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On Some Parameters of the Central Graphs of the Identity Graphs of Finite Cyclic Groups

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Abstract. The interplay of groups and graphs has been a subject of interest by mathematics researchers nowadays. One particular instance is the identity graph of a group introduced by Kandasamy [5]. Moreover, the concept of a central graph of any graph is widely used by many graph theorist. The central graph of a graph G denoted by $C(G)$ can be obtained by subdividing the edge of G exactly once and joining all the nonadjacent vertices of G in $C(G)$. In this paper, we construct the central graph of the identity graph of finite cyclic group and investigate some of its graph properties.

2020 Mathematics Subject Classifications: 05C07, 05C12, 05C25

Key Words and Phrases: Cyclic group, Identity graph of a group, Handshaking lemma, Central graph of a graph, Distance, Eccentricities, Radius, Diameter, Center of a graph, Periphery of a graph, Girth

1. Introduction

The interconnection between different fields of mathematics has been a subject of interest by mathematics researchers nowadays. One particular instance is the interplay of groups and graphs where the existence of the researches contribute to the productive area of mathematics.

The collaboration of the two areas of mathematics mentioned gain attention to the mathematical community because of its elegant results. Some of which are order divisor graphs of finite groups [7], graphs and classes of finite groups [1], the power graph of a finite group [2], commuting graphs of dihedral type groups [6] and many more.

In 2009, Kandasamy and Smarandache [5], wrote a short book entitled "Groups as Graphs". They represented every finite group in the form of graph and choose to call these graphs as identity graphs since the main role of obtaining the graph is played by

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the identity element of the group. In this paper, we construct the central graph of the identity graph of finite cyclic group and find its properties. Graph parameters like distance, eccentricities, radius, diameter, girth, center, periphery are included.

2. Preliminaries

In this paper, all groups considered are finite cyclic groups.

Definition 2.1. A ***group*** is a nonempty set $\mathscr{G}$ together with a binary operation

$$(a, b) \mapsto a \cdot b : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$$

satisfying the following properties:

G1: (associativity) for all $a, b, c \in \mathscr{G}$,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c);$$

G2: (existence of an identity element) there exists an element $e \in \mathscr{G}$ such that

$$a \cdot e = a = e \cdot a$$

for all $a \in \mathscr{G}$;
G3: (existence of inverse element) for each $a \in \mathscr{G}$, there exists an $a' \in \mathscr{G}$ such that

$$a \cdot a' = e = a' \cdot a$$

Note that the notation $(a, b) \mapsto a \cdot b : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ means that for any two elements $a, b \in \mathscr{G}$, $a \cdot b$ also belongs to $\mathscr{G}$. This is called *closure property*. This group $\mathscr{G}$ together with the binary operation $\cdot$ is written as $(\mathscr{G}, \cdot)$ but in this paper we will abbreviate $(\mathscr{G}, \cdot)$ to $\mathscr{G}$. Also, we usually write ab for $a \cdot b$ and 1 for e; alternatively, we write $a + b$ for $a \cdot b$ and 0 for e. In the first case, the group is said to be multiplicative, and in the second, it is said to be additive. In some standard Group Theory books, $\cdot$ is usually written as * but in this paper we use $\cdot$ as the binary operation for the group $\mathscr{G}$.

Example 1. Let $\mathscr{G} = \{1, -1\}$. $\mathscr{G}$ is a group under usual multiplication since $\mathscr{G}$ satisfices all the properties of a group; that is, $\mathscr{G}$ is nonempty, closed under usual multiplication; i.e. $\{(1 \times 1 = 1, 1 \times -1 = -1, -1 \times -1 = 1\}$. Since usual multiplication is associative $\mathscr{G}$ is also associative. $1 \in \mathscr{G}$ is the identity element of $\mathscr{G}$ and 1 and -1 are self-inverse elements.

Definition 2.2. A group $\mathscr{G}$ is called **cyclic** if there exists $a \in \mathscr{G}$ such that

$$\mathscr{G} = \{a^n \mid n \in \mathbb{Z}\}.$$

Such an element a is called a generator of $\mathscr{G}$. We may indicate that $\mathscr{G}$ is a cyclic group generated by a by writing $\mathscr{G} = \langle a \rangle$. We denote this group as C_n of order n.

Example 2. Consider the group $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ under usual addition modulo 6. Note that $1 \in \mathbb{Z}_6$ such that $\langle 1 \rangle = \mathbb{Z}_6$. We can also verify that $\langle 5 \rangle = \mathbb{Z}_6$, that is,

$$\begin{aligned} \langle 5 \rangle &= \{1 \cdot 5 \equiv 5,\ 2 \cdot 5 \equiv 4,\ 3 \cdot 5 \equiv 3,\ 4 \cdot 5 \equiv 2,\ 5 \cdot 5 \equiv 1,\ 6 \cdot 5 \equiv 0\} \\ &= \{0\ ,\ 1,\ 2,\ 3,\ 4,\ 5\} = \mathbb{Z}_6 \end{aligned}$$

Definition 2.3. [5] Given a group $\mathscr{G}$ with e as the identity element, define the **identity graph** $\Gamma_{\mathscr{G}}$ to have the vertex set $\mathscr{G}$ and the edge set $E(\Gamma_{\mathscr{G}})$ satisfying two conditions:
(i) For every $x, y \in \mathscr{G}$ $(x \neq e, y \neq e, x \neq y)$, x and y are adjacent in $\Gamma_{\mathscr{G}}$ if and only if

$$x \cdot y = e\ ;$$

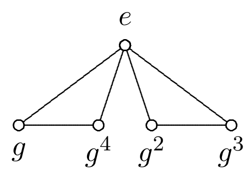
(ii) For each $x \in \mathscr{G}$ $(x \neq e)$, x and e are adjacent in $\Gamma_{\mathscr{G}}$.

Definition 2.4. [8] Given a group $\mathscr{G}$, a **line** in the identity graph $\Gamma_{\mathscr{G}}$ is an edge (x, e) such that the degree of a vertex $x \in \mathscr{G}$ is one. The number of lines in the identity graph $\Gamma_{\mathscr{G}}$ is denoted by $line(\mathscr{G})$.

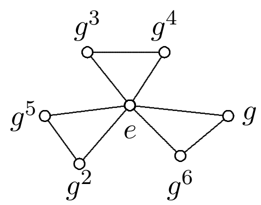
Definition 2.5. [8] A **triangle** in the identity graph $\Gamma_{\mathscr{G}}$ is a subgraph which is isomorphic to the cycle graph of length three. The number of triangles in the identity graph $\Gamma_{\mathscr{G}}$ is denoted by $tri(\mathscr{G})$.

Consider the identity graphs of cyclic groups below.

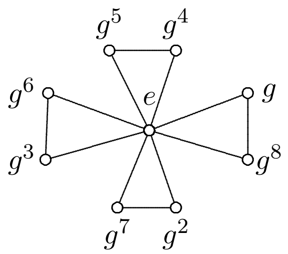
Γ_{C_5} :



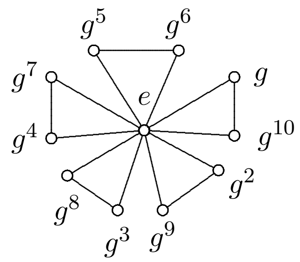
Γ_{C_7} :



Γ_{C_9} :



$\Gamma_{C_{11}}$:



Lemma 1. [8] For a group $\mathscr{G}$ of order n, we have

$$line(\mathscr{G}) + 2tri(\mathscr{G}) = n - 1.$$

Corollary 1. [3] If $\mathscr{G}$ is a cyclic group of odd order, then $\mathscr{G}$ has the identity graph $\Gamma_{\mathscr{G}}$ which is formed only by triangles with no lines.

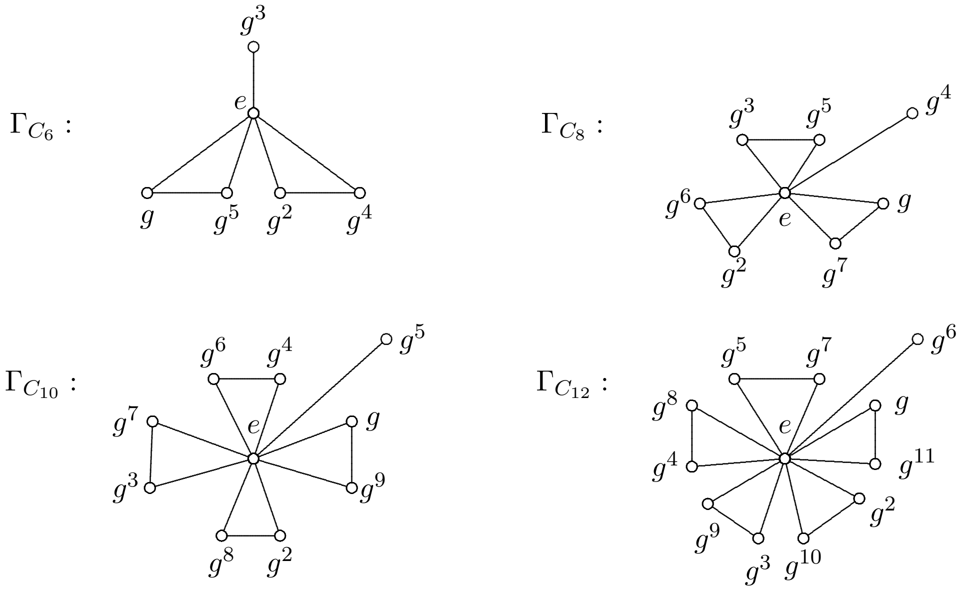


Figure 1: The identity graphs of cyclic groups C_5, C_6 C_7, C_8 C_9, C_{10}, C_{11}, and C_{12}

.

Theorem 1. [5] If C_n is a cyclic group of order n (n is odd), then the identity graph Γ_{C_n} of C_n is formed by $\frac{n-1}{2}$ triangles.

Theorem 2. [5] If C_n is a cyclic group of order n (n is even), then the identity graph Γ_{C_n} of C_n is formed by $\frac{n-2}{2}$ triangles and a line.

Theorem 3. [8] For a cyclic group C_n of order n, we have

$$line(C_n) = \begin{cases} 0, & n = 2k+1 \\ 1, & n = 2k+2 \end{cases}$$

and

$$tri(C_n) = \begin{cases} \frac{n-1}{2}, & n = 2k+1 \\ \frac{n-2}{2}, & n = 2k+2. \end{cases}$$

Theorem 4. [4] Let C_n be the cyclic group of order n and Γ_{C_n} be the identity graph associated with C_n. The size of Γ_{C_n} is

$$|E(\Gamma_{C_n})| = \begin{cases} \frac{3n-3}{2}, & n \text{ is odd}, \\ \frac{3n-4}{2}, & n \text{ is even}. \end{cases}$$

Lemma 2. ***(Handshaking Lemma)***
In a graph G, we have

$$\sum_{x \in V(G)} deg(x) = 2|E| \quad \text{where } |E| \text{ is the total number of edges.}$$

The distance $d(u,v)$ between $u,v \in V(G)$ is the length of a shortest $u-v$ path in the graph G. The eccentricity of a vertex $u \in V(G)$ is $e(u) = max\{d(u,v) \mid u \in V(G)\}$. The diameter of a graph G is $diam = max\{e(u) \mid u \in V(G)\}$. The radius of a graph G is $rad = min\{e(u) \mid u \in V(G)\}$. If $e(u) = diam(G)$, then u is a peripheral vertex. The set of all such vertices make the periphery of G. If $e(u) = rad(G)$, the vertex u is a central vertex. The set of all such vertices make the center of G. The girth of a graph G denoted by $gir(G)$ is the length of the shortest cycle (if any) in G. These graph parameters will be considered as the focus of the study.

3. Definition of Central Graph of Γ_{C_n}.

Consider the cyclic group C_n of order n (≥ 2). By definition 2.3, Γ_{C_n} is the identity graph associated with the group C_n. In this section, central graph of Γ_{C_n} will be discussed.

Definition 3.1. [9] Let C_n be a finite cyclic group of order n (≥ 2) and Γ_{C_n} be the identity graph of C_n. The central graph of Γ_{C_n} denoted by $C(\Gamma_{C_n})$ is obtained by subdividing the edges of Γ_{C_n} exactly once and joining all the non-adjacent vertices of Γ_{C_n} in $C(\Gamma_{C_n})$.

Throughout this paper, we fix a notation for the vertex-set and the edge-set of $C(\Gamma_{C_n})$. For any integer $n \geq 2$, let Γ_{C_n} be the identity graph of cyclic group C_n and let $V(\Gamma_{C_n}) = \{v_0, v_1, \cdots, v_{n-1}\}$. Consider its central graph $C(\Gamma_{C_n})$. The vertex-set and edge-set of $C(\Gamma_{C_n})$ are $V(C(\Gamma_{C_n})) = V(\Gamma_{C_n}) \bigcup \mathbf{C}$, where $\mathbf{C} = \{c_{ij} : (v_i, v_j) \in E(\Gamma_{C_n})\}$ and $E(C(\Gamma_{C_n})) = \{(v_i, c_{ij}), (v_j, c_{ij}) : (v_i, v_j) \in E(\Gamma_{C_n})\} \bigcup \{(v_i, v_j) : (v_i, v_j) \notin E(\Gamma_{C_n})\}$, respectively.

We consider two cases:

- central graph of the identity graph of odd cyclic group;
- central graph of the identity graph of even cyclic group.

Definition 3.2. Let $C(\Gamma_{C_n})$ be the central graph of Γ_{C_n} for any odd integer n. The vertex-set and edge-set of $C(\Gamma_{C_n})$ is given by

$$V(C(\Gamma_{C_n})) = \{v_0, v_i, c_{0i} : 1 \leq i \leq n-1\} \bigcup \left\{c_{(2i-1)(2i)} : 1 \leq i \leq \frac{n-1}{2}\right\}$$

$$\begin{aligned} E(C(\Gamma_{C_n})) = &\{(v_0, c_{0i}), (v_i, c_{0i}) : 1 \leq i \leq n-1\} \\ &\bigcup \left\{(v_{2i-1}, c_{(2i-1)(2i)}), (v_{2i}, c_{(2i-1)(2i)}) : 1 \leq i \leq \frac{n-1}{2}\right\} \\ &\bigcup \{(v_i, v_j) : 1 \leq i \leq n-3, \ i+2 \leq j \leq n-1\} \\ &\bigcup \left\{(v_{2i}, v_{2i+1}) : 1 \leq i \leq \frac{n-3}{2}\right\}, \end{aligned}$$

respectively.

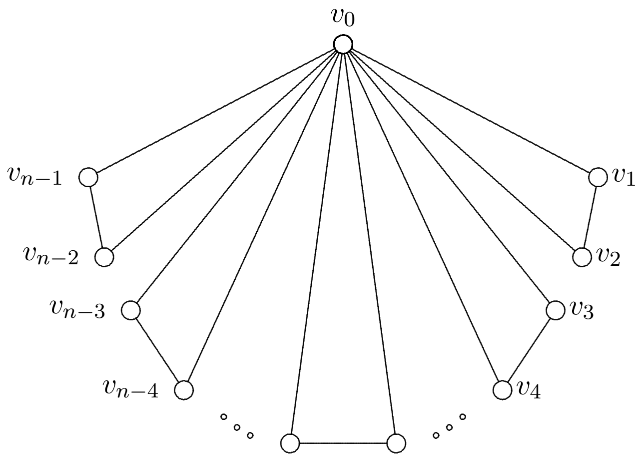


Figure 2: The identity graph of odd cyclic group

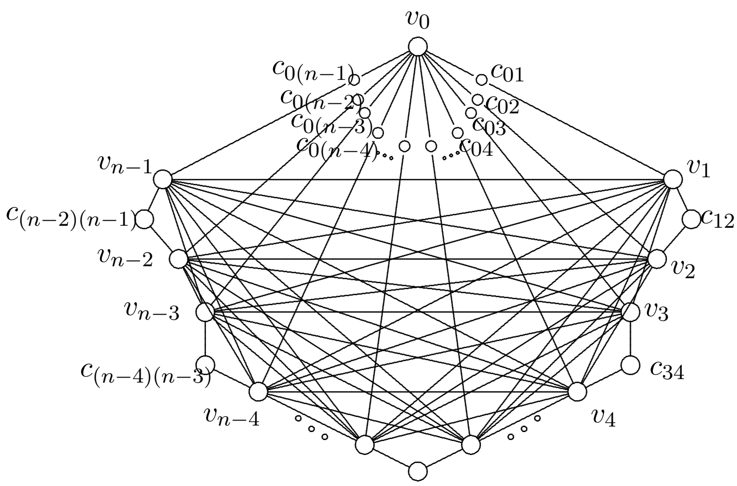


Figure 3: The Central Graph of Γ_{C_n} (n is odd).

The degrees of each vertex in $C(\Gamma_{C_n})$ is summarized below:

$$\begin{aligned} deg(v_0) &= n-1 \\ deg(v_i) &= n-1\ ;\ 1 \leq i \leq n-1 \\ deg(c_{0i}) &= 2\ ;\ 1 \leq i \leq n-1 \\ deg(c_{(2i-1)(2i)}) &= 2\ ;\ 1 \leq i \leq \frac{n-1}{2}. \end{aligned}$$

We will sum up all the degrees of vertices in $C(\Gamma_{C_n})$ and get

$$\begin{aligned} \sum_{x \in V(C(\Gamma_{C_n}))} deg(x) &= (n-1) + (n-1)(n-1) + 2(n-1) + 2(\frac{n-1}{2}) \\ &= 4(n-1) + n^2 - 2n + 1 \end{aligned}$$

$$\begin{aligned} &= 4n - 4 + n^2 - 2n + 1 \\ &= n^2 + 2n - 3. \end{aligned}$$

By Handsaking Lemma,

$$|E(C(\Gamma_{C_n}))| = \frac{1}{2}\Big(\sum_{x \in V(C(\Gamma_{C_n}))} deg(x)\Big) = \frac{n^2 + 2n - 3}{2}$$

From this result, we can characterize the order and size of $C(\Gamma_{C_n})$ (n is odd).

Proposition 1. Let C_n be a cyclic group of order n and Γ_{C_n} be the identity graph of C_n. If n is odd, then $|V(C(\Gamma_{C_n}))| = \frac{5n-3}{2}$ and $|E(C(\Gamma_{C_n}))| = \frac{n^2+2n-3}{2}$.

Proof. Let C_n be a cyclic group of order n (n is odd) and Γ_{C_n} be the identity graph of C_n. By Theorem 4, the size of Γ_{C_n} is $|E(\Gamma_{C_n})| = \frac{3n-3}{2}$. Thus, the order and size of $C(\Gamma_{C_n})$ are

$$|V(C(\Gamma_{C_n}))| = n + \left(\frac{3n-3}{2}\right) = \frac{2n + 3n - 3}{2} = \frac{5n-3}{2}$$

and

$$|E(C(\Gamma_{C_n}))| = \frac{n(n-1)}{2} + \left(\frac{3n-3}{2}\right) = \frac{n^2 - n + 3n - 3}{2} = \frac{n^2 + 2n - 3}{2},$$

respectively.

Illustration 1. The identity graph of C_9 and its central graph is in Figure 4.

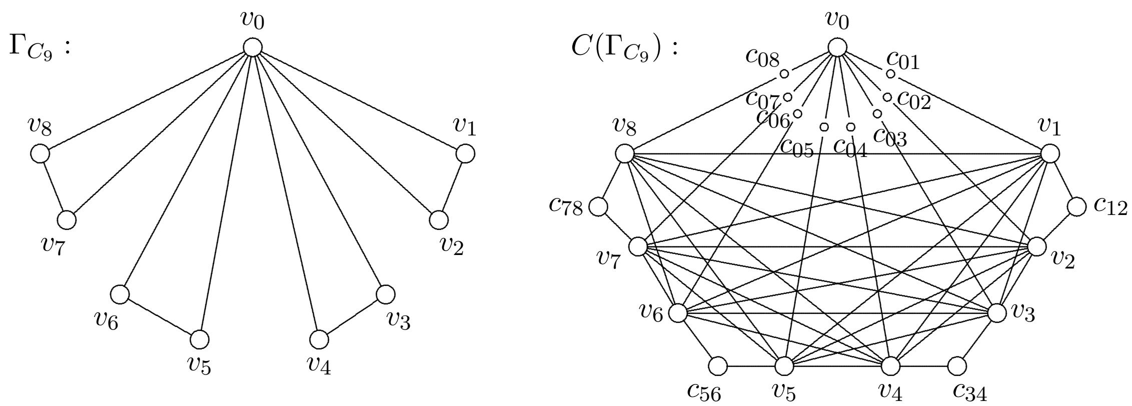


Figure 4: The Central Graph of Γ_{C_9}.

Definition 3.3. Let $C(\Gamma_{C_n})$ be the central graph of Γ_{C_n} for any even integer n. The vertex-set and edge-set of $C(\Gamma_{C_n})$ is given by

$$V(C(\Gamma_{C_n})) = \{v_0, v_i, c_{0i} : \ 1 \leq i \leq n-1\} \ \bigcup \ \left\{c_{(2i-1)(2i)} : \ 1 \leq i \leq \frac{n-2}{2}\right\}$$

$$\begin{aligned} E(C(\Gamma_{C_n})) = & \{(v_0, c_{0i}), (v_i, c_{0i}) : \ 1 \leq i \leq n-1\} \\ & \bigcup \ \left\{(v_{2i-1}, c_{(2i-1)(2i)}), (v_{2i}, c_{(2i-1)(2i)}) : \ 1 \leq i \leq \frac{n-2}{2}\right\} \\ & \bigcup \ \{(v_i, v_j) : \ 1 \leq i \leq n-3, \ i+2 \leq j \leq n-1\} \\ & \bigcup \ \left\{(v_{2i}, v_{2i+1}) : \ 1 \leq i \leq \frac{n-2}{2}\right\}, \end{aligned}$$

respectively.

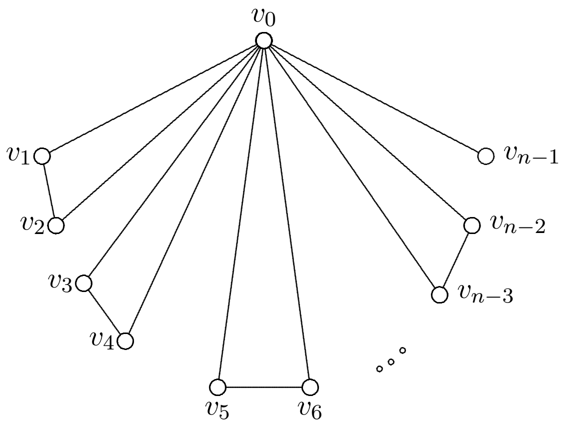


Figure 5: The Identity Graph of Even Cyclic Group

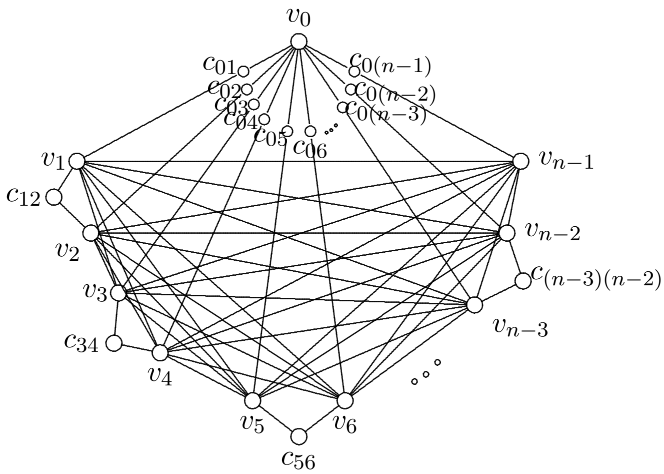


Figure 6: The Central Graph of Γ_{C_n} (n is even)

The degrees of each vertex in $C(\Gamma_{C_n})$ is summarized below:

$$\begin{aligned} deg(v_0) &= n-1 \\ deg(v_i) &= n-1\ ;\ 1 \leq i \leq n-1 \\ deg(c_{0i}) &= 2\ ;\ 1 \leq i \leq n-1 \\ deg(c_{(2i-1)(2i)}) &= 2\ ;\ 1 \leq i \leq \frac{n-2}{2}. \end{aligned}$$

We will sum up all the degrees of vertices in $C(\Gamma_{C_n})$ and get

$$\begin{aligned} \sum_{x \in V(C(\Gamma_{C_n}))} deg(x) &= (n-1)+(n-1)(n-1)+2(n-1)+2(\frac{n-2}{2}) \\ &= (n-1)+(n-2)+(2n-2)+(n^2-2n+1) \\ &= n^2+2n-4. \end{aligned}$$

By Handshaking Lemma,

$$|E(C(\Gamma_{C_n}))| = \frac{1}{2}(\sum_{x \in V(C(\Gamma_{C_n}))} deg(x)) = \frac{n^2+2n-4}{2}$$

From this result, we can characterize the order and size of $C(\Gamma_{C_n})$ (n is even).

Proposition 2. Let C_n be a cyclic group of order n and Γ_{C_n} be the identity graph of C_n. If n is even, then $|V(C(\Gamma_{C_n}))| = \frac{5n-4}{2}$ and $|E(C(\Gamma_{C_n}))| = \frac{n^2+2n-4}{2}$.

Proof. Let C_n be a cyclic group of order n (n is even) and Γ_{C_n} be the identity graph of C_n. By Theorem 4, the size of Γ_{C_n} is $|E(\Gamma_{C_n})| = \frac{3n-4}{2}$. Thus, the order and size of $C(\Gamma_{C_n})$ are

$$|V(C(\Gamma_{C_n}))| = n + \left(\frac{3n-4}{2}\right) = \frac{2n+3n-4}{2} = \frac{5n-4}{2}$$

and

$$|E(C(\Gamma_{C_n}))| = \frac{n(n-1)}{2} + \left(\frac{3n-4}{2}\right) = \frac{n^2-n+3n-4}{2} = \frac{n^2+2n-4}{2},$$

respectively.

4. Main Results

This section presents some graph parameters such as distance, eccentricities, radius, diameter, center, periphery, and girth of $C(\Gamma_{C_n})$.

Proposition 3. *Let Γ_{C_n} be the identity graph of finite cyclic group C_n $(n \geq 2)$ and $C(\Gamma_{C_n})$ be the central graph of Γ_{C_n}. If $u, v \in V(C(\Gamma_{C_n}))$, then $d(u,v) \leq 3$, where $d(u,v)$ is the distance between u and v.*

Proof. For any integer $n \geq 2$, let Γ_{C_n} be the identity graph of finite cyclic group C_n and let $V(\Gamma_{C_n}) = \{v_0, v_1, \cdots, v_{n-1}\}$. Consider its central graph $C(\Gamma_{C_n})$. The vertex-set and edge-set of $C(\Gamma_{C_n})$ are $V(C(\Gamma_{C_n})) = V(\Gamma_{C_n}) \bigcup \mathbf{C}$, where $\mathbf{C} = \{c_{ij} : (v_i, v_j) \in E(\Gamma_{C_n})\}$ and $E(C(\Gamma_{C_n})) = \{(v_i, c_{ij}), (v_j, c_{ij}) : (v_i, v_j) \in E(\Gamma_{C_n})\} \bigcup \{(v_i, v_j) : (v_i, v_j) \notin E(\Gamma_{C_n})\}$, respectively. Let $x \in V(C(\Gamma_{C_n}))$ for odd n.
If $x = v_0$, $d(x,u) \leq 3$ for any $u \in V(C(\Gamma_{C_n}))$.
If $x \in \{c_{0i} : 1 \leq i \leq (n-1)\}$, $d(x,u) \leq 3$ for any $u \in V(C(\Gamma_{C_n}))$.
If $x \in \{v_i : 1 \leq i \leq (n-1)\}$, $d(x,u) \leq 3$ for any $u \in V(C(\Gamma_{C_n}))$.
If $x \in \{c_{(2i-1)(2i)} : 1 \leq i \leq \frac{n-1}{2}\}$, $d(x,u) \leq 3$ for any $u \in V(C(\Gamma_{C_n}))$.
The proof is analogous to the first case if n is even. Therefore, $d(x,u) \leq 3$ for any $u, v \in V(C(\Gamma_{C_n}))$.

Illustration 2. The central graph of Γ_{C_n} (n is odd) is given in Figure 7.

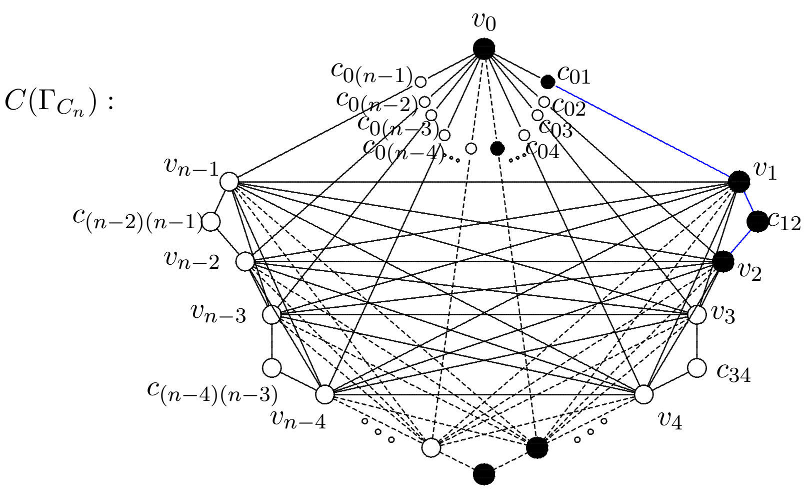


Figure 7: Central Graph of Γ_{C_n}, n is odd

Figure 7 shows that any arbitrary vertex $x \in V(C(\Gamma_{C_n}))$ has the maximum distance of 3. The same argument for the central graph of the identity graph of C_n (n is even). The only difference is that $d(v_{n-1}, x) \leq 2$, for any other $x \in V(C(\Gamma_{C_n}))$.

The next proposition determines the eccentricities of the vertices of $C(\Gamma_{C_n})$.

Proposition 4. *Let $C(\Gamma_{C_n})$ $(n \geq 2)$ be the central graph of Γ_{C_n}. The eccentricities of the vertices of $C(\Gamma_{C_n})$ are as follows:*
i. If n is odd, then $e(u) = 3$ for all $u \in V(C(\Gamma_{C_n}))$.
ii. If n is even, then $e(u)$ is either 2 or 3, $u \in V(C(\Gamma_{C_n}))$.

Proof. Let $C(\Gamma_{C_n})$ $(n \geq 2)$ be the central graph of Γ_{C_n}.
(i). If n is odd, (see Figure 3), every vertex $u \in V(C(\Gamma_{C_n}))$ has a maximum distance of 3, that is, $e(v_0) = 3$, $e(c_{0i}) = 3$, $1 \leq i \leq (n-1)$, $e(v_i) = 3$, $1 \leq i \leq (n-1)$, $e(c_{(2i-1)(2i)}) = 3$, $1 \leq i \leq \frac{n-1}{2}$. Thus, for any $u \in V(C(\Gamma_{C_n}))$, $e(u) = 3$.
(ii). If n is even, (see Figure 6), $e(v_0) = 3$, $e(c_{0i}) = 3$ $(1 \leq i \leq (n-1))$, $e(v_i) = 3$ $(1 \leq i \leq (n-2))$, $e(c_{(2i-1)(2i)}) = 3$ $(1 \leq i \leq \frac{n-2}{2})$, $e(v_{(n-1)}) = 2$.
Therefore, $e(u)$ is either 2 or 3..

Illustration 3. Consider the eccentricities of the vertices of $C(\Gamma_{C_4})$ and $C(\Gamma_{C_5})$ in Figure 8. Notice that all the vertices of $C(\Gamma_{C_4})$ have eccentricities 3 except the vertex v_3 which is 2. On the other hand, all the vertices of $C(\Gamma_{C_5})$ have eccentricities 3.

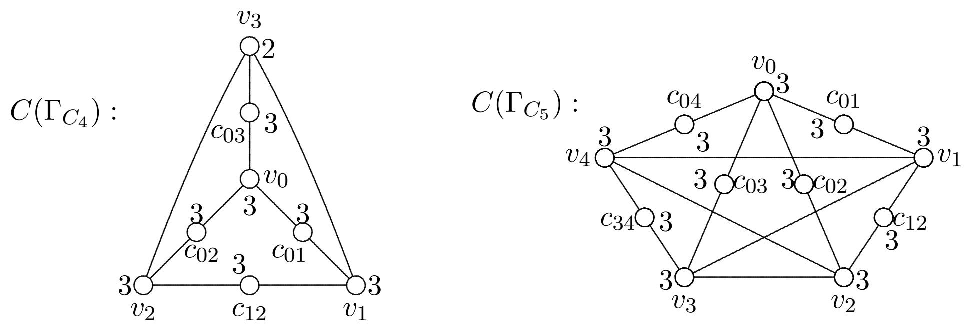


Figure 8: The Central Graphs of $C(\Gamma_{C_4})$ and Γ_{C_5}.

The radius and diameter of $C(\Gamma_{C_n})$ will be discussed in the next proposition.

Proposition 5. *Let $C(\Gamma_{C_n})$ be the central graph of Γ_{C_n} for any integer $n \geq 3$.*
i. If n is odd, then $rad(C(\Gamma_{C_n})) = diam(C(\Gamma_{C_n}))$.
ii. If n is even, then $rad(C(\Gamma_{C_n})) = 2$ and $diam(C(\Gamma_{C_n})) = 3$.

Proof. Let $C(\Gamma_{C_n})$ be the central graph of Γ_{C_n} for any integer $n \geq 3$.
(i). By Proposition 4-(i), for any $u \in V(C(\Gamma_{C_n}))$, $e(u) = 3$. Hence, $rad(C(\Gamma_{C_n})) = 3 = diam(C(\Gamma_{C_n}))$.
(ii). It is straightforward to prove case ii by using Proposition 4-(ii) since $min\ \{e(u) : \exists u \in V(C(\Gamma_{C_n}))\} = 2$ and $max\ \{e(u) : \exists u \in V(C(\Gamma_{C_n}))\} = 3\}$. Hence, $rad(C(\Gamma_{C_n})) = 2$ and $diam(C(\Gamma_{C_n})) = 3$.

Illustration 4. In Figure 8, $rad(C(\Gamma_{C_5})) = diam(C(\Gamma_{C_5})) = 3$ and $rad(C(\Gamma_{C_4})) = 2$ and $diam(C(\Gamma_{C_4})) = 3$.

In the next two propositions below, the center of $C(\Gamma_{C_n})$ is given.

Proposition 6. *For any even integer $n \geq 4$, let $C(\Gamma_{C_n})$ be the central graph of Γ_{C_n}. The center of $C(\Gamma_{C_n})$ denoted by $Cen(C(\Gamma_{C_n}))$ is a complete graph K_1.*

Proof. By Proposition 5-(ii), $rad(C(\Gamma_{C_n})) = 2$. And, by Proposition 4-(ii), the only vertex of $(C(\Gamma_{C_n}))$ that has of eccentricity 2 is the vertex v_{n-1}. Hence, v_{n-1} is the only central vertex of $C(\Gamma_{C_n})$. Thus, $Cen(C(\Gamma_{C_n}))$ is K_1 (consists of a single vertex v_{n-1}).

Illustration 5. Let $C(\Gamma_{C_8})$ be the central graph of Γ_{C_8} given below.

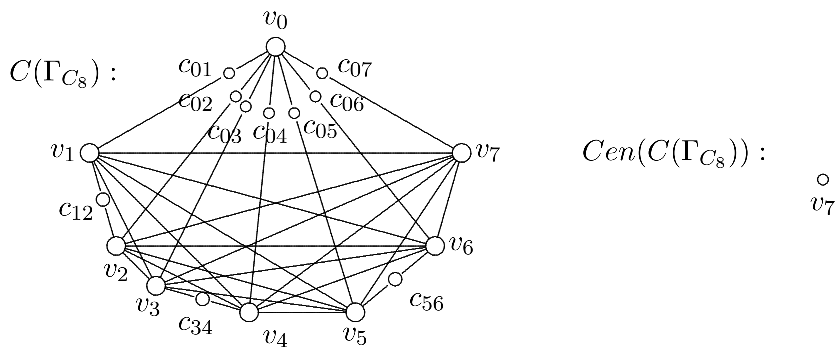


Figure 9: The Central Graph of Γ_{C_8} and its center.

The center of Γ_{C_8} is a subgraph induced by a vertex v_7.

If n is odd, the center of the graph $C(\Gamma_{C_n})$ is given in the next proposition.

Proposition 7. *Let $C(\Gamma_{C_n})$ be the central graph of Γ_{C_n}. If n is odd (≥ 3), then $C(\Gamma_{C_n})$ is a self-centered graph.*

Proof. Let C_n be a cyclic group of odd order $n \geq 3$. We can associate an identity graph (see Figure 2) and its central graph (Figure 3). Proposition 4-(i), tells us that for any $u \in V(C(\Gamma_{C_n}))$, $e(u) = 3$. Hence, $rad(C(\Gamma_{C_n})) = 3 = e(u)$. Thus, all $u \in V(C(\Gamma_{C_n}))$ are central vertices of $C(\Gamma_{C_n})$. Therefore a subgraph induced by the central vertices of $C(\Gamma_{C_n})$ is $C(\Gamma_n)$ itself.

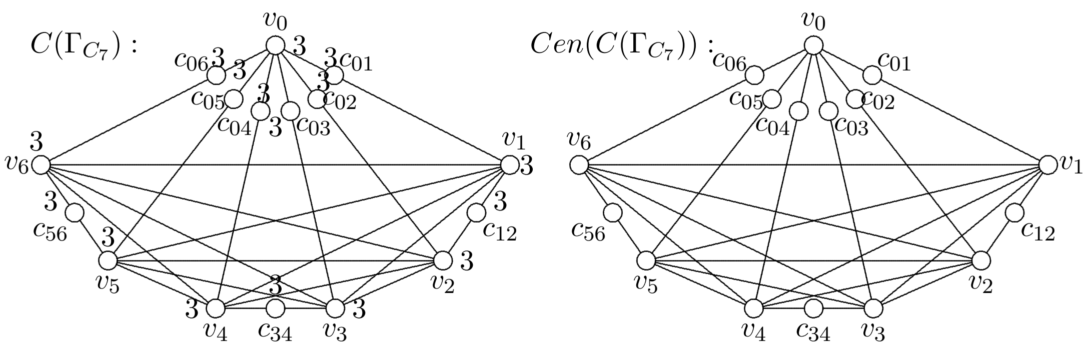


Figure 10: The Central Graph of Γ_{C_7}.

Illustration 6. For the central graph of Γ_{C_7} of Figure 10, the eccentricities of all the vertices of $C(\Gamma_{C_7})$ are all the same. Thus, all the vertices are central vertices of $C(\Gamma_{C_7})$. Therefore, the center of $C(\Gamma_{C_7})$ is $C(\Gamma_{C_7})$ itself.

Proposition 8. *All vertices of $C(\Gamma_{C_n})$ (odd integer $n \geq 3$) are peripheral vertices.*

Proof. Proposition 5-(i) tells us that for all $v \in V(C(\Gamma_{C_n}))$, $diam(C(\Gamma_{C_n})) = 3 = e(v)$. Therefore, by definition of peripheral vertex of a graph, all vertices $v \in V(C(\Gamma_{C_n}))$ are peripheral vertices.

Proposition 9. *Let $C(\Gamma_{C_n})$ be the central graph of Γ_{C_n} (even integer $n \geq 4$). The set of vertices $A = \{v_i : 0 \leq i \leq (n-2)\} \bigcup \{c_{0i} :\ 1 \leq i \leq (n-1)\} \bigcup \{c_{(2i-1)(2i)} :\ 1 \leq i \leq \frac{n-2}{2}\}$ are peripheral vertices of $C(\Gamma_{C_n})$.*

Proof. The proof follows from Propositions 4-(ii) and 5-(ii), respectively.

Proposition 10. *For any odd integer $n \geq 3$, $Per(C(\Gamma_{C_n})) = C(\Gamma_{C_n})$.*

Proof. By Proposition 7, if n is odd, $V(C(\Gamma_{C_n}))$ is the set of peripheral vertices of $C(\Gamma_{C_n})$. Therefore, $Per(C(\Gamma_{C_n})) = C(\Gamma_{C_n})$.

Proposition 11. *Let $C(\Gamma_{C_n})$ (even integer $n \geq 4$) with the vertex set $V(C(\Gamma_{C_n})) = \{v_0,\ v_i,\ c_{0i} :\ 1 \leq i \leq (n-1),\ c_{(2i-1)(2i)} :\ 1 \leq i \leq \frac{n-2}{2}\}$. Take $K = \{v_{n-1}\}$. The subgraph $C(\Gamma_{C_n})\backslash K$ of $C(\Gamma_{C_n})$ is the periphery of $C(\Gamma_{C_n})$.*

Proof. The proof follows from Proposition 4-(ii) and Proposition 8.

Proposition 12. *The graph $C(\Gamma_{C_n})$ is an eccentric graph if and only if n is odd.*

Proof. Let $C(\Gamma_{C_n})$ be the central graph of Γ_{C_n} where n is odd. Then, $C(\Gamma_{C_n})$ is an eccentric graph since every vertex of $C(\Gamma_{C_n})$ is a peripheral vertex and so is an eccentric vertex of the other in $C(\Gamma_{C_n})$. Conversely, let $C(\Gamma_{C_n})$ be an eccentric graph. Thus, every vertex of $C(\Gamma_{C_n})$ is an eccentric vertex of the other. There are only two cases for n. For the case that n is even, $v_{n-1} \in V(C(\Gamma_{C_n}))$ is not an eccentric vertex of any other vertex of $C(\Gamma_{C_n})$ and thus $C(\Gamma_{C_n})$ for even n is not an eccentric graph. So n must be odd since in this case every vertex is an eccentric vertex of the other.

Proposition 13. *The girth of $C(\Gamma_{C_n})$ is*

$$gir(C(\Gamma_{C_n})) = \begin{cases} 6, & if\ n = 3 \\ 4, & if\ n = 4, 5 \\ 3, & if\ n > 5. \end{cases}$$

Proof. Let Γ_{C_n} be the identity graph associated with a cyclic group C_n of order n. For $n = 3$, $C(\Gamma_{C_3})$ is isomorphic to a cycle graph of length 6. Thus, $gir(C(\Gamma_{C_3})) = 6$. For $n = 4$, let $V(C(\Gamma_{C_4})) = \{v_0, v_i, c_{0i}, c_{12} : 1 \leq i \leq 3\}$ and $E(C(\Gamma_{C_4})) = \{(v_0, c_{0i}), (v_i, c_{0i}) : 1 \leq i \leq 3\} \bigcup \{(v_1, c_{12}), (v_1, v_3), (c_12, v_2), (v_2, v_3)\}$. Clearly, the cycle $\{v_1, c_{12}, v_2, v_3, v_1\}$ is the smallest cycle of $C(\Gamma_{C_4})$. Hence, $gir(C(\Gamma_{C_4})) = 4$. For $n = 5$, let $V(C(\Gamma_{C_5})) = \{v_0, v_i, c_{0i}, c_{12} : 1 \leq i \leq 4\}$ and $E(C(\Gamma_{C_5})) = \{(v_0, c_{0i}), (v_i, c_{0i}) : 1 \leq i \leq 4\} \bigcup \{(v_1, c_{12}), (v_1, v_3), (v_1, v_4), (c_{12}, v_2), (v_2, v_3), (v_2, v_4), (v_3, c_{34}), (c_{34}, (v_4))\}$. Clearly, the cycle $\{v_1, c_{12}, v_2, v_3, v_1\}$ is one of the smallest cycles of $C(\Gamma_{C_4})$. Thus, $gir(C(\Gamma_{C_5})) = 4$. For $n \geq 6$, the cycle $\{v_1, v_3, v_{n-1}, v_1\}$ is always in $C(\Gamma_{C_n})$. In fact, there are many cycle graph of lenght 3 in $C(C(\Gamma_{C_n}))$. Thus, $gir(C(\Gamma_{C_n})) = 3$.

5. Summary and Conclusion

In this paper, we investigated the structures and some properties of the central graphs of the identity graphs of finite cyclic groups.

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