aligned}P(t, r_t; t+s)G^{\mu_c}(t+s, X_{t+s}) = G^{\mu_c}(t, x) \\ + \int_0^\tau \mathbb{L}P(t, r_t; t+s)G^{\mu_c}(t+s, X_{t+s})ds \\ + M_s\end{aligned} \tag{46}$$

where $M_s = \int_0^\tau \sigma_1 x \frac{\partial G^{\mu_c}}{\partial x} dW_s + \int_0^\tau \sigma_2 \sqrt{r_s} G^{\mu_c} \frac{\partial P}{\partial r_s} d\bar{W}_s$ defines a continuous martingale for $s \in [0, T-t]$ with $t \in [0, T]$. By Lemma 5.2 and the fact that $G^{\mu_c} \in C^{1,2}$ (see [11]), the infinitesimal generator $\mathbb{L}P(t, r_u; t+s)G^{\mu_c}(t, x)$ is continuous with respect to $(t, x) \in [0, T] \times (0, \infty)$. Thus, there exists an open neighborhood $U \times V \subset [0, T] \times (0, \infty)$ of (t, x) such that $\mathbb{L}_X V(t, r_t, x) > 0$ for all $(s, r_s, y) \in U \times V \times 0$. Let

$$\tau_U = \inf\left\{\tau : (t+\tau, r_{t+\tau}, X_{t+\tau}) \in U \times V \times 0, X_t = x, r_t = r\right\}.$$

By Optimal Sampling Theorem, the relation equation (46) with $s = \tau_U$ shows that

$$\begin{aligned} E\left[P(t, r_t; t+s)G^{\mu_c}(t+s, X_{t+s})\right] &= G^{\mu_c}(t, x) \\ &\quad + E\left[\int_0^T \mathbb{L}P(t, r_t; t+s)G^{\mu_c}(t+s, X_{t+s})ds\right]. \end{aligned} \tag{47}$$

Since $\mathbb{L}_X V(t, r_t, x) > 0$, the right hand side of (47) is strictly greater than G^{μ_c}, while from (19), we have

$$V(t, r_t, x) \geq E[P(t, r_t; t+s)G^{\mu_c}(t+s, X_{t+s})|\mathcal{F}_t]$$

showing that $V(t, r_t, x) > G^{\mu_c}(t, x)$, which implies that $(t, r_t, x) \in C$. This completes the proof. □

We now proceed to the solution of the free boundary problem in (21). Going back to the option price in (17), note that it can be written as

$$V = \sup_{0 \leq \tau \leq T} \tilde{E}\left[P(0, r_t; \tau)E^{\mu_c}(X_T - K)^+ |\mathcal{F}_\tau\right] \tag{48}$$

Similarly, (20) can be written as

$$V = \sup_{0 \leq \tau \leq T-t} E\left[P(0, r_t; \tau)G^{\mu_c}(t+\tau, xX_\tau)|X_t = x, r_t = r\right] \tag{49}$$

Now, let

$$\Gamma(s, r, x) = P(0, r; s)G^{\mu_c}(s, x). \tag{50}$$

Then

$$\frac{\partial \Gamma}{\partial s} = P_0^s \frac{\partial G^{\mu_c}}{\partial s} + G^{\mu_c} \frac{\partial P_0^s}{\partial s} R, \quad \frac{\partial^2 \Gamma}{\partial x^2} = P_0^s \frac{\partial^2 G^{\mu_c}}{\partial x^2}, \quad \frac{\partial \Gamma}{\partial r} = G^{\mu_c} \frac{\partial P_0^s}{\partial r},$$

$$\frac{\partial \Gamma}{\partial x} = P_0^s \frac{\partial G^{\mu_c}}{\partial x}, \qquad \frac{\partial^2 \Gamma}{\partial x \partial r} = \frac{\partial G^{\mu_c}}{\partial x} \frac{\partial P_0^s}{\partial r} \qquad \text{and } \frac{\partial^2 \Gamma}{\partial r^2} = G^{\mu_c} \frac{\partial^2 P_0^s}{\partial r^2}.$$

By Itö's formula,

$$d\Gamma = \frac{\partial \Gamma}{\partial s} ds + \frac{\partial \Gamma}{\partial r} dr + \frac{\partial \Gamma}{\partial x} dx + \frac{1}{2}\sigma_1^2 \frac{\partial^2 \Gamma}{\partial r^2} + \rho \sigma_1 \sigma_2 \sqrt{r} \frac{\partial^2 \Gamma}{\partial x \partial r} + \frac{1}{2}\sigma_2 r \frac{\partial^2 \Gamma}{\partial x^2} ds. \tag{51}$$

Thus,

$$
\begin{aligned}
d\left[P\left(0,r;s\right)G^{\mu_c}(s,X_s)\right] = P_0^s\frac{\partial G^{\mu_c}}{\partial s} + G^{\mu_c}\frac{\partial P_0^s}{\partial s}ds + [a\theta-(a+\lambda\sigma)r]G^{\mu_c}\frac{\partial P_0^s}{\partial r}ds \\
+ rX_sP_0^s\frac{\partial G^{\mu_c}}{\partial x}ds + \frac{1}{2}\sigma_1^2X_s^2P_0^s\frac{\partial^2 G^{\mu_c}}{\partial x^2}ds \\
+ \rho\sigma_1\sigma_2\sqrt{r}X_s\frac{\partial G^{\mu_c}}{\partial x}\frac{\partial P_0^s}{\partial r}ds + \frac{1}{2}\sigma_2^2rG^{\mu_c}\frac{\partial^2 P_0^s}{\partial r^2}ds \\
+ \sigma_1X_s\frac{\partial G^{\mu_c}}{\partial x}dW_s + \sigma_2\sqrt{r}G^{\mu_c}\frac{\partial P_0^s}{\partial r}\overline{dW_s}.
\end{aligned}
\tag{52}
$$

Integrating both sides from 0 to τ:

$$
\begin{aligned}
&P(t,r_t;t+s)G^{\mu_c}(t+s,X_{t+s}) \\
&= G^{\mu_c}(t,X_t) + \int_0^\tau \Bigg\{P(t,r_t;s)\Big[\frac{\partial G^{\mu_c}}{\partial s} + r_sX_s\frac{\partial G^{\mu_c}}{\partial x} + \frac{1}{2}\sigma_1^2X_s^2\frac{\partial^2 G^{\mu_c}}{\partial x^2}\Big] \\
&\quad + \frac{1}{P(0,r_0;t)}\Big[G^{\mu_c}(s,X_s)\frac{\partial P}{\partial s} + [a\theta-(a+\lambda\sigma)r_{t+u}]G^{\mu_c}(s,X_s)\frac{\partial P}{\partial r} \\
&\quad + \rho\sigma_1\sigma_2\sqrt{r_s}X_s\frac{\partial G\mu_c}{\partial x}(s,X_s)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_2^2r_sG^{\mu_c}(s,X_s)\frac{\partial^2 P}{\partial r^2}\Big]\Bigg\}d \\
&\quad + \int_0^\tau \frac{1}{P(0,r_0;t)}\left[\sigma_1X_s\frac{\partial G^{\mu_c}}{\partial x}dW_s\sigma_2\sqrt{r_s}G^{\mu_c}\frac{\partial P}{\partial r}\overline{dW_s}\right] \\
&\quad + \int_0^\tau \frac{1}{P(0,r_0;t)}\left[\sigma_1X_s\frac{\partial G^{\mu_c}}{\partial x}dW_s + \sigma_2\sqrt{r_s}G^{\mu_c}\frac{\partial P}{\partial r}\overline{dW_s}\right]
\end{aligned}
\tag{53}
$$

Since $P(t,r_t;t+\tau)G^{\mu_c}(t+\tau,X_{t+\tau})$ is a martingale, we have

$$
\begin{aligned}
&P(t,r_t;t+u)\left[\frac{\partial G^{\mu_c}}{\partial s} + r_sX_s\frac{\partial G^{\mu_c}}{\partial x} + \frac{1}{2}\sigma_1^2x^2\frac{\partial^2 G^{\mu_c}}{\partial x^2}\right] + \frac{G^{\mu_c}}{P(0,r_0;t)}\frac{\partial P}{\partial t} \\
&\quad + [a\theta-(a+\lambda\sigma)r]\frac{G^{\mu_c}}{P(0,r_0;t)}\frac{\partial P}{\partial r} \quad + \rho\sigma_1\sigma_2\sqrt{r_s}X_s\frac{x}{P(0,r_0;t)}\frac{\partial G^{\mu_c}}{\partial x}\frac{\partial P}{\partial r} \\
&\quad + \frac{1}{2}\sigma_2^2r_s\frac{G^{\mu_c}}{P(0,r_0;t)}\frac{\partial^2 P}{\partial r^2} = 0.
\end{aligned}
$$

Since the supremum in (19) is taken overall stopping time $\tau \in [0, T-t]$, we have

$$
\begin{aligned}
\mathbb{L}_xV(t,r_t,x) = \Bigg[&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2X_t^2\frac{\partial^2 V}{\partial x^2} + \rho\sigma_1\sigma_2\sqrt{r_t}X_t\frac{\partial^2 V}{\partial x\partial r} + \frac{1}{2}\sigma_2r_t\frac{\partial^2 V}{\partial r^2} + \frac{\partial V}{\partial x}r_tX_t \\
&- [a\theta-(a+\lambda\sigma)r]\frac{\partial V}{\partial r}\Bigg]V(t,r_t,x) = 0.
\end{aligned}
$$

This shows that there is a continuous (smooth) function $h : [0,T] \times \mathbb{R} \to \mathbb{R}^+$ such that

$$\mathbb{L}_X P(t, r_t; t+\tau) G^{\mu_c}(t+\tau, h(t, r_t) X_\tau) = 0 \tag{54}$$

for all $t \in [0,T]$. Linearity of $\mathbb{L}_X P(t, r_t; t+\tau) G^{\mu_c}(t+\tau, X_{t+\tau})$ in terms of $X_{t+\tau} > 0$ shows that $\mathbb{L}_X P(t, r_t; t+\tau) G^{\mu_c}(t+\tau, X_{t+\tau}) > 0$ when $X_{t+\tau} < h(t+\tau, r_t)$ and $\mathbb{L}_X P(t, r_t; t+\tau) G^{\mu_c}(t+\tau, X_{t+\tau}) < 0$ when $X_{t+\tau} > h(t+\tau, r_t)$. This shows in particular that it is optimal to exercise immediately when $x > h(t, r_t)$ and $t < T$ is sufficiently close to maturity T for the same reason mentioned in [10]. From here, we define the optimal stopping boundary (the early-exercise premium representation) as follows:

$$b_D(t, r_t) = \sup \{ x \in (0, \infty) : (t, r_t, x) \in D \}.$$

The result below characterizes the stopping set in terms of the boundary function $b_D(t, r_t)$.

Proposition 3. *The boundary function $b_D(t, r_t)$ is continuous in $t \in [0,T]$ for all $r_t \in \mathbb{R}$.*

Proof.

Let $r_t \in \mathbb{R}$ be fixed. We first show the right continuity. Suppose that $b_D(t, r_t)$ is not continuous at $t = t_0$. We consider two cases:

Case 1: $b_D(t_0, r_{t_0}) < b_D(t_{0^+}, r_{t_{0^+}})$

Let $(t', r_t.x') \in (t_0, T) \times \mathbb{R} \times (b_D(t_0, r_{t_0}), b_D(t_{0^+}, r_{t_{0^+}}))$ be a point in the stopping set D with t' close to t_0 and $t' \downarrow t_0$. By Newton-Leibniz formula and Lemma 3 we have

$$\begin{aligned} 0 &< \int_{b_D(t_0, r_t)}^{x'} [V_x(t', r_t, u) - G_x^{\mu_c}(t'u)] du \\ &= V(t', r_t, x') - G^{\mu_c}(t', x') \end{aligned}$$

as $t' \to t_0$. This implies that $V(t_0, r_t, x') - G^{\mu_c}(t_0, x') > 0, i.e., (t_0, r_t, x') \in D$, which contradicts the fact that $x' > b_D(t_0, r_t), i.e., (t_0, r_t, x') \in C$.

Case 2: $b_D(t_0, r_{t_0}) > b_D(t_{0^+}, r_{t_{0^+}})$

Let $(t*, r_t.x*) \in (t_0, T) \times \mathbb{R} \times (b_D(t_0, r_{t_0}), b_D(t_{0^+}, r_{t_{0^+}}))$ be a point in the continuation set D with $t*$ close to t_0 and $t* \downarrow t_0$. By Newton-Leibniz formula and Lemma 3 we have

$$\begin{aligned} 0 &< \int_{x*}^{b_D(t_0, r_t)} [V_x(t*, r_t, u) - G_x^{\mu_c}(t*, u)] du \\ &= G^{\mu_c}(t*, x*) - V(t*, r_t, x*) \end{aligned}$$

as $t* \to t_0$. This implies that $V(t_0, r_t, x*) > G^{\mu_c}(t_0, x') > 0, i.e., (t_0, r_t, x') \in D$, which contradicts the fact that $(t_0, r_t, x*) \in D$. $\square$

7. The Arbitrage-free Price and the Rational Exercise Boundary

In this section, we present the main result which is a derivation of the closed form expression for the arbitrage-free price $V(t, r_t, x)$ of the British call option in terms of the early-exercise premium over the European option counterpart.

First, we introduce the following functions:

$$F\,(t, r_t, x) := G^{\mu_c}(t, x) - P(t, r_t; T)G^{r_t}(t, x) \tag{55}$$

$$J(t, r_r, x, v, z) := -\int_z^{\infty} \mathbb{L}[P(t, r_t; v)G^{\mu_c}(v, y)]f(v - t, x, y)dy \tag{56}$$

for $t \in [0, T], x > 0, v \in [t, T]$ and $z > 0$, where $y \mapsto f(v - t, r_t, x, y)$ is the probability density function of $xZ^{r_t}_{v-t}$ (see [3], [13]) given by

$$f(v-t,x,y) = \frac{2a}{\sigma^2\left(1-e^{-a(v-t)}\right)}\left(\frac{ye^{a(v-t)}}{x}\right)^{\frac{a\theta}{\sigma^2}-\frac{1}{2}}\exp\left\{\frac{2a(x+ye^{a(v-t)})}{\sigma^2(1-e^{a(v-t)})}\right\}$$
$$\times\, I_q\left(\frac{-4a\sqrt{xye^{a(v-t)}}}{\sigma^2(1-e^{a(v-t)})}\right)$$

$I_q(\cdot)$ is the modified Bessel function of the first kind of order $q = \frac{2a\theta}{\sigma^2} - 1$. Note that $f(v - t, x, y)$ can be transformed as

$$f(v - t, x, y) = e^H\delta(x - y) \tag{57}$$

where

$$H = \frac{\partial}{\partial r}a(\theta - r_t) + \frac{\partial^2}{\partial r}\frac{\sigma_2^2}{2}r_t$$

and $\delta(x-y)$ is the delta function (see [13]). Thus, $J(t, r_r, x, v, z) > 0$ for all $(t, r_t, x) \in D$. The function $\mathbb{L}[P(t, r_t; v)G^{\mu_c}(v, y)]$ is given by

$$\begin{aligned}\mathbb{L}[P(t, r_t; v)G^{\mu_c}(v, y)] = [2r_t - \mu_c + \rho\sigma_1\sigma_2\sqrt{r_t}\lambda(0, t + u)]xP(t, r_t, t + u)\\ e^{\mu_c(T-(t+u))}\Phi(d_1) - r_tP(t, r_t, t + u)K\Phi(d_2).\end{aligned}$$

It can be verified that (see Appendix)

$$\begin{aligned}J(t, r_t, x, T, z) = r_te^{\frac{-xH}{z}}P(t, r_t, T)K\Phi(d_2)\\ - [2r_t - \mu_c + \rho\sigma_1\sigma_2\sqrt{r_t}\lambda]xe^{\frac{-xH+\mu_c}{z}}P(t, r_t, T)\Phi(d_1)\end{aligned} \tag{58}$$

where

$$d_1 = \frac{\ln\frac{x}{K} + A(t,T) + \frac{\sigma_1^2}{2}(T)}{\sigma_1\sqrt{T}}$$
$$d_2 = \frac{\ln\frac{x}{K} + A(t,T) - \frac{\sigma_1^2}{2}(T)}{\sigma_1\sqrt{T}}$$
$$A(t,T) = \frac{r_0-\theta}{a}\big(1-e^{-aT}\big) + \theta(T)$$
$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-y^2}{2}}dy$$
$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$$P(t,r_t,T) = \Bigg[\Big[x\Phi(d_1) - Ke^{-A(t,T)}\Phi(d_2)\Big] + \sigma C_0\Big[x\Phi(d_1) - Ke^{-A(t,T)}\Big(\phi(d_2) -$$
$$\sigma_1\sqrt{T-t}\Phi(d_2)\Big)\Big] + \sigma C_1\Big[d_2 x\phi(d_1) - d_1 Ke^{-A(t,T)})\phi(d_2)\Big] + 0(\sigma)\Bigg]$$
$$\text{where} \quad 0(\sigma) \quad \text{is a zero-mean error} \quad \text{(see [6])}$$

Theorem 1. *The arbitrage free-price of the British call option with stochastic interest rate admits the following early-exercise premium representation*

$$V(t,r_t,x) = P(t,r_t;T) + \int_t^T J(t,r_t,x,v,b_D(v,r_v))dv \tag{59}$$

for all $(t,r_t,x) \in [0,T]\times\mathbb{R}\times(0,\infty)$, where the first term is the arbitrage-free price of the European call option under stochastic interest rate and the second term is the early-exercise premium.
The rational exercise boundary of the British call option can be characterized as the unique continuous solution $b_D : [0,T]\times\mathbb{R}\to\mathbb{R}^+$ to the nonlinear integral equation

$$F(t,r_t,b_D(t,r_t)) = \int_t^T J(t,r_t,x,v,b_D(v,r_v))dv \tag{60}$$

which satisfies $b_D(t,r_t) \geq h(t,r_t)$ for all $t\in[0,T]$ where h is defined as a continuous (smooth) function $h:[0,T]\times\mathbb{R}\to\mathbb{R}^+$ such that $\mathbb{L}P(t,r_t;t+u)G^{\mu_c}(t+u,X_{t+u}) = 0$ for all $t\in[0,T]$.

Proof :
For any $(t,r_t,x)\in C$, we have

$$V(t, r_t, x) = E\left[P(t, r_t; t + \tau_D)G^{\mu_c}(t + \tau_D, X_{t+\tau_D})\right] \tag{61}$$

with $X_t = x \in (0, \infty)$ and $r_t = r \in \mathbb{R}$ where $\tau_D = \tau_D(t, r_t, x)$ is the optimal stopping defined as $\tau_D(t, r_t, x) := inf\{s \in [0, T - t] : (t + s, r_{t+s}, X_{t+s}) \in D\}$. It can easily be verified from (7) that $P(t, r_t, x)$ has continuous $\frac{\partial^2 P}{\partial r^2}(t, r_t, x)$. This implies that $V(t, r_t, x)$ has also continuous $\frac{\partial^2 V}{\partial r^2}(t, r_t, x)$. Moreover, it is well-known from the theory of Markov processes that V in (61) is $C^{1,2}$; hence is $C^{1,2,2}$, and it solves the Cauchy-Dirichlet free boundary problem

$$\mathbb{L}_X V(t, r_t, X_t) = 0 \quad (t, r_t, X_t) \in C \tag{62}$$

$$V(t, r_t, X_t) = G^{\mu_c}(t, X_t)(t, r_t, X_t) \in \partial C \tag{63}$$

where $\partial C \subset D$ denotes the open set C. By applying the change of variable formula with local time on surfaces in [9] to

$$(s, r_t, y) \mapsto P(t, r_t; t + s)V(t + s, r_{t+s}, y)$$

with $t \in [0, T]$ and $X_t = x \in (0, \infty)$ given and fixed, we have

$$\begin{aligned}
&E[P(t, r_t; t + s)V(t + s, r_{t+s}, X_{t+s})|r_t] = V(t, r_t, x)\\
&\quad + E[\int_0^s \mathbb{L}_x P(t, r_t; t + v)V(t + v, r_{t+v}, xX_v)I(X_{t+v} \neq b(t + v, r_{t+v}))dv|r_t]\\
&\quad + E[M_s^b|r_t]\\
&\quad + \frac{1}{2}E[\int_0^s P(t, r_t; t + v)[V_x(t + v, r_{t+v}, X_{t+v}+) - V_x(t + v, r_{t+v}, X_{t+v}-)]\\
&\qquad I(X_{t+v} = b(t + v, r_{t+v}))d\ell_v^b(X^x)|r_t]
\end{aligned} \tag{64}$$

where $M_s^b = \int_0^s \left[\sigma_1 X_{t+v} P\frac{\partial G^{\mu_c}}{\partial x}dW_{t+v} + \sigma_2 G^{\mu_c}\frac{\partial P}{\partial r}d\bar{W}_{t+v}\right]$ defines a continuous local martingale for $s \in [0, T - t]$ and $\ell^b = (\ell_v^b)_{0 \le v \le s}$ is the local time of $X^x = (X_{t+v})_{0 \le v \le s}$ on the curve b_D for $s \in [0, T - t]$. Since the coefficients of the respective Wiener processes of M_s^b are finite (and so are their respective squares) and that G^{μ_c} and P are F_t-adapted, hence we have $E\left[M_s^b\right] = 0$. By the smooth-fit property [11] or convexity of V, the last term in (64) vanishes. Hence,

$$\begin{aligned}
&E\left[P(t, r_t; t + s)V(t + s, r_{t+s}, X_{t+s})|r_t\right] = V(t, r_t, x)\\
&\quad + \int_0^s E\left[\mathbb{L}_x P(t, r_t; t + v)G^{\mu_c}(t + v, X_{t+v})I(X_{t+v} > b(t + v, r_{t+v}))|r_t\right]dv,
\end{aligned} \tag{65}$$

where we use (62) above and the fact that $V = G$ in the stopping set D to obtain the second term above. By replacing s by $T - t$, we have

$$\begin{aligned}
&E\left[P(t, r_t; T)(X_T - K)^+\right] = V(t + r_t, x)\\
&\quad + \int_0^{T-t} E\left[\mathbb{L}P(t, r_t; t + v)G^{\mu_c}(t + v, X_{t+v})I(X_{t+v} > b(t + v, r_{t+v}))|r_t\right]dv,
\end{aligned} \tag{66}$$

where we used the fact that $V(T, r_T, x) = G^{\mu_c}(T, x) = (x - K)^+$. Recognizing the left-hand side of (66) above as the price $p(t, r_t; T)$ of the European call option under stochastic interest rate, we have

$$V(t, r_t, x) = p(t, r_t; T) + \int_t^T J(t, r_t, x, v, b_D(v, r_v))dv.$$

Moreover, since $V(t, r_t, x) = G^{\mu_c}(t, x)$ for all $(t, r_t, x) \in D$, we have $V(t, r_t, b_D(t, r_t)) = G^{\mu_c}(t, b_D(t, r_t))$. This implies that the boundary functiom b_D solves equation (60). This establishes the existence of the solution to (60).

The uniqueness of this solution can be shown parallel to the proof in [11]. □

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References

[1] Laura Ballotta and Gianluca Fusai. Tools from stochastic analysis for mathematical finance: a gentle introduction. *Available at SSRN 3183712*, 2018.

[2] Tomas Björk. *Arbitrage theory in continuous time.* Oxford university press, 2009.

[3] John C Cox, Jonathan E Ingersoll, and Stephen A Ross. A theory of the term structure ofinterest rates. *Research Paper, Graduate School of Business, Stanford Univ*, 1978.

[4] Samuel H Cox and Gennady Medvedev. The market price of risk for affine interest rate term structures. In *6th International AFIR-Colloquium*, pages 913–924, 1996.

[5] John C Hull. *Options futures and other derivatives.* Pearson Education India, 2003.

[6] Yong-Jin Kim. Option pricing under stochastic interest rates: an empirical investigation. *Asia-Pacific Financial Markets*, 9(1):23–44, 2002.

[7] Elvira P De Lara-Tuprio and Felipe R Sumalpong. British put option on stocks under stochastic interest rate. *Model Assisted Statistics and Applications*, 12(4):321–334, 2017.

[8] Rogemar S Mamon. Three ways to solve for bond prices in the vasicek model. *Advances in Decision Sciences*, 8(1):1–14, 2004.

[9] Goran Peskir. A change-of-variable formula with local time on surfaces. In *Séminaire de probabilités XL*, pages 70–96. Springer, 2007.

[10] Goran Peskir and Farman Samee. The british put option. *Applied Mathematical Finance*, 18(6):537–563, 2011.

[11] Goran Peskir and Farman Samee. The british call option. *Quantitative Finance*, 13(1):95–109, 2013.

[12] Goran Peskir and Albert Shiryaev. *Optimal stopping and free-boundary problems.* Springer, 2006.

[13] Andras Vanyolos, Maxx Cho, and Scott Alan Glasgow. Probability density of the cir model. *Available at SSRN 2508699*, 2014.

[14] Duc Hong Vo, Son Van Huynh, Anh The Vo, and Dao Thi-Thieu Ha. The importance of the financial derivatives markets to economic development in the world's four major economies. *Journal of Risk and Financial Management*, 12(1):35, 2019.

[15] David D Yao, Qing Zhang, and Xun Yu Zhou. A regime-switching model for european options. In *Stochastic processes, optimization, and control theory: applications in financial engineering, queueing networks, and manufacturing systems*, pages 281–300. Springer, 2006.