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Restrained 2-Resolving Dominating Sets in the Join, Corona and Lexicographic Product of two Graphs

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Abstract. Let G be a connected graph. An ordered set of vertices $\{v_1, ..., v_l\}$ is a 2-resolving set for G if, for any distinct vertices $u, w \in V(G)$, the lists of distances $(d_G(u, v_1), ..., d_G(u, v_l))$ and $(d_G(w, v_1), ..., d_G(w, v_l))$ differ in at least 2 positions. A set $S \subseteq V(G)$ is a *restrained 2-resolving dominating set* in G if S is a 2-resolving dominating set in G and $S = V(G)$ or $\langle V(G)\backslash S\rangle$ has no isolated vertex. The *restrained 2R-domination number* of G, denoted by $\gamma_{r2R}(G)$, is the smallest cardinality of a restrained 2-resolving dominating set in G. Any restrained 2-resolving dominating set of cardinality $\gamma_{r2R}(G)$ is referred to as a γ_{r2R}*-set* in G. This study deals with the concept of restrained 2-resolving dominating set of a graph. It characterizes the restrained 2-resolving dominating set in the join, corona and lexicographic product of two graphs and determine the bounds or exact values of the restrained 2-resolving domination number of these graphs.

2020 Mathematics Subject Classifications: 05C62

Key Words and Phrases: Restrained 2-resolving set, restrained 2-resolving dominating set, join, corona, lexicographic product of two graphs

1. Introduction

The problem of uniquely determining the location of an intruder in a network was the principal motivation of introducing the concept of metric dimension in graphs by Slater [9], where the metric generators were called locating sets. The concept of metric dimension of a graph was also introduced independently by Harary and Melter in [4] where metric generators were called resolving sets.

Bailey and Yero in [1] demonstrated a construction of error-correcting codes from graphs by means of k-resolving sets, and present a decoding algorithm which makes use

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of covering designs.

The distance between two vertices u and v of a graph is the length of a shortest path between u and v, and we denote this by $d_G(u,v)$. In recent years, much attention has been paid to the *metric dimension* of graphs: this is the smallest size of a subset of vertices (called a *resolving set*) with the property that the list of distances from any vertex to those in the set uniquely identifies that vertex and is denoted by $\dim(G)$.

According to the paper of Saenpholphat et al. [8], for an ordered set of vertices $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex v in G, the k-vector (ordered k-tuple)

$$r(v/W) = (d_G(v,w_1), d_G(v,w_2), ..., d_G(v,w_k))$$

is referred to as the *(metric) representation of v with respect to W*. The set W is called a *resolving set* for G if distinct vertices have distinct representation with respect to W. Hence, if W is a resolving set of cardinality k for a graph G of order n, then the set $\{r(v/W) : v \in V(G)\}$ consists of n distinct k-vectors. A resolving set of minimum cardinality is called a *minimum resolving set* or a *basis*, and the cardinality of a basis for G is the *dimension* $\dim(G)$ of G.

In the paper of Rara and Cabaro [5], an ordered set of vertices $W = \{w_1, ..., w_l\}$ is a 2-*resolving set* for G if, for any distinct vertices $u, v \in V(G)$, the (metric) representations $r(u/W)$ and $r(v/W)$ of u and v, respectively differ in at least 2 positions. Then W is said to be a 2-resolving set for G. If G has a 2-resolving set, the minimum cardinality $\dim_2(G)$ is called the 2-*metric dimension* of G. If $k = 2$ is the largest integer for which G has a 2-resolving set, then we say that G is a 2-*metric dimensional* graph. In the paper of Cabaro and Rara [6], the concept of restrained 2-resolving set in the join, corona and lexicographic product of two graphs was discussed.

In this paper, the concept of restrained 2-resolving dominating set in the join, corona and lexicographic product of two graphs is discussed.

2. Preliminary Results

In this study, we consider finite, simple and connected undirected graphs. For basic graph-theoretic concepts, we refer readers to [3].

Proposition 1. [2] Let G be a connected graph of order $n \geq 2$. Then $\dim_2(G) = 2$ if and only if $G \cong P_n$.

Remark 1. For any connected graph G of order $n \geq 2$, $\gamma_{2R}(G) \leq \gamma_{r2R}(G)$.

Remark 2. For $n \geq 2$, $\gamma_{2R}(K_n) = n = \gamma_{r2R}(K_n)$.

Remark 3. Let G be a connected graph of order $n \geq 2$. Then $\gamma_{r2R}(G) = n$ if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$ or $G \cong C_4$.

The following remark follows from Proposition 1.

Remark 4. Let G be a connected graph of order $n \geq 2$. Then $\gamma_{r2R}(G) = 2$ if and only if $G \cong K_2$ or $G = P_4$.

Proof. Proof follows immediately from Proposition 1. $\square$

3. Restrained 2-Resolving Dominating Sets in the Join of Graphs

Theorem 1. Let G and H be nontrivial connected graphs. A set $S \subseteq V(G+H)$ is a restrained 2-resolving set in $G+H$ if and only if $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in G and H, respectively where S_G or S_H is a $(2,2)$-locating set or S_G and S_H are $(2,1)$-locating sets and one of the following holds:

(i) $S_G = V(G)$ and S_H is a restrained 2-locating set in H;

(ii) $S_H = V(H)$ and S_G is a restrained 2-locating set in G;

(iii) $S_G \neq V(G)$ and $S_H \neq V(H)$.

Theorem 2. [7] Let G be a connected non-trivial graph and let $K_1 = \{v\}$. Then $S \subseteq V(K_1 + G)$ is a restrained 2-resolving set of $K_1 + G$ if and only if either $v \notin S$ and S is a $(2,2)$-locating set in G with $V(G) \neq S$ or $S = \{v\} \cup T$, where T is a restrained $(2,1)$-locating set in G.

Theorem 3. Let G and H be nontrivial connected graphs. A proper subset S of $V(G+H)$ is a restrained 2-resolving dominating set in $G+H$ if and only if S is a restrained 2-resolving set in $G+H$.

Proof. Let $S \subseteq V(G+H)$ be a restrained 2-resolving dominating set in $G+H$. Then S is a restrained 2-resolving set in $G+H$.

Conversely, if S is a restrained 2-resolving set in $G+H$, then by Theorem 1, conditions (i), (ii) and (iii) hold. Since S is a 2-resolving set, $S_G \neq \varnothing$ and $S_H \neq \varnothing$. Thus, $S = S_G \cup S_H$ is a dominating set in $G+H$. Therefore, S is a restrained 2-resolving dominating set in $G+H$. $\square$

Corollary 1. Let G and H be a connected non-trivial graphs of order m and n, respectively. Then

$$\gamma_{r2R}(G+H) = r\dim_2(G+H).$$

The set consisting of the shaded vertices in Figure 1 is a restrained 2-resolving dominating set of the join $P_5 + P_6$.

$P_5 + P_6$:

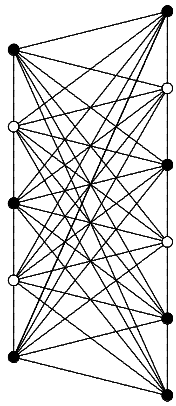

Figure 1: A graph $P_5 + P_6$ with $\gamma_{r2R}(P_5 + P_6) = 7$

Theorem 4. Let G be a connected non-trivial graph and let $K_1 = \{v\}$. Then $S \subseteq V(K_1 + G)$ is a restrained 2-resolving dominating set in $K_1 + G$ if and only if it is a restrained 2-resolving set in $K_1 + G$.

Proof. Suppose S is a restrained 2-resolving dominating set in $K_1 + G$. Then S is a restrained 2-resolving in $K_1 + G$.

Conversely, if S is a restrained 2-resolving set in $K_1 + G$, then by Theorem 2 either $v \in S$ and S is a $(2,2)$-locating set in G with $V(G) \neq S$ or $S = \{v\} \cup T$, where T is a restrained $(2,1)$-locating set in G. Hence, S is a restrained 2-resolving dominating set in $K_1 + G$. $\square$

Corollary 2. Let G be a connected nontrivial graph of order m. Then

$$\gamma_{r2R}(K_1 + G) = \begin{cases} 1 + m, \text{ if } ln_{(2,2)}(G) = m \text{ and } rln_{(2,1)}(G) = m \\ \min\left\{ln_{(2,2)}(G), rln_{(2,1)}(G) + 1\right\}, \text{ otherwise }. \end{cases}$$

4. Restrained 2-Resolving Dominating Sets in the Corona of Graphs

Theorem 5. [7] Let G and H be nontrivial connected graphs. A set $S \subseteq V(G \circ H)$ is a restrained 2-resolving set in $G \circ H$ if and only if $S = A \cup (\bigcup_{v \in V(G)} S_v)$ satisfying the following conditions.

(i) $A \subseteq V(G)$

(ii) S_v is a 2-resolving set for each $v \in V(G) \backslash A$

(iii) S_v is a restrained 2-resolving set for each $v \in A$

(iv) $w \in N_G(V(G) \backslash A)$ for each $w \in V(G) \backslash A$ with $S_w = V(H^w)$.

Theorem 6. Let G and H be nontrivial connected graphs. Then $S \subseteq V(G \circ H)$ is a restrained 2-resolving dominating set in $G \circ H$ if and only if $S = A \cup (\bigcup_{v \in A} S_v)(\bigcup_{w \in V(G) \setminus A} S_w)$ satisfying the following conditions.

(i) $A \subseteq V(G)$

(ii) S_w is a 2-resolving dominating set in H^w for each $w \in V(G) \setminus A$

(iii) S_v is a restrained 2-resolving set in H^v for each $v \in A$

(iv) $w \in N_G(V(G) \setminus A)$ for each $w \in V(G) \setminus A$ with $S_w = V(H^w)$.

Proof. Suppose S is a restrained 2-resolving dominating set in $G \circ H$. By Theorem 5, conditions (i), (ii), (iii) and (iv) hold.

Conversely, suppose $S = A \cup (\bigcup_{v \in A} S_v)(\bigcup_{w \in V(G) \setminus A} S_w)$ satisfying the conditions (i),(ii),(iii) and (iv). Then by Theorem 5, S is a restrained 2-resolving set in $G \circ H$. Let $x \in V(G \circ H) \setminus S$ and let $w \in V(G)$ such that $x \in V(w + H^w)$. If $w \in S$, then $xw \in E(G \circ H)$. If $w \notin S$, then $x \notin S_w = S \cap V(H^w)$, where S_w is a 2-resolving dominating set in H^w. Thus, there exists $z \in V(H^w) \cap S_w$ such that $xz \in E(G \circ H)$. Therefore, S is a dominating set in $G \circ H$. Thus, S is a restrained 2-resolving dominating set in $G \circ H$. □

Corollary 3. Let G and H be nontrivial connected graphs, where $|V(G)| = n$. Then

$$\gamma_{r2R}(G \circ H) \leq min \{|V(G)| \cdot \gamma_{2R}(H), |V(G)|(1 + r\dim_2(H))\}.$$

The set consisting of the shaded vertices in Figure 2 is a restrained 2-resolving dominating set of the corona $P_4 \circ C_5$.

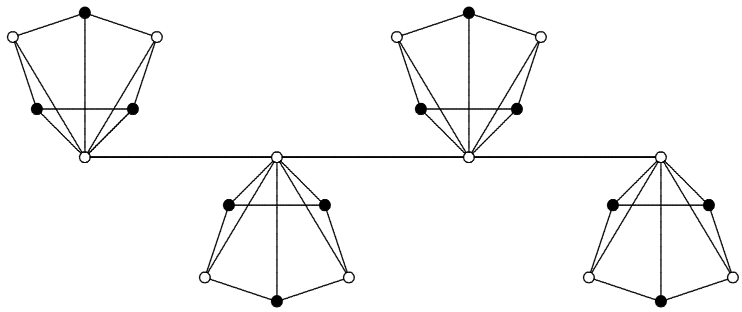

Figure 2: A graph $P_4 \circ C_5$ with $\gamma_{r2R}(P_4 \circ C_5) = 12$

5. Restrained 2-Resolving Dominating Sets in the Lexicographic Product of Graphs

Theorem 7. [7] Let G and H be non-trivial connected graphs. Then $W = \bigcup_{x\in S}\big[\{x\}\times T_x\big]$, where $S\subseteq V(G)$ and $T_x\subseteq V(H)$ for each $x\in S$, is a restrained 2-resolving set in $G[H]$ if and only if

(i) $S = V(G)$

(ii) T_x is a 2-locating set in H for all $x\in V(G)$;

(iii) T_x is a restrained 2-locating set for each x with $T_y = V(H)$, for all $y\in N_G(x)$;

(iv) T_x and T_y are $(2,1)$-locating sets or one of T_x and T_y is a $(2,2)$-locating set in H whenever $x, y\in EQ_1(G)$; and

(v) T_x and T_y are (2-locating) dominating sets in H or if one of T_x and T_y, say T_x is not dominating, then T_y is 2-dominating whenever $x, y\in EQ_2(G)$.

Theorem 8. Let G and H be non-trivial connected graphs. Then $W = \bigcup_{x\in S}\big[\{x\}\times T_x\big]$, where $S\subseteq V(G)$ and $T_x\subseteq V(H)$ for each $x\in S$, is a restrained 2-resolving dominating set in $G[H]$ if and only if it is a restrained 2-resolving set in $G[H]$.

Proof. Suppose $W = \bigcup_{x\in S}\big[\{x\}\times T_x\big]$ where $S\subseteq V(G)$ and $T_x\subseteq V(H)$ is a restrained 2-resolving dominating set in $G[H]$. Then W is a restrained 2-resolving set in $G[H]$.

For the converse, suppose that W is a restrained 2-resolving set in $G[H]$. Then by Theorem 7, (i)-(v) hold. Since W is a 2-resolving set, $T_x\neq\varnothing$ for every $x\in V(G)$. Thus, W is a dominating set in $G[H]$. Therefore, W is a restrained 2-resolving dominating set in $G[H]$. □

The following are the direct consequences of Theorem 8.

Corollary 4. Let G and H be nontrivial connected graphs such that G is not free-equidistant. Then,

$$\gamma_{r2R}(G[H]) = r\dim_2(G[H]).$$

The following result follows from Theorem 8.

Corollary 5. Let G and H be non-trivial connected graphs such that G is free-equidistant. Then

$$\gamma_{r2R}(G[H]) = r\dim_2(G[H]).$$

The set consisting of the shaded vertices in Figure 3 is a restrained 2-resolving dominating set of the lexicographic product $P_4[P_3]$.

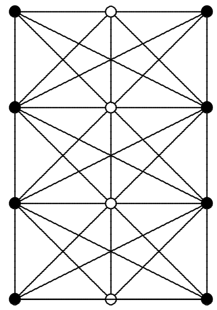

Figure 3: A graph $P_4[P_3]$ with $\gamma_{r2R}(P_4[P_3]) = 8$

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