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Neural Network of Multivariate Square Rational Bernstein Operators with Positive Integer Parameter

Ibtihal J. Mohammad[1,*], Ali J. Mohammad[1]

[1] *Department of Mathematics, College of Education for Pure Science, University of Basrah, Basrah, Iraq*

Abstract. This research is defined a new neural network(NN) that depends upon a positive integer parameter using the multivariate square rational Bernstein polynomials. Some theorems for this network are proved, such as the pointwise and the uniform approximation theorems. Firstly, the absolute moment for a function that belongs to Lipschitz space is defined to estimate the order of the NN. Secondly, some numerical applications for this NN are given by taking two test functions. Finally, the numerical results for this network are compared with the classical neural networks(NNs). The results turn out that the new network is better than the classical one.

2020 Mathematics Subject Classifications: 41A25, 41A30, 47A58

Key Words and Phrases: Multivariate neural network, Multivariate square rational Bernstein polynomials, Activation functions, Lipschitz Space

1. Introduction

In 2013, Costarelli and Spigler [3] introduced the artificial NN operators and studied the behavior of this neural network in univariate Bernstein polynomials as:

For a bounded function $f : [a, b] \longrightarrow \mathbb{R}$, the artificial neural networks $F_n(f; x)$, activated by the sigmoidal function σ and its acting on f, is defined as:

$$F_n(f;x) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi_\sigma(x-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_\sigma(x-k)}, x \in [a, b],$$

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Email addresses: pgs2206@uobasrah.edu.iq, ibtihaljas.moh@gmail.com (I.J. Mohammad), ali.mohammad@uobasrah.edu.iq, alijasmoh@gmail.com (A.J. Mohammad).

where the symbols $\lfloor . \rfloor, \lceil . \rceil$ denote taking the "floor" and the "ceiling" of a given number, respectively. And the case of multivariate in [4] it is given by the formula: The bounded function: $f : \mathcal{R} \longrightarrow \mathbb{R}$, activated by the sigmoidal function σ and acting on f, is defined as:

$$F_n^s(f;\mathbf{x}) = \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_s=\lceil na_s \rceil}^{\lfloor nb_s \rfloor} f\left(\frac{\mathbf{k}}{n}\right) \Psi_\sigma(n\mathbf{x}-\mathbf{k})}{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_s=\lceil na_s \rceil}^{\lfloor nb_s \rfloor} \Psi_\sigma(n\mathbf{x}-\mathbf{k})}, \tag{1}$$

where $\mathbf{x} \in \mathcal{R} = [a_1, b_1] \times \ldots \times [a_s, b_s]$, Ψ_σ is a density function that is built from a sigmoidal function σ and $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^+$.

In 2014, Costarelli and Spigler [5] extended formula (1) through the use of the Kantorovich operator type to introduce and studied approximation theorems to this multivariate NN operators.

In 2016, Costarelli and Vinti [6] introduced the structure of a NN operators of type multivariate max-product then studied the approximation theoremed and estimates the rate of convergence to this NN operators.

In 2017, Gavrea and Ivan [7] introduced definition to square Bernstein polynomials it is given by the formula:
For $x \in [0,1], f \in C[0,1]$,

$$B_{n,2}(f;x) = \frac{\sum_{k=0}^{n} b_{n,k}^2(x) f\left(\frac{k}{n}\right)}{\sum_{k=0}^{n} b_{n,k}^2(x)}, n = 1, 2, ..., \tag{2}$$

where $b_{n,k}^2(x) = (b_{n,k}(x))^2$.

In 2017, Mohammad and Mohammad [9] introduced a definition of the NN operators by using the type of summation-integral Bernstein, and then studied approximation theorems for this NN operators.

In 2018, Hassan [8] introduce and define the new modified of Bernstein operators that can use to build NNs.

In 2019, Bajpeyi and Kumar [1] introduced definition to the neural network of exponential type and studied behavior in two case one- dimensional and multi-dimensional.In 2019, Costarelli and others [2] introduced definition to the neural network of multivariate max-product NN of Kantorovich type.

In 2021, Mohammad and Mohammad [10] give a new modification to the formula (1) and studied approximation theorems for this NN operators,activated by the sigmoidal function σ and acting on f, it is given by the formula:

$$G_{n,m}(f;\mathbf{x}) = \frac{\sum_{\mathbf{k}} \Psi_\sigma(n\mathbf{x}-\mathbf{k}) f\left(\left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m - \mathbf{x}\right)}{\sum_{\mathbf{k}} \Psi_\sigma(n\mathbf{x}-\mathbf{k})}, \tag{3}$$

This paper gives extended to the NN operators in formula (3) by using formula of square Bernstein polynomials in formula (2) and studied approximation theorems for this neural network. In the end, we give some numerical examples for this NNs.

2. Preliminary Results

In this part recall some preliminary results.

A sigmoidal function is measurable functions satisfying $\lim_{x\to-\infty}\sigma(x) = 0$ and $\lim_{x\to+\infty}\sigma(x) = 1$, for example logistic function $\sigma_l(x) = (1+e^{-x})^{-1}$, hyperbolic tangent function $\sigma_h(x) = \frac{1}{2}[tanh(x)+1]$.

For every non-decreasing function σ satisfying assumptions:

(i) the function such that $g_\sigma(x) = \sigma(x) - 1/2$, is odd;

(ii) function $\sigma \in C^2(\mathbb{R})$ is concave for $x \geq 0$;

(iii) function σ satisfying $\sigma(x) = O(|x|^{-1-\alpha})$ as $x \longrightarrow -\infty$, for some $\alpha>0$.

Defined the function as: $\Phi_\sigma(x) = \frac{1}{2}[\sigma(x+1) - \sigma(x-1)], x \in \mathbb{R}$.

Now, gives some definitions that we will use:

Definition 1. *[4] A sigmoidal function is a measurable function satisfying the following two conditions:*

$\lim_{x\to-\infty}\zeta(x) = 0$;

$\lim_{x\to+\infty}\zeta(x) = 1$.

Definition 2. *[4]*

The Lipschitz classes are defined as follows:
Lip$(v) = \{f \in C^0(\mathcal{R})$ *such that there exist* $\gamma>0$, $C>0$ *so that, for each* $\mathbf{x} \in \mathcal{R}$, $|f(\mathbf{x}+\mathbf{t}) - f(\mathbf{x})| \leq C\|\mathbf{t}\|_2^v$ *for every* $\|\mathbf{t}\|_2 \leq \gamma$ *with* $(\mathbf{x}+\mathbf{t}) \in \mathcal{R}\}$.

Definition 3. *[5]*

The multivariate for the $\Phi_\sigma(x)$ *define as follows:* $\Psi_\sigma(\mathbf{x}) = \Phi_\sigma(x_1)\cdot\Phi_\sigma(x_2)\cdot\ldots\cdot\Phi_\sigma(x_s)$, *for every* $\mathbf{x} \in \mathbb{R}^s$.

Now, in the following lemmas set of properties for the functions $\Phi_\sigma(x)$ *will be studied.*

Lemma 1. *[5]*

To the function $\Phi_\sigma(x)$ *for* $x \in \mathbb{R}$, *then:*

(i) $\Phi_\sigma(x) \geq 0$ *for every* $x \in \mathbb{R}$ *and* $\lim_{x\to\pm\infty} \Phi_\sigma(x) = 0$*;*

(ii) $\Phi_\sigma(x)$ *is a symmetrical function about the y-axis;*

(iii) $\sum_{k\in\mathbb{Z}} \Phi_\sigma(x-k) = 1$*, For every* $x \in \mathbb{R}$ *;*

(iv) For $x<0$ *the function* $\Phi_\sigma(x)$ *is non-decreasing and for* $x \geq 0$ *it is non-increasing;*

(v) $\Phi_\sigma(x) = O(|x|^{-1-\alpha})$ *as* $x \longrightarrow \pm\infty$*;*

(vi) The sum $\sum_{k\in\mathbb{Z}} \Phi_\sigma(x-k)$ *converges uniformly on subsets compact of* $\mathbb{R}$*.*

The following lemmas set of properties for the functions $\Psi_\sigma(\mathbf{x}-\mathbf{k})$ will be studied.

Lemma 2. *[4]*
To the function $\Psi_\sigma(\mathbf{x}-\mathbf{k})$ *for* $\mathbf{x} \in \mathbb{R}^s$*, then:*

(i) $\sum_{\mathbf{k}} \Psi_\sigma(\mathbf{x}-\mathbf{k}) = 1$*,for every* $\mathbf{x} \in \mathbb{R}^s$*;*

(ii) On compact subsets of $\mathbb{R}^s$ *the series* $\sum_{\mathbf{k}} \Psi_\sigma(\mathbf{x}-\mathbf{k})$ *converges uniformly on compact subsets of* $\mathbb{R}^s$*;*

(iii) For every $\gamma>0$*, we get*

$$\lim_{x\to\infty} \sum_{\|\mathbf{x}-\mathbf{k}\|>\gamma n} \Psi_\sigma(\mathbf{x}-\mathbf{k}) = 0,$$

uniformly respect to $\mathbf{x} \in \mathbb{R}^s$*.In a special case, for every* $\gamma>0$ *and* $0<v<\alpha$ *,*

$$\sum_{\|\mathbf{x}-\mathbf{k}\|>\gamma n} \Psi_\sigma(\mathbf{x}-\mathbf{k}) = O(n^{-v}),\ n \longrightarrow +\infty,$$

where the constant $\alpha>0$ *as in condition (iii).*

Lemma 3. *[3], [4]*

(i) For $x \in [a,b] \subset \mathbb{R}, n \in \mathbb{N}^+$*,* $\lceil na \rceil \leq \lfloor nb \rfloor$*, then:*

$$\frac{1}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_\sigma(nx-k)} \leq \frac{1}{\Phi_\sigma(1)};$$

(ii) For $\mathbf{x} \in [a_1,b_1] \times ... \times [a_s,b_s] \subset \mathbb{R}^s$*,* $n \in \mathbb{N}^+$ *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$ *for every* $i = 1,...,s$*, then:*

$$\frac{1}{\prod\limits_{i=1}^{s} \sum\limits_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_\sigma(nx_i-k_i)} \leq \frac{1}{[\Phi_\sigma(1)]^s}.$$

3. Auxiliary Results

We will define and discuss multivariate NN operators $Q_m(f;x)$ as follows:

Definition 4. *For a continuous bounded function $f:\mathcal{R}\longrightarrow\mathbb{R}$, the NN operators of multivariate square rational Bernstein operators with positive integer parameter m, $Q_m(f;x)$ activated by the sigmoidal function σ acting on f, defined as the following:*

$$Q_m(f;\mathbf{x})=\frac{\sum_{\mathbf{k}}\Psi_\sigma^2(n\mathbf{x}-\mathbf{k})f\left(\left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m-\mathbf{x}\right)}{\sum_{\mathbf{k}}\Psi_\sigma^2(n\mathbf{x}-\mathbf{k})},m\in\mathbb{N}^+$$

$$\sum_{\mathbf{k}}=\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\cdots\sum_{k_s=\lceil na_s\rceil}^{\lfloor nb_s\rfloor}.$$

for sufficiently large $n\in\mathbb{N}$, $\mathbf{x}\in\mathcal{R}$, $Q_m(1;\mathbf{x})=1$.

Definition 5. *For $v>0$, the discrete absolutely moment of the function $\Phi_\sigma^2(x)$ of order v is defined as*

$$m_v(\Phi_\sigma^2)=\sup_{x\in\mathbb{R}}\sum_{k\in\mathbb{Z}}\Phi_\sigma^2(x-k)|x-k|^v.$$

We will need to give some properties of the functions $\Phi_\sigma^2(x)$ and $\Psi_\sigma^2(x)$ in the following lemmas:

Lemma 4. *Some properties to the function $\Phi_\sigma^2(x)$ defined on $x\in\mathbb{R}$, then:*

(i) $\Phi_\sigma^2(x)\geq 0$ *for every* $x\in\mathbb{R}$ *and* $\lim_{x\to\pm\infty}\Phi_\sigma^2(x)=0$;

(ii) $\Phi_\sigma^2(x)$ *is a symmetrical function about the y-axis;*

(iii) $\sum_{k\in\mathbb{Z}}\Phi_\sigma^2(x-k)\approx 0.156517$, *For every* $x\in\mathbb{R}$;

(iv) *For* $x<0$ *the function* $\Phi_\sigma^2(x)$ *is non-decreasing and for* $x\geq 0$ *it is non-increasing;*

(v) $\Phi_\sigma^2(x)=O(|x|^{2(-1-\alpha)})$ *as* $x\longrightarrow\pm\infty$;

(vi) *The sum* $\sum_{k\in\mathbb{Z}}\Phi_\sigma^2(x-k)$ *converges uniformly on subsets compact of* $\mathbb{R}$.

Proof. *By applying Lemma 1, we can prove (i), (ii), (iv),(v) and (vi) immediately, the consequence (iii) can be claimed by using Maple software.* □

The next lemma gives some properties for the function $\Psi_\sigma^2(n\mathbf{x}-\mathbf{k})$.

Lemma 5. *To the function $\Psi_\sigma^2(\mathbf{x}-\mathbf{k})$ for $x\in\mathbb{R}^s$, then:*

(i) $\sum_{\mathbf{k}} \Psi_\sigma^2(\mathbf{x}-\mathbf{k}) \approx (0.156517)^s$, *for every* $\mathbf{x} \in \mathbb{R}^s$*;*

(ii) *On compact subsets of* $\mathbb{R}^s$ *the series* $\sum_{\mathbf{k}} \Psi_\sigma^2(\mathbf{x}-\mathbf{k})$ *converges uniformly on compact subsets of* $\mathbb{R}^s$*;*

(iii) *For every* $\gamma>0$*, we get*

$$\lim_{x\to\infty} \sum_{\|\mathbf{x}-\mathbf{k}\|>\gamma n} \Psi_\sigma^2(\mathbf{x}-\mathbf{k}) = 0,$$

uniformly respect to $\mathbf{x} \in \mathbb{R}^s$*.In a special case, for every* $\gamma>0$ *and* $0<v<\alpha$*,*

$$\sum_{\|\mathbf{x}-\mathbf{k}\|>\gamma n} \Psi_\sigma^2(\mathbf{x}-\mathbf{k}) = O(n^{-v}),\ as\ n \longrightarrow +\infty$$

where the constant $\alpha>0$ *as in condition (iii).*

Proof. *Using the Definition 3 and Lemma 2, the consequence (i),(ii),(iii) gets immediate.* □

Lemma 6. *(i)* *For* $x \in [a,b] \subset \mathbb{R}, n \in \mathbb{N}^+$*,* $\lceil na \rceil \le \lfloor nb \rfloor$*, then:*

$$\frac{1}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_\sigma^2(nx-k)} \le \frac{1}{\Phi_\sigma^2(1)};$$

(ii) *For* $\mathbf{x} \in [a_1,b_1] \times ... \times [a_s,b_s] \subset \mathbb{R}^s$*,* $n \in \mathbb{N}^+$ *so that* $\lceil na \rceil \le \lfloor nb \rfloor$ *for every* $i = 1,...,s$*, then:*

$$\frac{1}{\prod\limits_{i=1}^{s} \sum\limits_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_\sigma^2(nx_i-k_i)} \le \frac{1}{[\Phi_\sigma^2(1)]^s}.$$

Proof. *Using the properties of Lemma 3 the proof of this Lemma follows immediately.* □

The following theorem studies the pointwise and the uniform convergence for the NN operators, $Q_m(f;x)$.

Theorem 1. *For a bounded function* $f : \mathcal{R} \longrightarrow \mathbb{R}$*, and continuous at each point* $\mathbf{x} \in \mathcal{R}$*, then*

$$\lim_{n\to\infty} Q_m(f;\mathbf{x}) = f(\mathbf{x})$$

if $f \in C^0(\mathcal{R})$*, then*

$$\lim_{n\to\infty} \sup_{\mathbf{x}\in\mathcal{R}} |Q_m(f;\mathbf{x}) - f(\mathbf{x})| = \lim_{n\longrightarrow\infty} \|Q_m(f;.) - f(.)\|_\infty = 0.$$

Proof. *Suppose* $\mathbf{x} \in \mathcal{R}$ *is a point of continuity of* f *we have*

$$|Q_m(f;\mathbf{x}) - f(\mathbf{x})| = \left| \frac{\sum_{\mathbf{k}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k}) f\left(\left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m - \mathbf{x}\right)}{\sum_{\mathbf{k}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k})} - f(\mathbf{x}) \right|$$

$$= \left| \frac{\sum_{\mathbf{k}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k}) \left[f\left(\left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m - \mathbf{x}\right) - f(\mathbf{x}) \right]}{\sum_{\mathbf{k}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k})} \right|$$

by using Lemma 6(ii),we get:

$$|Q_m(f;\mathbf{x}) - f(\mathbf{x})| \leq \frac{1}{[\Phi_\sigma^2(1)]^s} \sum_{\mathbf{k}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k}) \left| f\left(\left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m - \mathbf{x}\right) - f(\mathbf{x}) \right|$$

For every $n \longrightarrow \infty$,$n \in N^+$,$\mathbf{x} \in \mathbb{R}^s$ *are arbitrary but fixed. Suppose for a fixed* ε>0, *and from the continuity of* f *at* $\mathbf{x}$, $\exists\gamma$>0 : $|f(\mathbf{y}) - f(\mathbf{x})|$<$\varepsilon$, $\forall \mathbf{y} \in \mathcal{R}$ *with* $\|\mathbf{y}-\mathbf{x}\|$<$\varepsilon$, *the symbol* $\|.\|_2$ *denote to Euclidean norm.*
Now we get

$$|Q_m(f;\mathbf{x}) - f(\mathbf{x})| \leq \frac{1}{[\Phi_\sigma^2(1)]^s} \sum_{\|(n^{-1}\mathbf{k}-\mathbf{x})^m\| < \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k}) \left| f\left((n^{-1}\mathbf{k}-\mathbf{x})^m - \mathbf{x}\right) - f(\mathbf{x}) \right| +$$

$$\frac{1}{[\Phi_\sigma^2(1)]^s} \sum_{\|(n^{-1}\mathbf{k}-\mathbf{x})^m\| \geq \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k}) \left| f\left((n^{-1}\mathbf{k}-\mathbf{x})^m - \mathbf{x}\right) - f(\mathbf{x}) \right|$$

$$:= \frac{1}{[\Phi_\sigma^2(1)]^s} (I_1 + I_2)$$

Now using the continuity of f *and Lemma 5 we get that*

$$\left\|\left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m - \mathbf{x}\right\|_2 \leq \sqrt{s} \left\|\left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m - \mathbf{x}\right\| \leq \gamma$$

So estimation I_1 *is,*

$$I_1 < \varepsilon \sum_{\|(n\mathbf{x}-\mathbf{k})^m\| < \frac{\gamma n}{\sqrt{s}}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k}) \leq \varepsilon.$$

$$I_2 \leq 2\|f\|_\infty \sum_{\|(n\mathbf{x}-\mathbf{k})^m\| \geq \frac{\gamma n}{\sqrt{s}}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k}) \leq \varepsilon.$$

uniformly $\forall \mathbf{x} \in \mathbb{R}^s$*.The first direction of the theorem holds because* ε *arbitrarily. When* $f \in C^0(\mathcal{R})$*, the prove of other direction is readily followed in the same way by exchange* $\gamma>0$ *with the parameter of the uniform continuity of* f *on* $\mathcal{R}$*.* □

Now, in the following, study the order of approximation of NN operators in $f \in Lip(v)$.

Theorem 2. *Suppose* $f \in Lip(v)$ *for some* v*, at* $0<v \leq 1$*, and let sigmoidal function* σ *satisfies the condition (iii) for* $\|Q_m(f;\mathbf{x}) - f(\mathbf{x})\|_\infty = O(n^{-vm})$*, as* $n \longrightarrow \infty$*.*

Proof. *Let* $f \in Lip(v)$*, for every* $\mathbf{x} \in \mathcal{R}$*, for some* v*, with* $v \in (0,1]$*, and Lemma 6 one can write as in the theorem 1*

$$|Q_m(f;\mathbf{x}) - f(\mathbf{x})| \leq \frac{1}{[\Phi_\sigma^2(1)]^s} \sum_{\mathbf{k}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k}) \left| f\left(\left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m - \mathbf{x}\right) - f(\mathbf{x}) \right|$$

Now by using the definition of $Lip(v)$*, where* $\gamma>0, C>0$ *are constants relative to* f *we obtain Let* $\mathbf{x} \in \mathcal{R}$ *the point of continuity of* f

$$|Q_m(f;\mathbf{x}) - f(\mathbf{x})| \leq$$

$$\frac{1}{[\Phi_\sigma^2(1)]^s} \sum_{\|(n^{-1}\mathbf{k}-\mathbf{x})^m\| < \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^2(n^{-1}\mathbf{k}-\mathbf{x}) \left| f\left((n^{-1}\mathbf{k}-\mathbf{x})^m - \mathbf{x}\right) - f(\mathbf{x}) \right|$$

$$+\frac{1}{[\Phi_\sigma^2(1)]^s} \sum_{\|(n^{-1}\mathbf{k}-\mathbf{x})^m\| \geq \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k}) \left| f\left((n^{-1}\mathbf{k}-\mathbf{x})^m - \mathbf{x}\right) - f(\mathbf{x}) \right|$$

$$:= \frac{1}{[\Phi_\sigma^2(1)]^s} (J_1 + J_2)$$

since $f \in Lip(v)$*, we get for* $\left\| \left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m - \mathbf{x} \right\|_2 \leq \sqrt{s} \left\| \left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m - \mathbf{x} \right\| \leq \gamma$ *and hence*

$$|f((n\mathbf{x}-\mathbf{k})^m - \mathbf{x}) - f(\mathbf{x})| < C \left\| \left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m \right\|_2^v \leq C s^{\frac{v}{2}} \left\| \left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m \right\|^v$$

$$J_1 < n^{-vm} C s^{\frac{v}{2}} \sum_{\|(n\mathbf{x}-\mathbf{k})^m\| < \frac{\gamma n}{\sqrt{s}}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k}) \left\| \left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m \right\|^v$$

for fixed $0<v_i<\alpha$ *, by using Lemma 5, for a compact subset* $K \subset \mathbb{R}^s$*, for every* $\mathbf{x} \in \mathbb{R}^s$*, if* $n \longrightarrow \infty$ *can write the following:*

$$J_1 \leq n^{-vm} C s^{\frac{v}{2}} \sum_{\|(n\mathbf{x}-\mathbf{k})^m\| < \frac{\gamma n}{\sqrt{s}}} \Psi_\sigma^2(n\mathbf{x}-\mathbf{k}) \left\| \left(n^{-1}\mathbf{k}-\mathbf{x}\right)^m \right\|^v$$

$$\leq n^{-vm} C s^{\frac{v}{2}} \sum_{j=1}^{s} \left\{ \sum_{k_j \in \mathbb{Z}} \Phi_\sigma^2(nx_j - k_j) \left| nx_j - k_j \right|^{vm} \left[\sum_{\mathbf{k}_{[j]} \in \mathbb{Z}^{s-1}} \Psi_\sigma^{2[j]}(n\mathbf{x}_{[j]} - \mathbf{k}_{[j]}) \right] \right\}$$

where $\Psi_\sigma^{2[j]}(n\mathbf{x}_{[j]} - \mathbf{k}_{[j]}) = \Phi_\sigma^2(nx_1 - k_1) \cdot ... \cdot \Phi_\sigma^2(nx_{j-1} - k_{j-1}) \cdot \Phi_\sigma^2(nx_{j+1} - k_{j+1}) \cdot ... \cdot \Phi_\sigma^2(nx_s - k_s)$, *notice for every* $j = 1, ..., s$ $\mathbf{x}_{[j]} = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_s) \in \mathbb{R}^{s-1}$, $\mathbf{k}_{[j]} = (k_1, ..., k_{j-1}, k_{j+1}, ..., k_s) \in \mathbb{Z}^{s-1}$. *Now* $k_{[j]} \subset \mathbb{R}$ *the set of the* j*-th projection of a compact set of all elements. By using Lemma 5 and for all sufficiently large* $N \in \mathbb{N}^+$ *one can obtain*

$$\leq (0.156517)^{s-1} n^{-vm} C s^{\frac{v}{2}} \sum_{j=1}^{s} \left\{ \sum_{k_j \in \mathbb{Z}} \Phi_\sigma^2(nx_j - k_j) \left| nx_j - k_j \right|^{vm} \right\}$$

$$\leq (0.156517)^{s-1} n^{-vm} C s^{1+\frac{v}{2}} m_{vm}(\Phi_\sigma^2)$$

note that $m_{vm}(\Phi_\sigma^2) \leq \infty$ *, where* $m_{vm}(\Phi_\sigma^2)$ *give in Definition 5, since* $v < \alpha$ *one can obtain:*

$$J_1 = O(n^{-vm}),\ as\ n \longrightarrow \infty,$$

Now, we estimate J_2 *by using the other direction of lemma 5,*

$$I_2 \leq 2\left\| f \right\|_\infty \sum_{\|(n\mathbf{x}-\mathbf{k})^m\| \geq \frac{\gamma n}{\sqrt{s}}} \Psi_\sigma^2(n\mathbf{x} - \mathbf{k}) = O(n^{-vm}),\ as\ n \longrightarrow \infty. \ \square$$

Theorem 3. *Let the function* σ *for some* $\alpha \in (0, 1]$ *satisfy the condition* (iii)*, and let* $f \in Lip(v)$ *for some* $v \in (0, 1]$*, then we have,*

(i) $\|Q_m(f; .) - f(.)\|_\infty = O(n^{-vm})$*, as* $n \longrightarrow \infty$ *If* $v<\alpha$*;*

(ii) $\|Q_m(f; .) - f(.)\|_\infty = O(n^{-(\alpha-\varepsilon)m})$*, as* $n \longrightarrow \infty$*, for every* $0<\varepsilon<\alpha$*, if* $\alpha \leq v<1$*.*

Proof.

(i) Using the same step of the "Theorm 2" we can obtain proving

$$\|Q_m(f; .) - f(.)\|_\infty = O(n^{-vm}), \text{ as } n \longrightarrow \infty$$

(ii) As a special case for all $f \in Lip(v)$ with $\alpha \leq v \leq 1$, with ε is fixed but arbitrary choose $\beta := \alpha - \varepsilon$, and get $0<\beta<\alpha$, by based on part (i) we obtain,

$$\|Q_m(f; .) - f(.)\|_\infty = O(n^{-\beta m}) = O(n^{-(\alpha-\varepsilon)m}), \text{ as } n \longrightarrow \infty$$

for function $f \in Lip(\beta)$, $0<\varepsilon<\alpha$. $\square$

4. Numerical Examples

This section gives numerical examples for the real value of $n = 10, 30$, $m = 1, 2, 3$ and the functions of testing $f(x, y) = \cos(9xy) + 2\sin(x + y)$ and $g(x, y) = (2x - 1)^2 - (2y - 1)^2, (x, y) \in [0, 1]^2$. For the NN operators $Q_m(.; x, y)$ with the NN operators $F_n(.; x, y)$. We analyze the results in the figures as examples of the convergence of NN operators $Q_m(.; x, y)$, $F_n(.; x, y)$ with test the functions $f(x, y)$, $g(x, y)$. Also, we give the table to maximum error function for $Q_m(.; x, y)$,$F_n(.; x, y)$ as follows:

Example 1. *For $n = 10, 30, m = 1, 2, 3$, the convergence of NN operators $Q_m(f; x, y)$, $F_n(f; x, y)$ to test function $f(x, y)$ can be descripted in the Figure 1.*

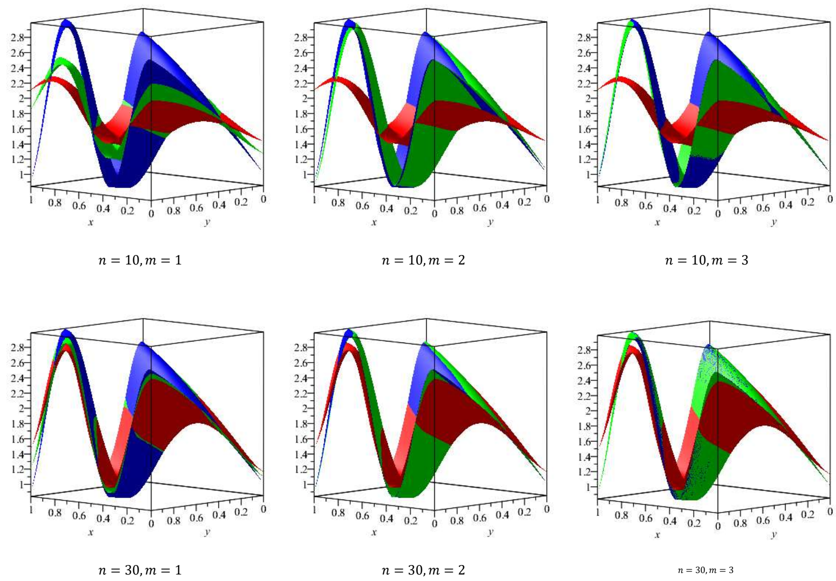

Figure 1: The numerical convergence of NN operators $F_n(f; x, y)$ (red) and $Q_m(f; x, y)$ (Lime) to $f(x, y)$ (blue).

Example 2. *Forn $= 10, 30, m = 1, 2, 3$, the convergence of NN operators $Q_m(g; x, y)$, $F_n(g; x, y)$to test function $g(x, y)$ can be descripted in the Figure 2.*

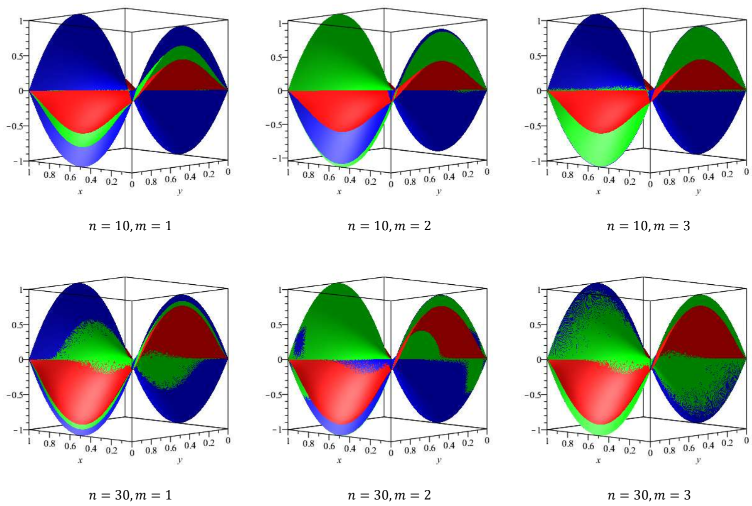

Figure 2: The numerical convergence of NN operators $F_n(g; x, y)$ (red) and $Q_m(g; x, y)$ (Lime) to $g(x, y)$ (blue).

Now, the maximum error values calculated by the flowing table between the test function $f(x, y), g(x, y)$ and NN operators in $\mathbb{R}^2$:

Table 1: Maximum error.

NN	n	$m = 1$	$m = 2$	$m = 3$
$F_n(f; x, y)$	10	0.757395453	0.757395451	0.757395451
$Q_m(f; x, y)$		0.490870074	0.187277077	0.019435682
$f_n(g; x, y)$		0.4733567674	0.4733567674	0.4733567674
$Q_m(g; x, y)$		0.2810197385	0.0785342121	0.0091303333
$F_n(f; x, y)$	30	0.185638151	0.185638151	0.185638151
$Q_m(f; x, y)$		0.091746328	0.024034970	0.000333905
$F_n(g; x, y)$		0.1551244367	0.1551244367	0.1551244367
$Q_m(g; x, y)$		0.0888466026	0.01026517101	0.0003333514

5. Conclusions

From Table 1 above and the two numerical examples, The NN operators $Q_m(.; x, y)$ better the classical NN operators $F_n(.; x, y)$ in terms of numerical results for the two test functions f and g.

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