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On the Cospectrality of Hermitian Adjacency Matrices of Mixed Graphs

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Abstract. A mixed graph D is a graph that can be obtained from a graph by orienting some of its edges. Let α be a primitive n^{th} root of unity, then the $\alpha-$Hermitian adjacency matrix of a mixed graph is defined to be the matrix $H_\alpha = [h_{rs}]$ where $h_{rs} = \alpha$ if rs is an arc in D, $h_{rs} = \overline{\alpha}$ if sr is an arc in D, $h_{rs} = 1$ if sr is a digon in D and $h_{rs} = 0$ otherwise. In this paper we study the cospectrality of the Hermitian adjacency matrix of a mixed graph.

2020 Mathematics Subject Classifications: 05C20, 05C50, 15A30

Key Words and Phrases: Adjacency Matrix, Mixed graphs, Hermitian Matrix, Spectrum, Cospectrality

1. Introduction

In this paper we consider graphs without loops or multiple edges. A mixed graph D is a set of vertices $V(D)$ together with a set of edges $E(D) \subset V(D) \times V(D)$ where, $(u, v) \in E(D)$ does not always imply $(v, u) \in E(D)$. An edge (u, v) of a mixed graph D is called oriented (resp. digon) if only one of (u, v) and (v, u) belongs to $E(D)$ (resp. both of (u, v) and (v, u) belong to $E(D)$). For simplicity, through this paper arcs and digon will be denoted by uv instead of (u, v). The underlying graph of a mixed graph D, denoted by $\Gamma(D)$, is the graph obtained from D after unorienting all of its edges. A cycle, (resp. a walk, a path) of a mixed graph D is just a cycle (resp. a walk, a path) of the underlying graph of D. A mixed graph D is called weakly connected if its underlying graph is connected.

One of the important branches of algebraic graph theory is the study of graphs and digraphs with respect to some graph matrix and its spectrum. For undirected graphs researchers focused on two kinds of adjacency matrices, the traditional adjacency matrix and the Laplacian adjacency matrix. On the other hand for directed graphs (digraphs

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) the traditional adjacency matrix is not symmetric in general. Therefore, it is very challenging to deal with such matrix. Recently, many researchers have proposed other Hermitian adjacency matrices of mixed graphs. For example in [2] and in [1] authors studied the singular values of the traditional adjacency matrix of a mixed graph D, in these papers it was very clear that the singular values of a mixed graph are related to the common out neighbors between vertices. Coincidentally, Guo and Mohar in [3] defined an interesting Hermitian adjacency matrix of mixed graphs as follows: For a mixed graph D, the $i-$Hermitian adjacency matrix of D is a $|V| \times |V|$ matrix $H_i(D) = [h_{uv}]$, where

$$h_{uv} = \begin{cases} 1 & \text{if } uv \text{ is a digon in } D, \\ i & \text{if } uv \text{ is an arc in } D, \\ -i & \text{if } vu \text{ is an arc in } D, \\ 0 & \text{otherwise.} \end{cases}$$

In [3] many interesting spectral properties of $H_i(D)$ have been demonstrated. Mohar in [5] extended the previously proposed adjacency matrix. In the new kind of Hermitian adjacency matrices the complex number i is replaced with the sixth root of unity $\omega = e^{\frac{\pi}{3}i}$. The new kind of Hermitian adjacency matrices is called$\omega-$Hermitian adjacency matrices. In this paper Mohar discussed many cases with general complex unit number $e^{i\theta}$ instead of ω. To be more precise, the definition of the general Hermitian adjacency matrix of a mixed graph, some times called α-Hermitian adjacency matrix, was as follows:
Let $\alpha = e^{i\theta}$ and D be a mixed graph. Then, the α-Hermitian adjacency matrix of D is a $|V| \times |V|$ matrix $H_\alpha(D) = [h_{uv}]$, where

$$h_{uv} = \begin{cases} 1 & \text{if } uv \text{ is a digon in } D, \\ \alpha & \text{if } uv \text{ is an arc in } D, \\ \overline{\alpha} & \text{if } vu \text{ is an arc in } D, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, these definitions can be considered as special cases of the complex unit gain graphs which was defined in [4]. The fact that these adjacency matrices (H_i and H_α) are Hermitian has opened a hot research topic nowadays.
Finally, let D be a mixed graph, H_α be its α-Hermitian adjacency matrix of D and u, v, r be three vertices of D. Then the α-weight of a walk W in D, say $W = r_1, r_2, ..., r_k$, is defined to be the product of the entries of H_α corresponds to the values of the arcs and digons in W. To be more formal,

$$h_\alpha(W) = h_{r_1 r_2} h_{r_2 r_3} \dots h_{r_{k-1} r_k}.$$

Furthermore, if W_{uv} is a walk from the vertex u to the vertex v and W_{vr} is a walk from the vertex v to the vertex r, then by the walk $W_{uv}W_{vr}$ we mean the walk from the vertex v to the vertex r through the walk W_{uv} then the walk W_{vr}. By $Rev(W_{uv})$ we mean the walk from the vertex v to the vertex u through the walk W_{uv}.

2. Cospectrality of Hermitian Adjacency Matrices of Mixed Graphs

Studying cospectrality of two graphs is one of the classical field of algebraic graph theory. However, the cospectrality of mixed graphs can be divided into two studies: The first one is studying the cospectrality of two different mixed graphs with same unit complex number α (α-cospectrality), and the second one is studying cospectrality of same mixed graphs but with different values of α.
In this section we study when the Hermitian adjacency matrices (with respect to different values of α) of a weakly connected mixed graph are cospectral. Also we study when α-spectrum of a weakly connected mixed graph D is completely determined by the underlying graph of the mixed graph D. To this end, fix a weakly connected mixed graph D and a unit complex number α. Let $u, v \in V(D)$ and let W be a walk in D, say $W = (u = r_1, ..., r_k = v)$. We now recursively define a function that assigns a value $f^{\alpha}_W(j)$ to the j^{th} vertex (i.e. to r_j) along W by

$$f^{\alpha}_W(1) = 1,$$

$$f^{\alpha}_W(j+1) = \left\{ \begin{array}{ll} f^{\alpha}_W(j) & \text{if } r_j r_{j+1} \text{ is a digon in } D \\ \alpha f^{\alpha}_W(j) & \text{if } r_j r_{j+1} \text{ is an arc in } D \\ \bar{\alpha} f^{\alpha}_W(j) & \text{if } r_{j+1} r_j \text{ is an arc in } D \end{array} \right\}$$

for $j = 1, ..., k$. We shall write $f^{\alpha}_W(*)$ for the final value $f^{\alpha}_W(k)$, see Figure 1

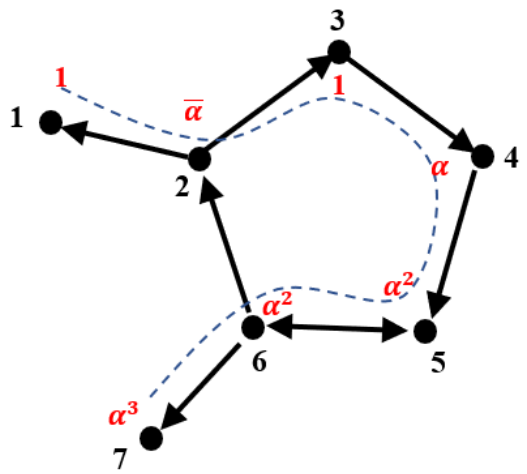


Figure 1: The values of the function $f^{\alpha}_W(j)$ along the walk $W = 1234567$

Let D be a mixed graph, $u \in V(D)$, then for a vertex $v \in V(D)$ define the α-store of the vertex v with respect to the vertex u by:

$$\mathbb{T}^{\alpha}_u(v) = \{f^{\alpha}_W(*) : W \text{ is a walk from } u \text{ to } v\},$$

and the store size of the vertex v with respect to the vertex u by $\mathfrak{t}^{\alpha}_u(v) = |\mathbb{T}^{\alpha}_u(v)|$. Clearly $1 \in \mathbb{T}^{\alpha}_u(u)$ and if D contains a cycle C of weight other than 1, then along any walk W that contains C, the store of u will contain values other than 1. In fact, if α is the primitive n^{th} root of unity, the weight of the cycle C is α^r, and $gcd(n, r) = 1$ then the store of the vertex u will contain all powers of α.

Theorem 1. *Let D be a weakly connected mixed graph, $u \in V(D)$ and $H_\alpha = [h_{ij}]$ be its α-Hermitian adjacency matrix. Then the following statements are equivalent:*

(1) $\mathfrak{t}_u^\alpha(v) = 1$ for a vertex v of D.

(2) $\mathfrak{t}_u^\alpha(m) = 1$ for every vertex m of D.

(3) For every cycle $C = (r_1, r_2, \ldots, r_{k-1}, r_1)$ in D, the weight of C

$$h_\alpha(C) = h_{r_1 r_2} h_{r_2 r_3} \ldots h_{r_{k-1} r_1}$$

in H_α equals one.

Proof. ((1)) $\to$ ((2)) Suppose that $\mathfrak{t}_u^\alpha(v) = 1$ for a vertex v of D and let m be a vertex of D with $\mathfrak{t}_u^\alpha(m) > 1$. Suppose further W_{uv} is a walk from u to v. Then, since $\mathfrak{t}_u^\alpha(m) > 1$, there are two walks W_{um} and W'_{um} from u to m with $f^\alpha_{W_{um}}(*) \neq f^\alpha_{W'_{um}}(*)$. Now, consider the walk $\mathbb{W} = W_{um} Rev(W'_{um}) W_{uv}$, where $Rev(W'_{um})$ is the walk from m to u through the walk W'_{um}. Since $f^\alpha_{W_{um}}(*) \neq f^\alpha_{W'_{um}}(*)$ we have:

$$\begin{aligned} f^\alpha_{\mathbb{W}}(*) &= f_{W_{um}}(*) f_{Rev(W'_{um}}(*) f_{W_{uv}}(*) && (1) \\ &\neq f_{W_{uv}}(*). && (2) \end{aligned}$$

Therefore, $\mathfrak{t}_u^\alpha(v) > 1$, which is a contradiction.

((2)) $\to$ ((3)) Suppose that C is a cycle in D and let v be a vertex of the cycle C. Then consider the closed walk $\mathbb{W} = P_{uv} C_v Rev(P_{uv})$, where P_{uv} is a path from u to v and C_v is the closed path from v to v along the cycle C. Then, using the definition of $f^\alpha_W(*)$ we have,

$$\begin{aligned} f^\alpha_{\mathbb{W}}(*) &= h_\alpha(P_{uv}) h_\alpha(C) h_\alpha(Rev(P_{uv})) && (3) \\ &= h_\alpha(C). && (4) \end{aligned}$$

Finally, using the definition of $f^\alpha_W(*)$ and the assumption that $\mathfrak{t}_u^\alpha(v) = 1$ for every vertex v of D, we have $h_\alpha(C)$ equals to 1.

((3)) $\to$ ((1)) Let W be a closed walk start from the initial vertex u. Then, W consists of a sequence of cycles $\{C_i\}_{i=1}^n$ and sequence of paths $\{P_j\}_{j=1}^m$. Further, using the assumption in (3) we have

$$f_{\mathbb{W}}^{\alpha}(*) = \prod_{i=1}^{n} h_{\alpha}(C_i) \prod_{j=1}^{m} h_{\alpha}(P_j) \prod_{j=1}^{m} h_{\alpha}(Rev(P_j)) \tag{5}$$

$$= \prod_{i=1}^{n} h_{\alpha}(C_i) \tag{6}$$

$$= 1. \tag{7}$$

Therefore, $\mathfrak{t}_u^{\alpha}(u) = 1$.

The above theorem paved the way to the following definition.

Definition 1. *Let D be a weakly connected mixed graph and u be a vertex of D. Then, D is called an α-monostore graph if $\mathfrak{t}_u^{\alpha}(v) = 1$ for a vertex v in D.*

Obviously, changing the initial vertex u of a mixed graph D will not change the store size of the vertices of D, but it may change the values in the store set of the vertices.

Theorem 2. *Let D be an α-monostore graph, $H_{\alpha} = [h_{st}]$ be its α-Hermitian adjacency matrix and for a vertex u in D, $\Delta_{\alpha} = diag\{t_v : t_v \in \mathbb{T}_u^{\alpha}(v)$ and $v \in V(D)\}$. Then, $\Delta_{\alpha} H_{\alpha} \Delta_{\alpha}^* = A(\Gamma(D))$ where $A(\Gamma(D))$ is the traditional adjacency matrix of the graph $\Gamma(D)$.*

Proof. Since the matrix Δ_{α} is a diagonal matrix, it is enough to prove that $\Delta_{\alpha} H_{\alpha} \Delta_{\alpha}^*$ is a $\{0, 1\}$-matrix. To do this, suppose that rs is an arc in D, then

$$(\Delta_{\alpha} H_{\alpha} \Delta_{\alpha}^*)_{rs} = t_r h_{rs} \overline{t_s}.$$

Since rs is an arc in D and D is α-monostore mixed graph, $t_s = h_{rs} t_r$. Therefore, $(\Delta_{\alpha} H_{\alpha} \Delta_{\alpha}^*)_{rs} = 1$.

Using the definition of $f_W^{\alpha}(*)$, Theorem 1 and Theorem 2, any vertex of an α-monostore graph D can be used to construct Δ_{α}. Also, Theorem 2 indicates that if D is an α-monostore mixed graph, then the orientation of D can be stored in a diagonal matrix Δ_{α}.

Definition 2. *If D is an α-monostore graph, then a diagonal matrix Δ_{α} that satisfies $\Delta_{\alpha} H_{\alpha} \Delta_{\alpha}^* = A(\Gamma(D))$ is called an orienting matrix of D.*

Corollary 1. *If D is an α-monostore mixed graph then D is cospectral with its underlying graph.*

Corollary 2. *If D is a tree mixed graph then the spectrum of D is completely determined by its underlying graph $\Gamma(D)$.*

Lemma 1. *Let H_α be the α-Hermitian adjacency matrix of a mixed graph D, $\Delta = diag(d_u : d_u \in \mathbb{C}, u \in V(D)$ and $|d_u| = 1)$ and $\mathbb{K} = [k_{uv}] = \Delta H_\alpha \Delta^*$. Then, for every cycle C in D the weight of the cycle C in H_α and in $\mathbb{K}$ are equal.*

Proof. Suppose that $C = u_1u_2 \dots u_nu_1$ is a cycle in D. Then, observing that $(\Delta H_\alpha \Delta^*)_{uv} = d_u h_{uv} \overline{d_v}$, we have

$$\begin{aligned} h_\alpha(C) &= h_\alpha(u_1u_2)h_\alpha(u_2u_3)\dots h_\alpha(u_nu_1) &(8)\\ &= d_{u_1}h_\alpha(u_1u_2)\overline{d_{u_2}}d_{u_2}h_\alpha(u_2u_3)\overline{d_{u_3}}\dots d_{u_n}h_\alpha(u_nu_1)\overline{d_{u_1}} &(9)\\ &= k(C). &(10)\end{aligned}$$

Example 1. *Let D be the mixed graph shown in Figure 2a, α is the primitive third root of unity $e^{\frac{2\pi}{3}i}$ and H_α its α-Hermitian adjacency matrix. Obviously, the tree mixed subgraph $\mathbb{T}$ illustrated in the solid lines is α-monograph with the orienting matrix $\Delta_\alpha = diag(t_v : t_v \in \mathbb{T}$ and $v \in V(D))$. Then, the matrix $\mathbb{K} = \Delta_\alpha H_\alpha \Delta_\alpha^*$ can be considered as the adjacency matrix of the complex weighted graph $\mathfrak{D}_\alpha$ (or the unit gain graph $\mathfrak{D}_\alpha$, see[4]) shown in Figure 2b with weight 1 for each edge of the tree $\Gamma(T)$. Furthermore according to Lemma 1, the weight of the edge 53 in $\mathfrak{D}_\alpha$, say $k(53)$, equals to the weight of the fundamental cycle $C_{53} = 32453$ in H_α. To be more formal,*

$$k(53) = t_5 h_{53} \overline{t_3}.$$

But,

$$t_5 = t_3 h_{32} h_{24} h_{45}$$

Therefore,

$$k(53) = \alpha^2 = \overline{k(35)}$$

Similar calculations can be done to get, $k(89) = \alpha^2$. Therefore, the matrix $\mathbb{K} = \Delta_\alpha H_\alpha \Delta_\alpha^*$ is the adjacency matrix of the complex weighted graph $\mathfrak{D}_\alpha$.

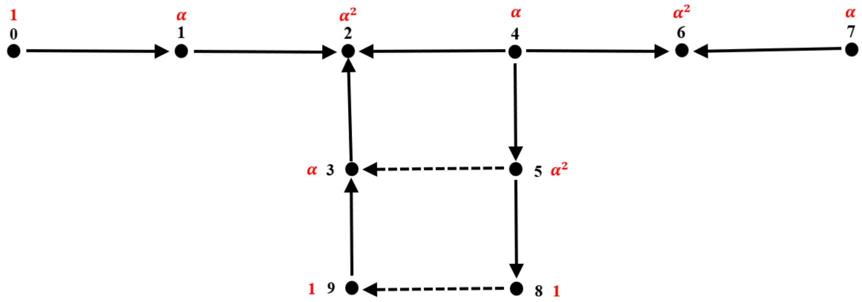
The above example illustrates that, for every mixed graph D and a spanning mixed tree $\mathbb{T}$ of D, there is a complex weighted mixed graph $\mathfrak{D}$ that is cospectral with the α-hermitian adjacency matrix of D. This idea will be used to prove some further results.

Definition 3. *Let D be a graph and α, γ are two unit complex numbers. If the α-Hermitian adjacency matrix and the γ-Hermitian adjacency matrix of D are cospectral, then D is called $\alpha - \gamma$ cospectral.*

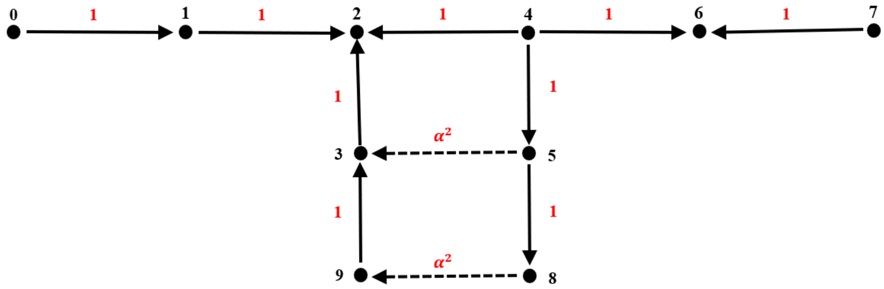
Theorem 3. *Let D be a mixed graph and α, γ be two unit complex numbers. If for all cycles C in D one of $h_\alpha(C) = h_\gamma(C)$ or $h_\alpha(C) = \overline{h_\gamma(C)}$ is hold then D is $\alpha - \gamma$ cospectral.*

Proof. Suppose that T is a spanning tree of D, u is a vertex of D and Δ_α (resp. Δ_γ) is the α-orienting (resp. the γ-orienting) matrix of T. Then,

$$\Delta_\alpha H_\alpha(T) \Delta_\alpha^* = \Delta_\gamma H_\gamma(T) \Delta_\gamma^* = A(\Gamma(T))$$



(a) The mixed graph D with the values of t_v in the spanning tree $\mathbb{T}$.



(b) The complex weighted graph $\mathfrak{D}_\alpha$ with the weight of the edges

Figure 2

which means that the matrices $\Delta_\alpha H_\alpha(D)\Delta_\alpha^*$ (resp. $\Delta_\gamma H_\gamma(D)\Delta_\gamma^*$) can be considered as a Hermitian adjacency matrices of the weight mixed graph $\mathfrak{D}_\alpha$ (resp. $\mathfrak{D}_\gamma$) with underlying graph $\Gamma(D)$, all edges of the tree T have weight 1. Furthermore, suppose that rs is an edge in D that is not in T, and C_{rs} be the fundamental cycle of D corresponding to rs. Then, using Lemma 1 we get that, the weight of rs in $\mathfrak{D}_\alpha$ (resp. $\mathfrak{D}_\gamma$) equals $h_\alpha(C_{rs})$ (resp. $h_\gamma(C_{rs})$). Therefore, either $\Delta_\alpha H_\alpha(D)\Delta_\alpha^* = \Delta_\gamma H_\gamma(D)\Delta_\gamma^*$ or $\Delta_\alpha H_\alpha(D)\Delta_\alpha^* = \Delta_\gamma \overline{H}_\gamma(D)\Delta_\gamma^*$, which means $H_\alpha(D)$ and $H_\gamma(D)$ are cospectral.

Corollary 3. *Let D be a mixed graph and α, γ are unit complex numbers. If D is α-monostore mixed graph then D is $\gamma - \alpha\gamma$ cospectral.*

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