EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 3, 2022, 1211-1216
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On the relationship between the integral of derivative and the derivative of the integral

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Abstract. We would like to study the relationship between the integral of the derivative and the derivative of the integral. This is the study of whether these two are the same or not. Also, this study began with the question of Kreyszig's assumptions presented in [8]. The research method used two tools: the Riemann integral and its generalized concept, the Lebesgue integral. The obtained research results give some answer to this question.

2020 Mathematics Subject Classifications: 35-02

Key Words and Phrases: Leibniz integral rule, interchange of integral and derivative, monotone convergence theorem

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1. Introduction

Among the various methods to find the solution of PDEs, there is a method using integral transform [3–9]. There is one problem in this solution process. That is, the solution is made under the assumption that the order of differentiation and integration can be changed. This problem appears frequently in the calculation process of $\pounds(u_{xx})$ in PDEs. This point was mentioned in [8], and it seems that Kreyszig was also aware of it(Sec. 12.11 of [8]). Is it possible to interchange the integral and derivative in solving PDEs by integral transform? This study intends to start with this question.

Since the solution by integral transform is almost similar, we would just like to approach by the Laplace transform.

Consider a function $w(x,t) = x^n + t^n$, where n is a positive integer. By simple calculation, we get

$$\frac{\partial^n}{\partial x^n}\int w(x,t)\ dt = \int \frac{\partial^n}{\partial x^n} w(x,t)\ dt$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v15i3.4424

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under the assumption that the difference as much as constant is accepted. On one hand, as a counter example, consider the following example: Let y be the positive integers, dy be the counting measure, and

$$f(x,y) = \{ \begin{array}{cc} x & (\frac{1}{y+1} < x < \frac{1}{y}) \\ 0 & \text{elsewhere} \end{array} .$$

It is well-known that the integral of the derivative of f is 0 at $x = 0$ and the derivative of the integral is 1 at $x = 0$.

It is necessary to find out more about to what extent equality holds and where equality does not hold. In the next section, consider a further approach to this topic.

2. On the relationship between the integral of derivative and the derivative of the integral

From the point of view of the Riemann integral, this interchange is closely related to the following Leibniz integral rule. From the point of view of the Lebesgue integral, this interchange relates to monotone convergence theorem, Beppo Levi's theorem and Lebesgue dominated convergence theorem. Therefore, let us take a look at these theorems first as lemmas.

Lemma 1. *(Leibniz integral rule) If f and f_x are continuous functions, then*

$$\frac{d}{dx}(\int_{a(x)}^{b(x)} f(x,t)\ dt) = f(x,b(x)) \cdot \frac{d}{dx}b(x) - f(x,a(x)) \cdot \frac{d}{dx}a(x)$$

$$+ \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\ dt$$

is valid for $a(x), b(x), a'(x)$ and $b'(x)$ are all continuous. If $a(x)$ and $b(x)$ are constants, we can easily see that

$$\frac{d}{dx}(\int_a^b f(x,t)\ dt) = \int_a^b \frac{\partial}{\partial x} f(x,t)\ dt.$$

Lemma 2. *(Monotone convergence theorem(MCT) [1, 2]). Let M^+ be the collection of all non-negative measurable functions and let μ be a measure. If (f_n) is a monotone increasing sequence of functions in M^+ which converges to f, then*

$$\int f\ d\mu = \lim \int f_n\ d\mu.$$

In other words, this theorem can be expressed as

$$\int f\ d\mu = \int \lim f_n\ d\mu = \lim \int f_n\ d\mu.$$

Putting $f_n = g_1 + \cdots + g_n$ and applying the MCT, then

$$\int \sum_{n=1}^{\infty} g_n \; d\mu = \int \lim \sum_{n=1}^{m} g_n \; d\mu = \lim \sum_{n=1}^{m} \int g_n \; d\mu = \sum_{n=1}^{\infty} \int g_n \; d\mu$$

for g_n is measurable. This is the famous Beppo Levi's theorem.

Lemma 3. *(Beppo Levi's theorem[2]). Let (g_n) be a sequence in M^+, then*

$$\int \sum_{n=1}^{\infty} g_n \; d\mu = \sum_{n=1}^{\infty} \int g_n \; d\mu.$$

Lemma 4. *(Lebesgue dominated convergence theorem(LDCT) [1, 2]). Let (f_n) be a sequence of integrable functions which converges almost everywhere to a real-valued measurable function f. If there exists an integrable function g such that $|f_n| \leq g$ for all n, then f is integrable and*

$$\int f \; d\mu = \lim \int f_n d\mu.$$

As an intuitive approach, consider whether

$$\frac{d}{dx} \int_a^b f(x,t) \; dt = \int_a^b f_x(x,t) \; dt$$

or not. Putting $g(x) = \int_a^b f(x,t) \; dt$, we have

$$\begin{aligned} g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} &= \lim_{h\to 0} \frac{\int_a^b f(x+h,t) \; dt - \int_a^b f(x,t) \; dt}{h} \\ &= \lim_{h\to 0} \frac{\int_a^b (f(x+h,t) - f(x,t))}{h} \; dt \\ &= \lim_{h\to 0} \int_a^b \frac{f(x+h,t) - f(x,t)}{h} \; dt \\ &= \int_a^b f_x(x,t) \; dt \end{aligned}$$

by LDCT and mean value theorem. Therefore,

$$\frac{d}{dx} \int_a^b f(x,t) \; dt = \int_a^b f_x(x,t) \; dt,$$

where f_x is the partial derivative with respect to x. In this regard, the following lemma can be constructed:

Lemma 5. *Let X be an open subset of R, and ω be a measure space. If $f : X \times \omega \to R$ satisfies the following conditions: (i)f is Lebesgue-integrable, (ii)f_x exists almost everywhere, and (iii) there is an integrable function g such that $|f_x| \le g$, then*

$$\frac{d}{dx}\int_{\omega} f = \int_{\omega} f_x.$$

Proof. The proof follows from the LDCT and the mean value theorem.

Regarding this problem, consider three places in Kreyszig's Advanced Engineering Mathematics [8]. One is shown in the proof of theorem 1 of section 11.1, and the other two are shown in the solution of example 1 of section 12.12.

Theorem 1. *(1)(Interchange of infinite series and integral)*

$$\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)\ dx$$

$$= \sum_{n=1}^{\infty}(a_n \int_{-\pi}^{\pi} cosnx\ dx + b_n \int_{-\pi}^{\pi} sinnx\ dx).$$

(2) (Interchange of limit and integral) Let w be the displacement of an elastic string such that

$$w(0,t) = f(t) = \{ \begin{array}{cc} sin\ t & (0 \le t \le 2\pi) \\ 0 & \text{otherwise} \end{array} .$$

Then

$$\lim_{x\to\infty}\int_0^{\infty} e^{-st}w(x,t)\ dt = \int_0^{\infty} e^{-st}\lim_{x\to\infty} w(x,t)\ dt$$

is valid for

$$\lim_{x\to\infty} w(x,t) = 0.$$

(3) (Interchange of derivative and integral) In the function of (2),

$$\int_0^{\infty} e^{-st}\frac{\partial^2 w}{\partial x^2}\ dt = \frac{\partial^2}{\partial x^2}\int_0^{\infty} e^{-st} w\ dt$$

is established.

Proof. (1) It suffices to show that the assumptions of Beppo Levi's theorem are satisfied. Since

$$\sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)$$

is nonnegative valued on $(-\pi, \pi)$ and measurable, the proof is complete.

(2) Let us check whether the assumption of LDCT is satisfied. Note that $|w(x,t)| \leq 1$ for all t and

$$\lim_{x \to \infty} w(x,t) = 0$$

for $t \geq 0$. Therefore, the result of (2) is naturally derived.

(3) From the point of view of the Riemann integral, e^{-st} and $-se^{-st}$ are continuous, so the result is obtained by the Leibniz integral rule. Next, let us check it in terms of the more general Lebesgue integral. Since e^{-st} is Lebesgue integrable on $[0,\infty)$, $-se^{-st}$ exists, and for some constants M and k it satisfies

$$|-se^{-st}| \leq Me^{kt},$$

by lemma 5, the proof is complete.

Based on the above results, Kreyszig's assumption (p 601 of [8]) can be considered unnecessary. The reason is that $e^{-st}w$ and its derivative are continuous.

Conflict of interest. The authors declare no conflicts of interest.

Acknowledgements

The corresponding author (Hj. Kim) acknowledges the support of Kyungdong University Research Fund, 2022.

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