The Fuglede-Putnam theorem and quasinormality for class $p$-$wA(s,t)$ operators

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Abstract. In this work, we demonstrate that (i) if $T$ is a class $p$-$wA(s,t)$ operator and $T(s,t)$ is quasinormal (resp., normal), then $T$ is also quasinormal (resp., normal) (ii) If $T$ and $T^*$ are class $p$-$wA(s,t)$ operators, then $T$ is normal; (iii) the normal portions of quasisimilar class $p$-$wA(s,t)$ operators are unitarily equivalent; and (iv) Fuglede-Putnam type theorem holds for a class $p$-$wA(s,t)$ operator $T$ for $0 < s,t, s+t = 1$ and $0 < p \leq 1$ if $T$ satisfies a kernel condition $\ker(T) \subset \ker(T^*)$.

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1. Introduction

On a complex Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators. Aluthge [2] investigated the $p$-hyponormal operator $T$, which is defined as $(T^*T)^p \geq (TT^*)^p$ with $0 \leq p \leq 1$ using the Furuta inequality [14]. When $p = 1$, $T$ is said to be hyponormal. As a result, $p$-hyponormality is a broadening of hyponormality. Following [2], several authors are looking towards novel hyponormal operator generalizations.

It is known that $p$-hyponormal operators have many interesting properties as hyponormal operators, for example, Putnam’s inequality, Fuglede-Putnam type theorem, Bishop’s property ($\beta$), Weyl’s theorem and polaroid. Let $T \in \mathcal{B}(\mathcal{H})$ and $|T| = (T^*T)^{\frac{1}{2}}$. By taking $U|T|x = Tx$ for $x \in \mathcal{H}$ and $Ux = 0$ for $x \in \ker|T|$, $T$ has a unique polar decomposition $T = U|T|$ with condition $\ker U = \ker|T|$. We say that $T = U|T|$ is the polar decomposition of $T$. In [2], Aluthge extended the class of hyponormal operators by introducing $p$-hyponormal operators and obtained some properties with the help of the transformation

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Moreover, for each nonnegative integer \( n \), the \( n \)-th generalized Aluthge transform \( \Delta^n(T(s, t)) \) of \( T(s, t) \) is defined as follows:

\[
\Delta^n(T(s, t)) = \Delta(\Delta^{n-1}(T(s, t))), \quad \Delta^0(T(s, t)) = T(s, t).
\]

**Definition 1.** Let \( T = U|T| \) be the polar decomposition of an operator \( T \in \mathcal{B(H)} \). Then the generalized Aluthge transform \( T(s, t) \) of \( T \) is defined as follows:

\[
T(s, t) = |T|^s U|T|^t.
\]

Moreover, for each nonnegative integer \( n \), the \( n \)-th generalized Aluthge transform \( \Delta^n(T(s, t)) \) of \( T(s, t) \) is defined as follows:

\[
\Delta^n(T(s, t)) = \Delta(\Delta^{n-1}(T(s, t))), \quad \Delta^0(T(s, t)) = T(s, t).
\]

**Definition 2.** Let \( 0 < s, t, \) and \( 0 < p \leq 1 \). An operator \( T \) is said to be a class

(i) \( p\)-\( wA(s, t) \) if

\[
(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}
\]

and

\[
|T|^{2sp} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{2t}{s+t}}.
\]

(ii) \( p\)-\( A(s, t) \) if

\[
(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}.
\]

(iii) \( p\)-\( A \) if \( |T|^2 \geq |T|^{2p} \).

(iv) \( (s, p)\)-\( w\)-hyponormal if \( |T(s, s)|^p \geq |T|^{2sp} \geq |(T(s, s)^*)|^p \).

It is known that \( p\)-hyponormal operators and \( log\)-\( hyponormal \) operators are class \( 1\)-\( wA(s, t) \) for any \( 0 < s, t \). Class \( p\)-\( wA(s, s) \) is called class \( (s, p)\)-\( w\)-hyponormal, class \( 1\)-\( wA(1, 1) \) is called class \( A \) and class \( 1\)-\( wA(\frac{1}{2}, \frac{1}{2}) \) is called \( w\)-hyponormal [13, 15, 18, 19, 33]. Hence class \( p\)-\( wA(s, t) \) operator is a generalization of class \( (s, p)\)-\( w\)-hyponormal, class \( A \) and \( w\)-hyponormal operators. C. Yang and J. Yuan [34–36] studied class \( wF(p, r, q) \) operator \( T \), i.e.,

\[
(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{2}} \geq |T^*|^{\frac{2(p+r)}{r}}
\]

and

\[
|T|^{2(p+r)(1-\frac{1}{q})} \geq (|T|^p|T^*|^{2r}|T|^p)^{1-\frac{1}{q}}
\]

where \( 0 < p, 0 < r, 1 \leq q \). If we take small \( p_1 \) such that \( 0 < p_1 \leq \frac{p+r}{qr} \) and \( p_1 \leq \frac{(p+r)(q-1)}{pq} \), then \( T \) is class \( p_1\)-\( wA(p, r) \). Hence class \( p_1\)-\( wA(p, r) \) is a generalization of class \( wF(p, r, q) \). We will use this property frequently.
It is known that $T = U|T|$ is class $p$-$wA(s,t)$ if and only if
$$|T(s,t)|^{2rp} \geq |T|^{2p}, \quad |T|^{2rp} \geq |(T(s,t))^\ast|^{2rp}$$
by [26]. Hence
$$|T(s,t)|^{2rp} \geq |T|^{2rp} \geq |(T(s,t))^\ast|^{2rp}$$
and $T(s,t)$ is $rp$-hyponormal for all $r \in (0, \min\{s,t\}]$.

The following is a breakdown of the paper’s structure: In section 2, we prove that if $T$ is a class of $p$-$wA(s,t)$ operators and its Aluthge transform $T(s,t)$ is quasinormal (respectively, normal), then $T$ is also quasinormal (resp., normal). The normal parts of quasisimilar class $p$-$wA(s,t)$ operators are unitarily equivalent in section 3. The major goal of Section 4 is to demonstrate that the Fuglede-Putnam theorem holds for a class $p$-$wA(s,t)$ operator $T$ with $0 < s, t, s+t = 1$ and $0 < p \leq 1$ if $T$ fulfills the kernel condition $\ker(T) \subset \ker(T^\ast)$.

## 2. Quasinormality

Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B(H)}$. $T$ is said to be quasinormal if $|T|U = U|T|$, or equivalently, $TT^\ast T = T^\ast TT$. S. M. Patel, K. Tanahashi, A. Uchiyama and M. Yanagida [27] proved that if $T$ is class $A(s,t)$ and $T(s,t)$ is quasinormal, then $T$ is quasinormal and $T = T(s,t)$ if $s + t = 1$. The following is a generalization of this result.

**Theorem 1.** Let $T$ be a class $p$-$wA(s,t)$ operator with the polar decomposition $T = U|T$. If $T(s,t) = |T|U|T|^\ast$ is quasinormal, then $T$ is also quasinormal. Hence $T$ coincides with its generalized Aluthge transform $T(s,t)$.

**Proof.** Since $T$ is a class $p$-$A(s,t)$ operator,
$$|T(s,t)|^{2rp} \geq |T|^{2rp} \geq |(T(s,t))^\ast|^{2rp}$$
for all $r \in (0, \min\{s,t\})$ by [19, Theorem 3] and Löwner-Heinz inequality. Then Douglas’s theorem [11] implies
$$\text{ran}(T(s,t)) = \overline{\text{ran}((T(s,t))^\ast)} \subset \text{ran}(|T|) = \overline{\text{ran}(|T(s,t)|)}$$
where $\overline{\mathcal{M}}$ denotes the norm closure of $\mathcal{M}$. Let $T(s,t) = W|T(s,t)|$ be the polar decomposition of $T(s,t)$. Then $E := W^\ast W = U^\ast U \geq WW^\ast =: F$. Put
$$|(T(s,t))^\ast|^{1\over 2rp} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, W = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}$$
on $\mathcal{H} = \overline{\text{ran}(T(s,t))} \oplus \ker((T(s,t))^\ast)$.

Then $X$ is injective and has a dense range. Since $T(s,t)$ is quasinormal, $W$ commutes with $|T(s,t)|$ and
$$|T(s,t)|^{2rp} = W^\ast W|T(s,t)|^{2rp} = W^\ast |T(s,t)|^{2rp} W$$

Theorem 1.

Let $T$ be a class $p$-$wA(s,t)$ operator with the polar decomposition $T = U|T$. If $T(s,t) = |T|U|T|^\ast$ is quasinormal, then $T$ is also quasinormal. Hence $T$ coincides with its generalized Aluthge transform $T(s,t)$. 

**Proof.** Since $T$ is a class $p$-$A(s,t)$ operator,
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Theorem 1.
\[
\geq W^*|T|^{2rp}W \geq W^*|(T(s, t))^*|^{2rp}W = |T(s, t)|^{2rp}.
\]

Hence
\[
|T(s, t)|^{2rp} = W^*|T(s, t)|^{2rp}W = W^*|T|^{2rp}W,
\]
and
\[
|(T(s, t))^*|^{2rp} = W^*|T(s, t)|^{2rp}W^* = WW^*|T|^{2rp}W^* = \begin{pmatrix} X^{2rp} & 0 \\ 0 & 0 \end{pmatrix}.
\]

Since \(WW^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\), (1), (2) and (3) imply that \(|T(s, t)|^{2rp}\) and \(|T|^{2rp}\) are of the forms
\[
|T(s, t)|^{2rp} = \begin{pmatrix} X^{2rp} & 0 \\ 0 & Y^{2rp} \end{pmatrix} \geq |T|^{2rp} = \begin{pmatrix} X^{2rp} & 0 \\ 0 & Z^{2rp} \end{pmatrix},
\]
where \(\text{ran}(Y) = \text{ran}(Z) = \text{ran}(|T|) \oplus \text{ran}(T(s, t)) = \ker((T(s, t))^*) \oplus \ker(T)\).

Since \(W\) commutes with \(|T(s, t)|\),
\[
\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}.
\]

So \(W_1X = WX_1\) and \(W_2Y = WX_2\), and hence \(\overline{\text{ran}(W_1)}\) and \(\overline{\text{ran}(W_2)}\) are reducing subspaces of \(X\). Since \(W^*W|T(s, t)| = |T(s, t)|\), we have \(W_1^*W_1 = 1\) and
\[
X^k = W_1^*W_1X^k = W_1^*X^kW_1,
\]
\[
Y^k = W_2^*W_2Y^k = W_2^*X^kW_2,
\]
for \(k = 1, 2, \ldots\).

Put \(U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\). Then \(T(s, t) = |T|^sU|T|^t = W|T(s, t)|\) implies
\[
\begin{pmatrix} X^s & 0 \\ 0 & Z^s \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} X^t & 0 \\ 0 & Z^t \end{pmatrix} = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X^{s+t} & 0 \\ 0 & Y^{s+t} \end{pmatrix}.
\]

Hence
\[
X^sU_{11}X^t = W_1X^{s+t} = X^sW_1X^t,
\]
\[
X^sU_{12}Z^t = W_2Y^{s+t} = X^{s+t}W_2
\]
and
\[
X^s(U_{11} - W_1)X^t = 0,
\]
\[
X^s(U_{12}Z^t - X^tW_2) = 0.
\]
Since $X$ is injective and has a dense range, $U_{11} = W_1$ is isometry and $U_{12}Z^t = X^tW_2$. Then
\[
U^*U = \begin{pmatrix}
U_{11}^*U_{11} + U_{21}^*U_{21} & U_{11}^*U_{12} + U_{21}^*U_{22} \\
U_{12}^*U_{11} + U_{22}^*U_{21} & U_{12}^*U_{12} + U_{22}^*U_{22}
\end{pmatrix}
\]
on $\mathcal{H} = \text{ran}(T(s, t)) \oplus \text{ker}((T(s, t))^*)$ is the orthogonal projection onto $\text{ran}([T^*]) \supset \text{ran}(T(s, t))$, we have $U_{21} = 0$ and
\[
U^*U = \begin{pmatrix}
1 & 0 \\
0 & U_{12}^*U_{12} + U_{22}^*U_{22}
\end{pmatrix}.
\]
Since $U_{12}Z^t = X^tW_2$, we have
\[
Z^{2t} = Z^tU_{12}^*U_{12}Z^t = W_2^tX^2W_2 = Y^{2t},
\]
and
\[
Z^{2rp} \geq (Z^tU_{12}^*U_{12}Z^t)^{2p} = (W_2^tX^tW_2)^{2p} = Y^{2rp} \geq Z^{2rp}
\]
by Löwner-Heinz inequality and (4). Hence
\[
(Z^tU_{12}^*U_{12}Z^t)^{2p} = Z^{2rp} = Y^{2rp},
\]
so $Z = Y$ and $|T(s, t)| = |T|^{s+t}$. Since
\[
Z^{2t} = Z^tU_{12}^*U_{12}Z^t \\
\leq Z^tU_{12}^*U_{12}Z^t + Z^tU_{22}^*U_{22}Z^t \leq Z^{2t}
\]
$Z^tU_{22}^*U_{22}Z^t = 0$ and $U_{22}Z^t = 0$. This implies $\text{ran}(U_{12}^*U_{12} \subset \text{ker}(Z))$. Since $\text{ran}(U_{12}^*U_{12} + U_{22}^*U_{22}) \subset \overline{\text{ran}(Z)}$ and $U_{22}^*U_{22} \leq U_{12}^*U_{12} + U_{22}^*U_{22}$, we have $\text{ran}(U_{22}^*) \subset \overline{\text{ran}(Z)}$. Hence
\[
U_{22} = 0, U = \begin{pmatrix}
W_1 & U_{12} \\
0 & 0
\end{pmatrix}
\]
and
\[
\text{ran}(U) \subset \overline{\text{ran}(T(s, t))} \subset \overline{\mathbb{R}([T])} = \text{ran}(E).
\]
Since $W$ commutes with $|T(s, t)| = |T|^{s+t}$, $W$ commutes with $|T|$ and
\[
|T|^s(W - U)|T|^t = W|T|^s|T|^t - |T|^sU|T|^t = W|T(s, t)| - T(s, t) = 0.
\]
Hence $E(W - U)E = 0$ and
\[
U = UE = EUE = EWE = WE = W.
\]
Thus $U = W$ commutes with $|T|$ and $T$ is quasinormal.

**Corollary 1.** Let $T = U|T|$ be a class $p$-$wA(s, t)$ operator. If $T(s, t) = |T|^sU|T|^t$ is normal, then $T$ is also normal.
Proof. Since $T(s, t)$ is normal, $T$ is quasinormal by Theorem 1. Hence $T(s, t) = |T|^s U |T|^t = U |T|^{s+t}$ and $(T(s, t))^* = |T|^{s+t} U^*$. Hence

$|T|^{2(s+t)} = |T(s, t)|^2 = |(T(s, t))^*|^2 = |T^*|^{2(s+t)}$.

This implies $|T| = |T^*|$ and $T$ is normal.

Theorem 2. [25] Let $s_1 > 0$, $s_2 > 0$, $t_1 > 0$, $t_2 > 0$ and $0 < p \leq 1$. If $T$ belongs to class $p_1-wA(s_1, t_1)$ for $0 < p_1 \leq p$ and $T^*$ belongs to class $p_2-wA(s_2, t_2)$ for $0 < p_2 \leq p$, then $T$ is normal.

To prove Theorem 2, we need the following results.

Lemma 1. ([21]) If $T$ is class $p-wA(s, t)$ and $0 < s \leq s_1$, $0 < t \leq t_1$, $0 < p_1 \leq p < 1$, then $T$ is class $p_1-wA(s_1, t_1)$.

Theorem 3 (Furuta theorem [14]). If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $(B^\frac{r}{2} A^p B^\frac{r}{2})^\frac{1}{2} \geq B^{\frac{r+p}{2}}$ and

(ii) $A^{r+p} \geq (A^r B^p A^\frac{s}{2})^\frac{1}{2}$

hold for $p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$.

Proposition 1. ([19]) Let $A \geq 0$ and $B \geq 0$. If

$B^\frac{r}{2} A B^\frac{r}{2} \geq B^2$ and $A^\frac{r}{2} B A^\frac{r}{2} \geq A^2$, \hspace{1cm} (5)

then $A = B$.

Proof. [Proof of Theorem 2] Let $r = \max\{s_1, s_2, t_1, t_2\}$ and let $q = \min\{p_1, p_2\}$.

Firstly, if $T$ belongs to class $p_1-wA(s_1, t_1)$, then $T$ belongs to class $q-wA(r, r)$ by Lemma 1. Hence we have

$(|T^*|^r |T|^{2r} |T^*|^r)^\frac{2}{r} \geq |T^*|^{2rq}$ and $|T|^{2rq} \geq (|T|^r |T^*|^{2r} |T|^r)^\frac{2}{r}$ \hspace{1cm} (6)

Secondly, if $T^*$ belongs to class $p_2-wA(s_2, t_2)$, then $T^*$ belongs to class $q-wA(r, r)$ by Lemma 1. Hence we have

$(|T|^r |T^*|^{2r} |T|^r)^\frac{2}{r} \geq |T|^{2rq}$ and $|T|^{2rq} \geq (|T|^r |T^*|^{2r} |T|^r)^\frac{2}{r}$ \hspace{1cm} (7)

Therefore

$|T^*|^r |T|^{2r} |T^*|^r = |T^*|^{4r}$ and $|T|^{4r} = |T|^r |T^*|^{2r} |T|^r$

hold by (6) and (7), and then $|T| = |T^*|$ by Proposition 1.

The following result is very important in the sequel
Theorem 4. [17, Jensen’s Operator Inequality (JOI)] Suppose that $f$ is a continuous function defined on an interval $I$. Then $f$ is operator convex on an interval $I$ containing $0$ with $f(0) \leq 0$ if and only if $f(a^*xa) \leq a^*f(x)a$ for every self-adjoint $x$ with spectrum in $I$ and every contraction $a$.

Theorem 5. ([11]) Let $A$ and $B$ be bounded linear operators on a Hilbert space $\mathcal{H}$. Then the following are equivalent:

(i) $\text{ran}(A) \subseteq \text{ran}(B);$  
(ii) $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0;$ and
(iii) there exists a bounded linear operator $C$ on $\mathcal{H}$ so that $A = BC$.

Lemma 2. Let $A, B$ and $C$ be positive operators. Then the following assertions hold for each $p \geq 0$, $r \in [0, 1]$ and $0 < q \leq 1$:

(i) If $(B^{r/2}A^pB^{r/2})^{\frac{r}{p+q}} \geq B^q$ and $B \geq C$, then $(C^{r/2}A^pC^{r/2})^{\frac{r}{p+q}} \geq C^q$.

(ii) If $A \geq B$, $B^q \geq (B^{r/2}C^pB^{r/2})^{\frac{r}{p+q}}$ and the condition

\[
\text{if } \lim_{n \to \infty} B^{1/2}x_n = 0 \text{ and } \lim_{n \to \infty} A^{1/2}x_n \text{ exists,}
\]

\[
\text{then } \lim_{n \to \infty} A^{1/2}x_n = 0 \text{ for any sequence of vectors } \{x_n\}
\]

(8)

hold, then $A^q \geq (A^{r/2}C^pA^{r/2})^{\frac{r}{p+q}}$.

Lemma 2 can be obtained as an application of the following results.

Theorem 6. ([11]) Let $A$ and $B$ be bounded linear operators on a Hilbert space $\mathcal{H}$. Then the following are equivalent:

(i) $\text{ran}(A) \subseteq \text{ran}(B);$  
(ii) $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0;$ and
(iii) there exists a bounded linear operator $C$ on $\mathcal{H}$ so that $A = BC$.

Moreover, if (i), (ii) and (iii) are valid, then there exists a unique operator $C$ so that

(a) $\|C\|^2 = \inf\{\mu : AA^* \leq \mu BB^*\};$

(b) $\text{ker}(A) = \text{ker}(C);$ and

(c) $\text{ran}(C) \subseteq \overline{\text{ran}(B^*)}$.

Theorem 7. ([16]) Let $X$ and $A$ be bounded linear operator on a Hilbert space $\mathcal{H}$. We suppose that $A \geq 0$ and $\|X\| \leq 1$. If $f$ is an operator monotone function defined on $[0, \infty)$, then

\[X^*f(A)X \leq f(X^*AX).\]
We remark that the condition (c) of Theorem 6 is equivalent to (c‘):
\[ \text{ran}(C) \subseteq \text{ran}(B^*). \]
Here we consider when the equality of (c‘) holds.

**Lemma 3. ([33])** Let \( A \) and \( B \) be operators which satisfy (i), (ii) and (iii) of Theorem 6 and \( C \) be the operator which is given in (iii) and determined uniquely by (a), (b) and (c) of Theorem 6. Then the following assertions are mutually equivalent:

(i) \( \text{ran}(C) = \text{ran}(B^*). \)

(ii) If \( \lim_{n \to \infty} A^* x_n = 0 \) and \( \lim_{n \to \infty} B^* x_n \) exists, then \( \lim_{n \to \infty} B^* x_n = 0 \) for any sequence of vectors \( \{x_n\}. \)

We also prepare the following lemma in order to give a proof of Lemma 2.

**Lemma 4. ([33])** Let \( S \) be a positive operator and \( 0 < q \leq 1 \). If \( \lim_{n \to \infty} S x_n = 0 \) and \( \lim_{n \to \infty} S^q x_n \) exists, then \( \lim_{n \to \infty} S^q x_n = 0 \) for any sequence of vectors \( \{x_n\}. \)

**Proof.** [Proof of Lemma 2] (i) The hypothesis \( B \geq C \) ensures then \( B^t \geq C^t \) for each \( t \in (0,1] \) by Löwner-Heinz theorem. By Theorem 6, there exists an operator \( X \) with \( \|X\| \leq 1 \) such that
\[ B^\frac{t}{2} X = X^* B^\frac{t}{2} = C^\frac{t}{2}. \]

Then we have
\[ (C^{r/2} A^p C^{r/2})^{\frac{r}{p+r}} = (X^* B^{r/2} A^p B^{r/2} X)^{\frac{r}{p+r}} \]
\[ \geq X^*(B^{r/2} A^p B^{r/2})^{\frac{r}{p+r}} X \quad \text{(by Theorem 7)} \]
\[ \geq X^* B^r X \quad \text{(by the hypothesis)} \]
\[ = (C^{\frac{r}{2}} C^{\frac{r}{2}})^q = C^{rq} \quad \text{(by Equation (9))}. \]

(ii) The hypothesis \( A \geq B \) ensures \( A^s \geq B^s \) for \( s \in (0,1] \) by Löwner-Heinz theorem. By Theorem 6, there exists an operator \( X \) with \( \|X\| \leq 1 \) such that
\[ A^{s/2} X = X^* A^{s/2} = B^{s/2}. \]

Then we have
\[ X^*(A^{r/2} C^p A^{r/2})^{\frac{r}{p+r}} X \leq (X^* A^{r/2} C^p A^{r/2} X)^{\frac{r}{p+r}} \quad \text{(by Theorem 7)} \]
\[ = (B^{r/2} C^p B^{r/2})^{\frac{r}{p+r}} \]
\[ \leq B^{rq} \quad \text{(by the hypothesis)} \]
\[ = (B^r)^q = (X^* A^{\frac{r}{2}} A^{\frac{r}{2}} X)^q \leq X^* A^{rq} X \quad \text{(by Theorem 4)} \]
so that \( A^{rq} \geq (A^{r/2} C^p A^{r/2})^{\frac{r}{p+r}} \) holds on \( \text{ran}(X) \). On the other hand, the hypothesis (8) implies the following (11)
\[ \text{If } \lim_{n \to \infty} B^{r/2} x_n = 0 \text{ and } \lim_{n \to \infty} A^{r/2} x_n \text{ exists,} \]
Then by Theorem 3 it follows that
\[ \text{ran}(X) = \text{ran}(A^{r/2}) \] by Lemma 3, hence we have
\[ \ker((A^{r/2}C^pA^{r/2})_{\frac{r^p}{n}}) = \ker(A^{r/2}C^pA^{r/2}) \]
\[ \supseteq \ker(A^{r/2}) = \ker(A^r) = \ker(X^r) = \ker(X^s), \]
so that \( A^{\frac{r^p}{n}} = (A^{r/2}C^pA^{r/2})_{\frac{r^p}{n}} = 0 \) holds on \( \ker(X^s) \). Consequently the proof is complete since \( H = \text{ran}(X) \oplus \ker(X^s) \).

**Lemma 5.** ([26]) Let \( T = U|T| \in B(H) \) be the polar decomposition of \( T \). Then \( T \) is class \( p-wA(s, t) \) if and only if \( |T(s, t)|^{\frac{2p}{p+1}} \geq |T|^{2p} \) and \( |T|^{2p} \geq \|T(s, t)\|^{\frac{2p}{p+1}} \).

**Lemma 6.** Let \( 0 < s, t, s + t \leq 1 \) and \( 0 < p \leq 1 \). Let \( T \in B(H) \) be class \( p-wA(s, t) \) and let \( M \) an invariant subspace of \( T \). Then the restriction \( T|_M \) is also class \( p-wA(s, t) \).

**Proof.** Let \( T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix} \) on \( H = M \oplus M^\perp \) and \( P \) the orthogonal projection onto \( M \). Let \( T_0 := TP = PTP = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \). Then
\[ |T_0|^{2t} = (P|T|^{2p})^t \geq P|T|^{2t} \]
for each \( 0 < t \leq 1 \)
by Hansen’s inequality, and
\[ |T^s|^2 = TT^* \geq TPT^* = |T_0|^2. \]
Hence
\[ T \text{ is class } p-A(s, t) \iff |T^s|^{2tp} \leq (|T^s|^t|T|^{2s}|T^s|^t)^{\frac{2p}{p+1}} \]
\[ \Rightarrow |T_0|^{2tp} \leq (|T_0^s|^t|T_0|^{2s}|T_0^s|^t)^{\frac{2p}{p+1}} \text{ (by Lemma 2)} \]
\[ \Rightarrow |T_0|^{2tp} \leq (|T_0^s|^t|T_0|^{2s}|T_0^s|^t)^{\frac{2p}{p+1}} \text{ (since } |T_0^s|^t = |T_0^s|^t P = P|T_0^s|^t \text{ for every } 0 < t \leq 1). \]
Now
\[ |T_0| = P|\widetilde{T}|P \geq P|T|P \geq P|\widetilde{T}|^s P = |T_0^s| |P = |T_0^s|. \]
Then by Theorem 3 it follows that
\[ |T_0|^{2tp} \geq (|T_0^s|^t|T_0^s|^{2t}|T_0^s|^s)^{\frac{2p}{p+1}}. \]
Therefore, \( T|_M \) is class \( p-A(s, t) \) operator.

The following example shows that there exists a class \( p-wA(s, t) \) operator \( T \) such that \( T|_M \) is quasinormal but \( M \) does not reduce \( T \).
Example 1. Let $T$ be a bilateral shift on $\ell^2(\mathbb{Z})$ defined by $Te_n = e_{n+1}$ and $\mathcal{M} = \bigvee_{n \geq 0} \mathbb{C}e_n$. Then $T$ is unitary and $T|_{\mathcal{M}}$ is isometry. However, $\mathcal{M}$ does not reduce $T$.

Lemma 7. Let $0 < s, t, s+t = 1$ and $0 < p \leq 1$. Let $T \in \mathcal{B}(\mathcal{H})$ be class $p$-$wA(s, t)$ operator, let $\mathcal{M}$ be an invariant subspace for $T$ and a reducing subspace for $T(s, t)$ such that $T(s, t)|_{\mathcal{M}}$ the restriction of $T(s, t)$ to $\mathcal{M}$ is an injective normal operator, then $T|_{\mathcal{M}} = T(s, t)|_{\mathcal{M}}$ and $\mathcal{M}$ reduces $T$.

Proof. Let

$$T(s, t) = \begin{pmatrix} T_0 & 0 \\ 0 & A \end{pmatrix}, \quad T = \begin{pmatrix} S & B \\ 0 & D \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$ 

Since $T$ is class $p$-$wA(s, t)$ we have $|T(s, t)|^{2rp} \geq |T|^{2rp} \geq |(T(s, t))^*|^{2rp}$ for $r \in \min\{s, t\}$. Let $P$ be the orthogonal projection onto $\mathcal{M}$. Then

$$|T_0| = P|T(s, t)|P \geq P|T|P \geq P|(T(s, t))^*|P = |T_0|^*.$$ 

By Löwner-Heinz theorem we get

$$|T_0|^{2rp} = P|T(s, t)|^{2rp}P \geq P|T|^{2rp}P \geq P|(T(s, t))^*|^{2rp}P = |T_0|^{2rp}. $$

Since $|T|^*T = T(s, t)|T|^*$ and $P|T|^*P = |T_0|^*$, we deduce that

$$|T_0|^*S = T_0|T_0|^*.$$ 

We have $T_0$ is an injective normal operator, then $S = T|_{\mathcal{M}} = T_0 = T(s, t)|_{\mathcal{M}}$, consequently

$$T = \begin{pmatrix} T_0 & B \\ 0 & D \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$ 

Hence

$$T^*T = \begin{pmatrix} T_0^*T_0 & T_0^*B \\ B^*T_0 & B^*B + D^*D \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$ 

So we can write

$$|T|^p = \begin{pmatrix} |T_0|^p & X \\ X^* & Y \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$ 

Since

$$P|T|^p|T|^pP = |T_0|^{2rp},$$

then $|T_0|^{2rp} = |T_0|^{2rp} + XX^*$, and thus $X = 0$.

It follows that $|T|^p = |T_0|^p + Y^2$ implying $|T|^{2rp} = |T_0|^{2rp} + Y^4$. Consequently we get $B^*B = 0$ it follows that $B = 0$ and hence $\mathcal{M}$ reduces $T$.

The next lemma is a simple consequence of the preceding one.
Lemma 8. Let $0 < s, t, s + t = 1$ and $0 < p \leq 1$. Let $T \in \mathcal{B}(\mathcal{H})$ be a class $p$-w$A(s, t)$ operator with $\ker(T) \subset \ker(T^*)$. Then $T = T_1 \oplus T_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $T_1$ is normal, $\ker(T_2) = \{0\}$ and $T_2$ is pure class $p$-w$A(s, t)$ i.e., $T_2$ has no non-zero invariant subspace $\mathcal{M}$ such that $T_2|_{\mathcal{M}}$ is normal.

Lemma 9. Let $0 < s, t, s + t = 1$ and $0 < p \leq 1$. Let $T = U[T] \in \mathcal{B}(\mathcal{H})$ be class $p$-w$A(s, t)$ and $\ker(T) \subset \ker(T^*)$. Suppose $T(s, t) = |T|sU|T|t$ be of the form $N \oplus T^*$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where $N$ is a normal operator on $\mathcal{M}$. Then $T = N \oplus T_1$ and $U = U_{11} \oplus U_{22}$ where $T_1$ is class $p$-w$A(s, t)$ with $\ker(T_1) \subset \ker(T_1^*)$ and $N = U_{11}|N|$ is the polar decomposition of $N$.

Proof. Since

$$|T(s, t)|^{2rp} \geq |T|^{2rp} \geq |(T(s, t))^*|^{2rp}$$

for $r \in \min\{s, t\}$, we have

$$|N|^{2rp} \oplus |T|^*|^{2rp} \geq |T|^{2rp} \geq |N|^{2rp} \oplus |T|^*|^{2rp}$$

by assumption. This implies that $|T|$ is of the form $|N| \oplus L$ for some positive operator $L$.

Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ be $2 \times 2$ matrix representation of $U$ with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Then the definition $T(s, t)$ means

$$\begin{pmatrix} N & 0 \\ 0 & T^* \end{pmatrix} = \begin{pmatrix} |N|s & 0 \\ 0 & L^* \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} |N|^t & 0 \\ 0 & L^t \end{pmatrix}$$

Hence, we have

$$N = |N|^sU_{11}|N|^t, \quad |N|^sU_{12}L^t = 0 \quad \text{and} \quad L^sU_{21}|N|^t = 0.$$ 

Since $\ker(T) \subset \ker(T^*)$,

$$\overline{\text{ran}(U)} = \overline{\text{ran}(T)} = \ker(T^*)^\perp \subset \ker(T)^\perp = \overline{\text{ran}(|T|)}.$$ 

Let $Nx = 0$ for $x \in \mathcal{M}$. Then $x \in \ker(|T|) = \ker(U)$, and

$$Ux = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}x \\ U_{21}x \end{pmatrix} = 0.$$ 

Hence

$$\ker(N) \subset \ker(U_{11}) \cap \ker(U_{21}).$$

Let $x \in \mathcal{M}$. Then

$$U \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}x \\ U_{21}x \end{pmatrix} \in \overline{\text{ran}(|T|)} = \overline{\text{ran}(|N| \oplus L)}.$$ 

Hence

$$\ker(U_{11}) \subset \ker(|N|), \quad \ker(U_{21}) \subset \overline{\text{ran}(L)}.$$
Similarly
\[ \text{ran}(U_{12}) \subset \text{ran}(|N|), \ \text{ran}(U_{22}) \subset \overline{\text{ran}(L)}. \]

Let \( Lx = 0 \) for \( x \in M^{\perp}. \) Then \( x \in \ker(|T|) = \ker(U) \) and
\[
U \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} U_{12}x \\ U_{22}x \end{pmatrix} = 0
\]

Hence
\[ \ker(L) \subset \ker(U_{12}) \cap \ker(U_{22}). \]

Let \( N = V|N| \) be the polar decomposition of \( N. \) Then
\[ (V|N|^* - |N|^*U_{11})|N|^t = 0. \]

Hence \( V|N|^* - |N|^*U_{11} = 0 \) on \( \overline{\text{ran}(|N|)}. \) Since \( \ker(N) \subset \ker(U_{11}), \) this implies \( 0 = V|N|^* - |N|^*U_{11} = |N|^*(V - U_{11}). \) Hence
\[ \text{ran}(V - U_{11}) \subset \ker(|N|) \cap \overline{\text{ran}(|N|)} = \{0\}. \]

Hence \( V = U_{11} \) and \( N = U_{11}|N| \) is the polar decomposition of \( N. \) Since \( |N|^*U_{12}L^t = 0, \)
\[ \text{ran}(U_{11}L^t) \subset \ker(|N|) \cap \overline{\text{ran}(|N|)} = \{0\}. \]

Hence \( U_{12}L^t \) and \( U_{12} = 0. \) Similarly we have \( U_{21} = 0 \) by \( L^*U_{21}|N|^t = 0. \) Hence \( U = U_{11} \oplus U_{22}. \) So we obtain
\[ T = U|T| = U_{11}|N| \oplus U_{22}L = N \oplus T_1, \]
where \( T_1 = U_{22}L. \)

3. Quasisimilarity

An operator \( X \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) is called quasiaffinity if \( X \) is both injective and has a dense range. For \( T \in \mathcal{B}(\mathcal{H}) \) and \( S \in \mathcal{B}(\mathcal{K}), \) if there exist quasiaffinities \( X \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) and \( Y \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) such that \( TX = XS \) and \( YT = SY, \) then we say that \( T \) and \( S \) are quasisimilar. The operator \( T \in \mathcal{B}(\mathcal{H}) \) is said to be pure if there exists no nontrivial reducing subspace \( \mathcal{M} \) of \( \mathcal{H} \) such that the restriction of \( T \) to \( \mathcal{M} \) is normal and is completely hyponormal if it is pure. Recall that every operator \( T \in \mathcal{B}(\mathcal{H}) \) has a direct sum decomposition \( T = T_1 \oplus T_2, \) where \( T_1 \) and \( T_2 \) are normal and pure parts, respectively. Of course in the sum decomposition, either \( T_1 \) or \( T_2 \) may be absent. The following lemma is due to Williams [32, Lemma 1.1].

Lemma 10. Let \( T \in \mathcal{B}(\mathcal{H}) \) and \( S \in \mathcal{B}(\mathcal{K}) \) be normal operators. It there exist injective operators \( X \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) and \( Y \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) such that \( TX = XS \) and \( YT = SY, \) then \( T \) and \( S \) are unitarily equivalent.
Corollary 2. Let $T \in \mathcal{B}(\mathcal{H})$ be class $p$-$wA(s,t)$ operator for $0 < s, t, s + t = 1$ and $0 < p \leq 1$. Then $T = T_1 \oplus T_2$ on the space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $T_1$ is normal and $T_2$ is pure and class $p$-$wA(s,t)$, i.e., $T_2$ has no invariant subspace $\mathcal{M}$ such that $T_2|_{\mathcal{M}}$ is normal.

The next result was proved for dominant operators in [28, Theorem 1], for $p$-hyponormal operators in [20] and for $w$-hyponormal operators in [22, Lemma 2.12].

Proposition 2. Let $T \in \mathcal{B}(\mathcal{H})$ be class $p$-$wA(s,t)$ operator for $0 < s, t, s + t = 1$ and $0 < p \leq 1$ such that $\ker(T) \subset \ker(T^*)$ and let $S \in \mathcal{B}(\mathcal{K})$ be a normal operator. If there exists a quasiaffinity $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ with dense range such that $TX = XS$, then $T$ is normal.

To prove Proposition 2, we need the following lemmas.

Lemma 11. [9] If $N$ is a normal operator on $\mathcal{H}$, then we have

$$\bigcap_{\lambda \in \mathbb{C}} (N - \lambda)\mathcal{H} = \{0\}.$$ 

Lemma 12. ([10]) Let $T \in \mathcal{B}(\mathcal{H})$, $D \in \mathcal{B}(\mathcal{H})$ with $0 \leq D \leq M(T - \lambda)(T - \lambda)^*$ for all $\lambda \in \mathbb{C}$, where $M$ is a positive real number. Then for every $x \in D\frac{1}{2}\mathcal{H}$ there exists a bounded function $f : \mathbb{C} \rightarrow \mathcal{H}$ such that $(T - \lambda)f(\lambda) \equiv x$.

**Proof.** [Proof of Proposition 2] $\ker(T) \subset \ker(T^*)$ implies $\ker(T)$ reduces $T$. Also $\ker(S)$ reduces $S$ since $S$ is normal. Using the orthogonal decompositions $\mathcal{H} = \overline{\text{ran}(T)} \oplus \ker(T)$ and $\mathcal{H} = \overline{\text{ran}(S)} \oplus \ker(S)$, we can represent $T$ and $S$ as follows: $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, $S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $T_1$ is an injective class $p$-$wA(s,t)$ operator on $\overline{\text{ran}(T)}$ and $S_1$ is injective normal on $\overline{\text{ran}(S)}$. The assumption $TX = XS$ asserts that $X$ maps $\text{ran}(S)$ to $\text{ran}(T) \subset \overline{\text{ran}(T)}$ and $\ker(S)$ to $\ker(T)$, hence $X$ is the form: $X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$, where $X_1 \in \mathcal{B}(\overline{\text{ran}(S)}, \overline{\text{ran}(T)})$, $X_2 \in \mathcal{B}(\ker(S), \ker(T))$. Since $TX = XS$, we have that $T_1 X_1 = X_1 S_1$. Since $X$ is injective with dense range, $X_1$ is also injective with dense range. Put $W_1 = [T_1^* X_1$, then $W_1$ is also injective with dense range and satisfies $T(s,t)W_1 = W_1 S$. Put $W_n = (\Delta^n(T(s,t))W_{n-1}$, then $W_n$ is also injective with dense range and satisfies $\Delta^n(T(s,t))W_n = W_n S$. From [26, Corollary 2.7] and [6], if there exists an integer $m$ such that $\Delta^m(T(s,t))$ is a hyponormal operator, then $\Delta^n(T(s,t))$ is a hyponormal operator for $n \geq m$. It follows from Lemma 12 that there exists a bounded function $f : \mathbb{C} \rightarrow \mathcal{H}$ such that $(\Delta^n(T_1(s,t))^* - \lambda)f(\lambda) \equiv x$, for every $x \in (\Delta^n(T_1(s,t))^* \Delta^n(T_1(s,t) - \Delta^n(T_1(s,t))(\Delta^n(T_1(s,t))^*) \frac{1}{2}\mathcal{H})$. Hence

$$W_n^* x = W_n^* (\Delta^n(T_1(s,t))^* - \lambda)f(\lambda)$$

$$= (S_1^* - \lambda)W_n^* f(\lambda) \in \text{ran}(S_1^* - \lambda)$$

for all $\lambda \in \mathbb{C}$. 

By Lemma 11, we have $W_n^* x = 0$, and hence $x = 0$ because $W_n^*$ is injective. This implies that $\Delta^n(T_1(s, t))$ is normal. By Corollary 1, $T_1$ is normal and therefore $T = T_1 \oplus 0$ is also normal.

**Theorem 8.** Let $T$ and $S^*$ be class $p$-$wA(s, t)$ operators with $0 < s, t, s + t = 1$ and $0 < p \leq 1$ such that $\ker(T) \subset \ker(T^*)$ and $\ker(S^*) \subset \ker(S)$. If there exist a quasi-affinity $X$ such that $TX = XS$, then $T$ and $S$ are unitarily equivalent normal operators.

**Proof.** First decompose $T$ and $S^*$ into their normal and pure parts by $T = T_1 \oplus T_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $S^* = S_1^* \oplus S_2^*$ on $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, where $T_1, S_1$ are normal and $T_2, S_2^*$ are pure. Let $X = [X_{ij}]_{i,j=1}^2$. Then $TX = XS$ implies that $T_2X_{21} = X_{21}S_1$ and $T_2X_{22} = X_{22}S_2$. Let $T_2 = U_2|T_2|$, $S_2 = V_2^*|S_2^*|$ be the polar decompositions of $T_2$ and $S_2^*$, respectively and

$$T_2(s, t) = |T_2|^s U_2|T_2|^t, \quad S_2(s, t) = |S_2|^s V_2^*|S_2^*|^t, \quad W = |T_2|^s X_{22}|S_2|^t.$$  

Then

$$T_2(s, t)W = |T_2|^s T_2X_{22}|S_2|^t = |T_2|^s X_{22}S_2|S_2^*|^t = W(S_2^*(s, t))^t.$$  

Since $\text{ran}(W^*)$ reduces $T_2(s, t)$ and $\ker(W)^\perp$ reduces $S_2^*(s, t)$, and $T_2(s, t)|_{\text{ran}(W^*)}$ and $S_2(s, t)|_{\ker(W)^\perp}$ are unitarily equivalent normal operators, and since $T_2, S_2$ are injective class $p$-$wA(s, t)$ operators, we have $T_2|_{\text{ran}(W^*)} = T_2(s, t)|_{\text{ran}(W^*)}$ and $S_2|_{\ker(W)^\perp} = S_2^*(s, t)|_{\ker(W)^\perp}$ by Lemma 9. Since $T_2, S_2^*$ are pure, it implies $W|T_2|^s X_{22}|S_2|^t = 0$. Hence $X_{22} = 0$. Similarly $X_{12} = 0, X_{21} = 0$. Hence $X = X_{11}$ and $S, T$ are unitarily equivalent normal operators.

The following lemma is due to Williams [32, Lemma 1.1]

**Lemma 13.** Let $N_1 \in \mathcal{B}(\mathcal{H})$ and $N_2 \in \mathcal{B}(\mathcal{K})$ be normal. If $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are injective such that $N_1X = XN_2$ and $YN_1 = N_2Y$, then $N_1$ and $N_2$ are unitarily equivalent.

Stampfli and Wadhwa [28] proved that the normal parts of quasisimilar dominant operators are unitarily equivalent. This result was generalized to classes of $p$-hyponormal operators in [12]. We prove that theses results hold for class $p$-$wA(s, t)$ operators.

**Theorem 9.** Suppose that $0 < s, t, s + t = 1$ and $0 < p \leq 1$. For each $i = 1, 2$, let $T_i \in \mathcal{B}(\mathcal{H}_i)$ be class $p$-$wA(s, t)$ operators such that $\ker(T_j) \subset \ker(T_j^*)$ and let $T_i = N_i \oplus V_i$ on $\mathcal{H}_i = \mathcal{H}_{i1} \oplus \mathcal{H}_{i2}$, where $N_i$ and $V_i$ are the normal and pure parts, respectively of $T_i$. If $T_1$ and $T_2$ are quasisimilar, then $N_1$ and $N_2$ are unitarily equivalent and there exist $X_s \in \mathcal{B}(\mathcal{H}_{22}, \mathcal{H}_{12})$ and $Y_s \in \mathcal{B}(\mathcal{H}_{12}, \mathcal{H}_{22})$ having dense range such that $V_1X_s = X_sV_2$ and $Y_sV_1 = V_2Y_s$. 

Proof. By hypothesis there exist quasiadjoints \( X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \) and \( Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) such that \( T_1 X = X T_2 \) and \( Y T_1 = T_2 Y \). Let

\[
X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}
\]

with respect to \( \mathcal{H}_2 = \mathcal{H}_{21} \oplus \mathcal{H}_{22} \) and \( \mathcal{H}_1 = \mathcal{H}_{11} \oplus \mathcal{H}_{12} \), respectively. A simple calculation shows that

\[
V_1 X_3 = X_3 N_2 \quad \text{and} \quad V_2 Y_3 = Y_3 N_1.
\]

We claim that \( X_3 = Y_3 = 0 \). Let \( \mathcal{M} = \text{ran}(X_3) \). Then \( \mathcal{M} \) is a non-trivial invariant subspace of \( V_1 \). Since \( V_1^* X_3 = X_3 N_2^* \) by Proposition 2, \( \mathcal{M} \) is an invariant subspace of \( V_1^* \). Hence \( \mathcal{M} \) reduces \( V_1, \sigma(V_1|_\mathcal{M}) \subset \sigma(V_1) \) and \( V_1|_\mathcal{M} \) is invertible. Let \( V_1' = V_1|_\mathcal{M} \) and define an operator \( X_3' : \mathcal{H}_{12} \rightarrow \mathcal{M} \) by \( X_3' x = X_3 x \) for each \( x \in \mathcal{H}_{12} \). Then \( V_1' \) is class \( p-wA(s,t) \) by Lemma 6, so that \( X_3' \) has dense range and satisfies \( V_1' X_3' = X_3' N_2 \). Hence \( V_1' \) is normal by Proposition 2. Since \( V_1 \) is pure, this implies that \( \mathcal{M} = \{0\} \) and \( X_3 = 0 \). Similarly, we have \( Y_3 = 0 \). Hence \( X_1 \) and \( Y_1 \) are injective.

Since \( N_1 X_1 = X_1 N_2 \) and \( Y_1 N_1 = N_2 Y_1 \), \( N_1 \) and \( N_2 \) are unitarily equivalent, by Lemma 13. Also, \( X_4 \) and \( Y_4 \) have dense ranges. Hence \( V_1 X_4 = X_4 V_2 \) and \( Y_4 V_1 = Y_4 V_4 \), so the proof is complete.

Corollary 3. Let \( T_1 \in \mathcal{B}(\mathcal{H}_1) \) and \( T_2 \in \mathcal{B}(\mathcal{H}_2) \) be quasisimilar class \( p-wA(s,t) \) operators for \( 0 < s, t, s + t = 1 \) and \( 0 < p \leq 1 \). If \( T_1 \) is pure, then \( T_2 \) is also pure.

Corollary 4. Let \( T_1 \in \mathcal{B}(\mathcal{H}_1) \) be class \( p-wA(s,t) \) operators for \( 0 < s, t, s + t = 1 \) and \( 0 < p \leq 1 \) and \( T_2 \in \mathcal{B}(\mathcal{H}_2) \) be normal. If \( T_1 \) and \( T_2 \) are quasisimilar, then \( T_1 \) and \( T_2 \) are unitarily equivalent normal operators.

4. The Fuglede-Putnam Theorem

We offer various results related to the Fuglede-Putnam theorem in this section. If \( T^* X = X S^* \) whenever \( TX = XS \) for every \( X \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \), a pair \((T, S)\) is said to have the Fuglede-Putnam property. In operator theory, the Fuglede-Putnam theorem is well-known. It claims that the pair \((T, S)\) possesses the Fuglede-Putnam property for any normal operators \( T \) and \( S \). There are several generalizations of this theorem, the majority of which loosen the normality of \( T \) and \( S \); see, for example, \([22–24, 27, 28]\), and some references therein and for more details (see [3],[5],[4]). The Fuglede-Putnam theorem is the subject of the next lemma, which we will require in the future.

Lemma 14. ([29]) Let \( T \in \mathcal{B}(\mathcal{H}) \) and \( S \in \mathcal{B}(\mathcal{K}) \). Then the following assertions equivalent.

(i) The pair \((T, S)\) has the Fuglede-Putnam property.

(ii) If \( TX = XS \), then \( \text{ran}(X) \) reduces \( T \), \( \ker(X) \) reduces \( S \), and \( T|_{\text{ran}(X)} \), \( S|_{\ker(X)\perp} \) are unitarily equivalent normal operators.
Remark 1. A necessary condition for the pair \((T, T^*)\) to satisfy Fuglede-Putnam’s theorem is \(\ker(T) \subseteq \ker(T^*)\). Since for a class \(p\)-\(w\)(s, t) operator this is not always true, class \(p\)-\(w\)(s, t) operator do not Fuglede-Putnam’s theorem. For example, if \(P\) is the orthogonal projection onto \(\ker(T)\), with \(T\) is class \(p\)-\(w\)(s, t), then \(TP = PT^*\) but \(T^*P \neq PT^*\). The following result (Corollary 6) prove that if \(T^*\), \(S\) are \(p\)-class \(A(s, t)\) operators for \(0 < s, t, s + t = 1\) and \(0 < p \leq 1\) such that \(\ker(T^*)\) reduces \(T^*\) and \(\ker(S)\) reduces \(S\), then the pair \((T, S)\) satisfy Fuglede-Putnam’s theorem.

Theorem 10. Let \(T \in B(H)\) be class \(p\)-\(w\)(s, t) operator for \(0 < s, t, s + t = 1\) and \(0 < p \leq 1\) and \(\ker(T) \subseteq \ker(T^*)\). If \(L\) is self-adjoint and \(TL = LT^*\), then \(T^*L = LT\).

Proof. Since \(\ker(T) \subseteq \ker(T^*)\) and \(TL = LT^*\), \(\ker(T)\) reduces \(T\) and \(L\). Hence

\[
T = T_1 \oplus 0, \quad L = L_1 \oplus L_2 \quad \text{on} \quad H = \overline{\text{ran}(T^*)} \oplus \ker(T),
\]

\(T_1L_1 = L_1T^*\) and \(\{0\} = \ker(T_1) \subseteq \ker(T_1^*)\). Since \(\text{ran}(L_1)\) is invariant under \(T_1\) and reduces \(L_1\),

\[
T = \begin{pmatrix} T_{11} & S \\ 0 & T_{22} \end{pmatrix}, \quad L_1 = L_{11} \oplus 0 \quad \text{on} \quad H = \overline{\text{ran}(T^*)} = \overline{\text{ran}(L_1)} \oplus \ker(L_1).
\]

\(T_{11}\) is an injective class \(p\)-\(w\)(s, t) operator by Lemma 6 and \(L_{11}\) is an injective self-adjoint operator (hence it has dense range) such that \(T_{11}L_{11} = L_{11}T_{11}^*\). Let \(T_{11} = V_{11}|T_{11}|\) be the polar decomposition of \(T_{11}\) and \(T_{11}(s, t) = |T_{11}|^sV_{11}|T_{11}|^t\), \(W = |T_{11}|^sL_{11}|T_{11}|^t\). Then

\[
T_{11}(s, t)W = |T_{11}|^sV_{11}|T_{11}|^t|T_{11}|^sL_{11}|T_{11}|^t
= |T_{11}|^sT_{11}L_{11}|T_{11}|^t
= |T_{11}|^sL_{11}T_{11}^*|T_{11}|^t
= |T_{11}|^sL_{11}|T_{11}|^t|V_{11}|T_{11}|^s
= W(T_{11}(s, t))^t.
\]

Since \(T_{11}(s, t)\) is \(\min\{sp, tp\}\)-hyponormal and \(\text{ran}(W)\) is dense (because \(\ker(W) = \{0\}\)), \(T_{11}(s, t)\) is normal by [12, Theorem 7]. Hence \(T_{11}\) is normal and \(T_{11} = T_{11}(s, t)\) by Corollary 1. Then \(\text{ran}(L_1)\) reduces \(T_1\) by Lemma 7 and \(T_{11}L_{11} = L_{11}T_{11}\) by Lemma 14. Hence

\[
T = T_{11} \oplus T_{22} \oplus 0, \quad L = L_{11} \oplus 0 \oplus L_2
\]

and

\[
T^*L = T_{11}^*L_{11} \oplus 0 \oplus 0 = L_{11}T_{11} \oplus 0 \oplus 0 = LT.
\]

Example 2. Let \(H = \bigoplus_{n=0}^{\infty} \mathbb{C}^2\) and define an operator \(R\) on \(H\) by

\[
R(\cdots \oplus x_{-2} \oplus x_{-1} \oplus x_0^{(0)} \oplus x_1 \oplus \cdots) = \cdots \oplus Ax_{-2} \oplus Ax_{-1}^{(0)} \oplus Bx_0 \oplus Bx_1 \oplus \cdots,
\]
where
\[ A = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]

Then \( R \) is a class \( p\)-\( w \)(\( s,t \)). Moreover, \( \text{ran}(E) = \ker(R) \), \( E \) is not a self-adjoint and \( \ker(R) \neq \ker(R^*) \), where \( E \) is the Riesz idempotent with respect to 0, see [31, Example 13]. Let \( T = R \) and \( L = P \) be the orthogonal projection onto \( \ker(T) \). Then \( T \) is a class \( p\)-\( w \)(\( s,t \)) operator and \( TL = 0 = LT^* \), but \( T^*L \neq LT \). Hence the kernel condition \( \ker(T) \subset \ker(T^*) \) is necessary for Theorem 10.

**Corollary 5.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be a class \( p\)-\( w \)(\( s,t \)) operator for \( 0 < s,t,s + t = 1 \) and \( 0 < p \leq 1 \) and \( \ker(T) \subset \ker(T^*) \). If \( TX = XT^* \) for some \( X \in \mathcal{B}(\mathcal{H}) \) then \( T^*X = XT \).

**Proof.** Let \( X = L + iJ \) be the Cartesian decomposition of \( X \). Then we have \( TL = LT^* \) and \( TJ = JT^* \) by the assumption. By Theorem 10, we have \( T^*L = LT \) and \( T^*J = JT \). This implies that \( T^*X = XT \).

If we use the \( 2 \times 2 \) matrix trick, we easily deduce the following result.

**Corollary 6.** Suppose that \( 0 < s,t,s + t = 1 \) and \( 0 < p \leq 1 \). Let \( T^* \in \mathcal{B}(\mathcal{H}) \) be a class \( p\)-\( w \)(\( s,t \)) operator and \( S \in \mathcal{B}(\mathcal{K}) \) be a class \( p\)-\( w \)(\( s,t \)) operator with \( \ker(T^*) \subset \ker(T) \) and \( \ker(S) \subset \ker(S^*) \). If \( X \in \mathcal{B}(\mathcal{H},\mathcal{K}) \) and \( XT = SX \), then \( XT^* = S^*X \).

**Proof.** Put \( A = \begin{pmatrix} T^* & 0 \\ 0 & S \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \) on \( \mathcal{H} \oplus \mathcal{K} \). Then \( A \) is a class \( p\)-\( w \)(\( s,t \)) operator on \( \mathcal{H} \oplus \mathcal{K} \) that satisfies \( BA^* = AB \) and \( \ker(A) \subset \ker(A^*) \). Hence we have \( BA = A^*B \), by Corollary 5, and so \( XT^* = S^*X \).

**Example 3.** Let \( S = T^* = R \) as in Example 2 and \( X = P \) be the orthogonal projection onto \( \ker(S) \). Then \( SX = 0 = XT \), but \( S^*X \neq XT^* \). Hence the kernel condition is necessary for Corollary 6.

As an application of Corollary 6, we establish the following result.

**Corollary 7.** Suppose that \( 0 < s,t,s + t = 1 \). Let \( T \in \mathcal{B}(\mathcal{H}) \) and \( S^* \in \mathcal{B}(\mathcal{K}) \) be class \( p\)-\( w \)(\( s,t \)) and \( \ker(T) \subset \ker(T^*) \), \( \ker(S^*) \subset \ker(S) \). Let \( TX = XS \) for some operator \( X \in \mathcal{B}(\mathcal{K},\mathcal{H}) \). Then \( \text{ran}(X) \) reduces \( T \), \( \ker(S^*) \) reduces \( S \) and \( T^{\dagger}_{\text{ran}(X)} \), \( S^{\dagger}_{\ker(X)\dagger} \) are unitarily equivalent normal operators.

**Proof.** By Corollary 6, \( T^*X = XS^* \). Therefore \( T^*TX = XS^*S \) and so \( |T|X = X|S| \). Let \( T = U|T|, S = V|S| \) be the polar decomposition. Then \( UX|S| = U|T|X = TX = XS = XV|S| \). Let \( x \in \ker(|S|) \). Then \( Vx = 0 \) and \( TXx = XSSx = 0 \). Hence \( Xx \in \ker(T) = \ker(U) \) and \( UXx = 0 \). Hence \( UX = XV \). Since \( \ker(U) \subset \ker(T^*) \subset \ker(U^*) \), \( UU^* \leq U^*U \). Hence \( U^*UU = U^*UUU^*U = UU^*U = U \). This implies \( U \) and \( V^* \) are quasinormal. Hence \( U^*X = XV^*, \text{ran}(X) \) reduces \( U \), \( |T| \), \( \ker(X)^\dagger \) reduces \( V \), \( |S| \). We may assume \( t < s \). Then \( T, S^* \) are class \( p\)-\( w \)(\( s,s \)) operators with reducing kernels.
Let $T(s, s) = |T|^*U|T|^s, S(s, s) = |S|^*V|S|^s$. Then $T(s, s), S^*(s, s) = |S|^*V^*|S^*|^s = VS(s, s)V^*$ are $\frac{1}{2}$-hyponormal. Also, since
\[ |S(s, s)^*| - |S(s, s)| = V^*|S^*(s, s)| - |S^*(s, s)^*|V \geq 0, \]
$S(s, s)^*$ is $\frac{1}{2}$-hyponormal, too. Then
\[ T(s, s)X = |T|^*U|T|^sX = |T|^*UX|S|^s = |T|^*XV|S|^s = XS(s, s), \]
hence $T(s, s)^*X = XS(s, s)^*$, $\overline{\text{ran}(X)}$ reduces $T(s, s)$, $\ker(X)^\perp$ reduces $S(s, s)$ and
\[ T|_{\overline{\text{ran}(X)}(s, s)} = T(s, s)|_{\overline{\text{ran}(X)}} = S(s, s)|_{\ker(X)^\perp} = S|_{\ker(X)^\perp}(s, s) \]
are unitarily equivalent normal operators. Hence $T|_{\overline{\text{ran}(X)}}, S|_{\ker(X)^\perp}$ are normal by Corollary 1, and that they are unitarily equivalent follows from the fact that if $N = U|N|$ and $M = W|M|$ are normal operators, then for a unitary operator $V$, $N = V^*MV$ if and only if $U = V^*WV$ and $|N|^s = V^*|M|^sV$ for any $s > 0$.

**Theorem 11.** Suppose that $0 < s, t, s + t = 1$. Let $T \in \mathcal{B}(H)$ be class $p$-wA$(s, t)$ and $N$ a normal operator. Let $TX = XN$. Then the following assertions hold.

(i) If the range $\text{ran}(X)$ is dense, then $T$ is normal.

(ii) If $\ker(X^*) \subseteq \ker(T^*)$, then $T$ is quasinormal.

**Proof.** Let $Z = |T|^sX$. Then
\[ T(s, t)Z = |T|^*U|T|^s|T|^sX = |T|^sTX = |T|^sXN = ZN. \]
Since $T(s, t)$ is $\min\{sp, tp\}$-hyponormal, we have
\[ T(s, t)^*Z = ZN^* \]
by [30]. Hence
\[ (T(s, t)^*T(s, t) - T(s, t)T(s, t)^*)|T|^sX = T(s, t)^*T(s, t)Z - T(s, t)T(s, t)^*Z = T(s, t)^*ZN - T(s, t)ZN^* = ZN^*N - ZNN^* = 0. \]

(i) If $\overline{\text{ran}(X)}$ is dense, then
\[ (T(s, t)^*T(s, t) - T(s, t)T(s, t)^*)|T|^s = 0. \]
Since
\[ \ker(|T|^s) \subseteq \ker(T(s, t)) \cap \ker(T(s, t)^*), \]
this implies \( T(s,t) \) is normal. Hence \( T \) is normal by Corollary 1.

(ii) Let \( X^*|T^*x = 0 \). Then \( |T^*x| \in \ker(X^*) \subset \ker(T^*) = \ker(U^*) \) and \( T(s,t)^*x = |T^*U^*|T^*x = 0 \). Hence \( \ker(X^*|T^*) \subset \ker(T(s,t)^*) \) and \( \text{ran}(T(s,t)) \subset \text{ran}(|T^*X|) \). Hence

\[
(T(s,t)^*T(s,t) - T(s,t)T(s,t)^*)T(s,t) = 0
\]

by (i). This implies \( T(s,t) \) is quasinormal, and \( T \) is quasinormal by Theorem 1.

**Theorem 12.** Suppose that \( 0 < s, t, s + t = 1 \) and \( 0 < q \leq 1 \). Let \( T \in B(H) \) be such that \( T^* \) is \( p \)-hyponormal or log-hyponormal. Let \( S \in B(K) \) be class \( q \)-\( wA(s,t) \) with \( \ker(S) \subset \ker(S^*) \). If \( XT = SX \), for some \( X \in B(H,K) \). Then \( XT^* = S^*X \).

**Proof.** Let \( T^* \) be a \( p \)-hyponormal operator for \( p \geq \frac{1}{2} \) and let \( T = U|T| \) be the polar decomposition of \( T \). Then the generalized Aluthge transform \( T^*(s,t) \) of \( T^* \) is hyponormal and satisfies

\[
|T^*(s,t)|^2 \geq |T|^2 \geq \langle (T^*(s,t))^* \rangle^2,
\]

\[
X'T(s,t) = SX'
\]

where \( X' = XU|T| \). Using the decompositions \( H = \ker(X')^\perp \oplus \ker(X') \) and \( K = \text{ran}(X')^\perp \oplus \text{ran}(X')^1 \), we see that \( T(s,t), S \) and \( X' \) are of the form

\[
T^*(s,t) = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}, \quad X' = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}
\]

where \( T_1 \) is hyponormal, \( S_1 \) is class \( q \)-\( wA(s,t) \) with \( \ker(S_1) \subset \ker(S_1^*) \) and \( X_1 \) is a one-one operator with dense range. Since \( X'T(s,t) = SX' \), we have

\[
X_1T_1 = S_1X_1.
\]

Hence \( T_1 \) and \( S_1 \) are normal by Corollary 6, so that \( T_2 = 0 \), by Lemma 12 of [30] and \( S_2 = 0 \) by Lemma 7. Then \( |T| = |T_1| \oplus P \), for some positive operator \( P \), by (12) and \( U = \begin{pmatrix} U_1 & U_2 \\ 0 & U_3 \end{pmatrix} \) by Lemma 13 of [30]. Let \( X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \) be a \( 2 \times 2 \) matrix representation of \( X \) with respect to the decomposition \( H = \ker(X')^\perp \oplus \ker(X') \) and \( K = \text{ran}(X')^\perp \oplus \text{ran}(X')^1 \). Then \( X' = XU|T| \) implies that \( X_1 = X_{11}U_1|T_1| \) and hence \( \ker(T_1) \subset \ker(X_1) = \{0\} \). This shows that \( T_1 \) is one-one and hence it has dense range, so that \( U_2 = 0 \) and \( T = T_1 \oplus T_4 \) for some hyponormal operator \( T_4 \) by [30, Lemma 13]. Since

\[
\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} = X' = XU|T| = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} U_1|T_1| & 0 \\ 0 & U_3|T_4| \end{pmatrix}
\]

we deduce the following assertions.

\[
X_{12}U_2|T_4| = 0; \quad \text{hence } X_{12}T_3 = 0 \text{ because } T_4 = U_3|T_4|.
\]
Since $T_1$ and $S_1$ are normal, we have $X_{11}T_1^* = S_1^*X_{11}$, by Fuglede-Putnam theorem. The $p$-hyponormality of $T_1^*$ shows that $\text{ran}(T_1^*) \subset \text{ran}(T_4)$. Also, we have $\ker(S_3) \subset \ker(S_3^*)$. Hence, we also have $X_{12}T_4^* = S_1^*X_{12} = 0$ and $X_{22}T_4^*S_3^*X_{22} = 0$. This implies that $XT^* = X_{11}T_1^* \oplus 0 = S_1^*X_{11} \oplus 0 = S_*X$.

Next, we prove the case where $T^*$ is $p$-hyponormal for $0 < p \leq \frac{1}{2}$. Let $X'$ be as above. Then $T^*(s, t)$ is $(p + \frac{1}{2})$-hyponormal and satisfies $X'T(s, t) = SX'$. Use the same argument as above. We obtain $X_{21} = 0, X_{11}T_1^* = S_1^*X_{11}, X_{12}T_4^* = S_1^*X_{12} = 0$ and $X_{22}T_4^* = S_3^*X_{22} = 0$. Hence we have $XT^* = S_*X$.

Finally, we assume that $T^*$ is log-hyponormal. Let $T(s, t)$ and $X'$ be as above. Then $X'T(s, t) = SX'$ and $T^*(s, t)$ is semi-hyponormal and satisfies

$$|T^*(s, t)| \geq |T^*| \geq |(T^*(s, t))^*|.$$ 

By the same argument as above, we have $T(s, t) = T_1 \oplus T_3$ on $H = \ker(X') \oplus \ker(X')$ and $S = S_1 \oplus S_3$ on $K = \text{ran}(X') \oplus \text{ran}(X')^\perp$, where $T_1$ is an injective normal operator and $S_1$ is normal, $T_3^*$ is invertible semi-hyponormal and $S_3$ is class $q-wA(s, t)$ with $\ker(S_3) \subset \ker(S_3^*)$. By Lemma 13 of [30], we have that $T$ is of the form $T = T_1 \oplus T_4$, for some log-hyponormal $T_4^*$. Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$. Then $X' = UX[T]^t$ implies that $X_{12} = 0, X_{21} = 0$ and $X_{22} = 0$. The assumption $XT = SX$ implies that $X_{11}T_1 = S_1X_{11}$, hence $X_{11}T_1^* \oplus 0 = S_1^*X_{11}$ by Fuglede-Putnam theorem. Thus we have $XT^* = X_{11}T_1^* \oplus 0 = S_1^*X_{11} \oplus 0 = S_*X$. Therefore, the proof of the theorem is achieved.

**Example 4.** Let $R$ be an operator such that $\ker(R)$ does not reduce $R$ and let $P$ be the orthogonal projection onto $\ker(R)$. Then $P$ does not commute with $T$; otherwise $\text{ran}(R) = \ker(R)$ reduce $T$. Hence $PR \neq 0 = RP$. It is easy to see that $RP = PR^* = 0$ but $R^*P \neq PR(\neq 0)$ because $\text{ran}(R^*) \subset \text{ran}(R) \subset \ker(R^*) = I - P$. If we put $T = R$, then the assertion of Theorem 10 does not hold for such $T$. Also, if we put $T = R^*, S = I - P$ and $X = P$, then $XT = PR^* = 0 = (I - P)P = SX$. However, $XT^* = PR \neq 0 = (I - P)P = S_*X$. Hence the assertion of Theorem 12 does not hold for such $T$. 


Theorem 13. Let $T \in \mathcal{B}(\mathcal{H})$ be such that $T^*$ is an injective class $p$-$wA(s, t)$ for $0 < s, t, s + t$ and $0 < p \leq 1$. Let $S \in \mathcal{B}(\mathcal{K})$ be dominant. If $XT = SX$, for some $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $XT^* = S^*X$.

Proof. Assume that $T^*$ is an injective $p$-$w$-hyponormal and let $T = U|T|$ be the polar decomposition of $T$. Let $T(s, t)$ be the aluthge transform of $T$ and $X' = XU|T|^t$. Then $X'T(s, t) = SX'$ and $T^*(s, t)$ is $rp$-hyponormal and satisfies

$$|T^*(s, t)|^{2rp} \geq |T^*|^{2rp} \geq |(T^*(s, t))^*|^{2rp}$$

for $r \in \min\{s, t\}$. By the same argument in the proof of Theorem 12, we conclude that $T^*(s, t) = T_1 \oplus T_3$ on $\mathcal{H} = \ker(X') \oplus \ker(X')$ and $S = S_1 \oplus S_3$, where $T_1$ is an injective normal operator and $S_1$ is also normal, $T^*_3$ is invertible class $p$-$wA(s, t)$ and $S_3$ is dominant. Hence by Lemma 7, we have that $T$ is of the form $T = T_1 \oplus T_4$ for some class $p$-$wA(s, t)$ $T_4^*$. Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$ 

Then $X' = XU|T|^t$ implies that $X_{12} = 0$, $X_{21} = 0$ and $X_{22} = 0$. The assumption $XT = SX$ implies that $X_{11}T_1 = S_1X_{11}$, hence $X_{11}T_1^* = S_1^*X_{11}$ by Fuglede-Putnam theorem. Thus we have $XT^* = X_{11}T_1^* \oplus 0 = S_1^*X_{11} \oplus 0 = S^*X$. Therefore, the proof of the theorem is achieved.

Example 5. Let $T^* = R$ as in Example 2. Let $X = P$ be the orthogonal projection onto $\ker(T^*)$ and $S = I - P$. Then $SX = 0 = XT^*$, but $0 \neq S^*X = XT^*$. Hence the injectivity condition is necessary for Theorem 13.

References


[14] T. Furuta. $A \geq B \geq O$ assures $(B^r A^p B^r)^{1/2} \geq B^{r+2r} \geq B^{p+2r}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1 + 2r)q \geq (p + 2r)$. *Proc. Amer. Math. Soc.*, 101:85–88, 1987.


